# History Without Technical Change. Distribution, Productive Inheritance, and the Order of Reproduction in Sraffian Systems



*Published version:*

DOI: 10.2139/ssrn.7387458



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# History Without Technical Change

## Distribution, Productive Inheritance, and the Order of Reproduction in Sraffian Systems

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SSRN Working Paper — September 2026

### Abstract

Can an economy have a genuine history even when its technical coefficients do not change? This paper develops a fixed-coefficient theory of productive path dependence. It distinguishes known production methods from the productive structure inherited from the past. Temporary distributive conditions can alter which renewal activities are profitable and therefore the material composition of productive capacity when the same distributive environment returns. History can enter reproduction without entering technology.

In a Sraffian framework, the paper separates three meanings of recovery: feasible reconstruction, first restoration under the quantity rule actually followed, and durable restoration. Exact constructions show that these dates can diverge sharply and that inherited capital composition, not aggregate capital alone, governs recovery.

The paper also links distributive history to the order of reproduction. Conditional on a distributive sequence and stated closures, a fixed activity catalogue determines competitive supports, wages, physical state maps, and inherited composition. Reversing the sequence can leave aggregate terminal stock unchanged yet alter its composition; under a stated recurrent-capacity rather than maximum-profit criterion, a terminal activity problem can support a different non-cyclic ordering of the same recurrent phases, with different Sraffian Standard properties. The mechanism holds throughout the regular two-parameter distributive region.

The result is path dependence based on productive inheritance rather than technological change, learning, or increasing returns. Demand, saving, utilization, finance, and the distributive sequence itself remain exogenous. **Keywords:** productive inheritance; path dependence; capital composition; reproduction; distribution; productive capacity; Sraffian economics; joint production. **JEL:** B51; B52; C67; D24; D33.

*I thank Mauro Maltagliati for independently checking the mathematical proofs and computational certificates. Generative AI assisted drafting, code generation, symbolic and numerical cross-checks, and editing, but was not treated as an authority for mathematical or bibliographical claims. Responsibility for the final text and any remaining errors is mine.

## Contents

# PROLOGUE — WHY HISTORY MATTERS EVEN WHEN TECHNOLOGY DOES NOT CHANGE

### P.1 The economic question before the Sraffian language

The paper begins from a question that belongs to heterodox economics more broadly than to any one school. Suppose an economy's engineering knowledge is unchanged. The same production methods are known before and after a disturbance. Can the economy nevertheless emerge from that disturbance with a different set of economically relevant productive possibilities?

The answer developed here is yes. Knowing how to produce something and inheriting the productive structure needed to reproduce it are different facts. Firms may know the same blueprints, recipes, maintenance routines, and input coefficients, while the economy inherits a different composition of machines, intermediate stocks, productive stages, and complementary capacities. The historical object is therefore not technical knowledge alone. It is *productive inheritance*: the dated composition of means of production with which the next round of production must actually begin.

This distinction matters because production is sequential. What is operated today changes what exists tomorrow. If profitability conditions alter which assets are renewed, which processes are run, or which intermediate stages are maintained, a temporary distributive episode can leave a material residue after the episode itself has ended. The same profit rate can return without restoring the same productive state. In that precise sense, fixed technical coefficients do not imply a historically fixed economy.

This is the central claim of the paper. The formal apparatus is built to keep it from becoming a metaphor. The question is not whether "history matters" in some unrestricted sense. It is which part of history can matter in a fixed-coefficient production system, through what mechanism, for how long, and under which explicit closure rules.

### P.2 A simple mechanism: profitability changes what is renewed

Consider a sector whose productive capacity is distributed across complementary stages or cohorts. Some processes renew an old machine; some run it down. Some preserve an intermediate stage; others use it to obtain current output. Some activities are profitable under one distributive position and not under another. Nothing in this description requires a change in technical coefficients. It requires only that different distributive conditions support different uses of the same catalogue of activities.

The sequence of distributive positions can then matter. If an environment favourable to renewal comes first, firms may preserve or rebuild the productive links that will be needed later. If the same environments occur in the opposite order, those links may be depleted before the favourable environment arrives. The difference is not that agents forgot a technique. It is that the economy arrives at the comparison date with a different productive composition.

The formal chain is:

$$\begin{aligned}\text{distributive sequence} &\longrightarrow \text{competitive activity support}\\ &\longrightarrow \text{physical transformation of stocks}\\ &\longrightarrow \text{inherited productive composition.}\end{aligned}$$

The integrated construction in this paper goes one step further. The inherited composition can change which recurrent organization of production is supported afterwards. Thus the chain can continue:

$$\begin{aligned}\text{inherited composition} &\longrightarrow \text{supported recurrent organization}\\ &\longrightarrow \text{long-period properties of that organization.}\end{aligned}$$

This is why the paper treats distribution as historically consequential without making technology itself a function of distribution. Distribution affects the path through the catalogue; the path transforms the stocks; the stocks condition subsequent reproduction.

### P.3 Why composition matters more than a scalar "capital stock"

One of the strongest exact results is also one of the easiest to state without specialized terminology. Two histories can finish with exactly the same aggregate amount of productive stock and still leave different productive possibilities because the composition is different.

In the integrated witness, the histories conventionally labelled $PE$ and $EP$ use the same two distributive positions once each, in opposite order. They begin from the same state. At the comparison date they have exactly the same aggregate stock, $66/25$, but its two components are reversed. A terminal activity problem then supports one recurrent ordering of production phases after $PE$ and another after $EP$.

The point is not the particular fractions. It is the ablation behind them. If the two-dimensional inherited state is replaced by its scalar aggregate, the historical distinction disappears. The mechanism therefore cannot be represented by an aggregate quantity of capital alone. Productive history is carried by composition.

This observation should be recognizable from several heterodox traditions. A Marxian reader may think of the material composition through which accumulation and valorization have to be reproduced. An institutionalist may think of historically accumulated productive capabilities and complementarities that survive a policy or profitability episode. A structuralist may think of sectoral bottlenecks and interdependent capacities that a scalar aggregate conceals. An evolutionary economist may recognize path dependence without needing innovation or increasing returns as the primitive source of the path effect. The model does not identify these traditions with one another. It isolates a production-theoretic mechanism they can all interpret: inherited productive composition constrains what can be reproduced next.

### P.4 Recovery is not one event but three different questions

Once history has changed the productive state, saying that an economy has "recovered" is ambiguous. The paper separates three questions.

The first is technological and organizational: *when does a route back become feasible?* There may exist some admissible sequence of activities that reconstructs the reference productive position even if the economy is not actually following it.

The second is behavioural or closure-dependent: *when does the quantity rule actually followed by the economy first restore the reference position?* A feasible road need not be the road taken.

The third is stronger: *when is the reference position restored and never subsequently lost again?* First passage and durable restoration are different. A rule can cross a benchmark once and then destroy the very composition it has just rebuilt.

These distinctions produce the three formal clocks used later in the paper. Their symbols are useful in proofs, but the economic content is simply:

$$\boxed{\text{possible recovery} \neq \text{first actual recovery} \neq \text{durable recovery.}}$$

The exact examples show all three possibilities. In one construction the best feasible reconstruction takes three periods, while the stated myopic rule never realizes it. In another, the rule crosses the reference after two periods but later falls below it again. In the proper-cone multidimensional construction, feasible reconstruction occurs at date 37 whereas the realized accumulation rule restores the reference only at date 55, after which restoration persists. The numbers are examples; the conceptual separation is the portable result.

This also clarifies the use of the word "irreversibility". Some mechanisms genuinely destroy a required productive support and create an absorbing boundary. Others merely delay reconstruction. Still others allow first passage without persistent restoration. The paper keeps these cases separate rather than describing every long delay as lock-in.

### P.5 The counterfactual matters: restoration is not the same as catching up

Historical statements require a comparison path. The paper therefore distinguishes two counterfactual questions.

One can freeze the productive capacity that a no-disturbance economy possessed at the comparison date and ask when the historical economy restores that dated benchmark. This is a question about reconstruction of a lost position.

Alternatively, one can allow the no-disturbance economy to continue moving and ask when the historical branch catches it. This is a question about relative catch-up. The second benchmark is not automatically more demanding or more informative. If both branches are subsequently pushed through the same positive linear transformation and one starts componentwise behind the other, no overtaking is possible by monotonicity alone. Calling that failure "historical persistence" would add an interpretation to a simple order-preserving property.

For a broad heterodox audience, the distinction is familiar in other language. Rebuilding the productive capacity lost during a crisis is not the same as closing the gap with the path that would have occurred without the crisis. Restoring an industrial base is not the same as catching a moving productivity frontier. The paper formalizes the analogous distinction inside a fixed-coefficient reproduction model.

### P.6 Two meanings of order, connected through productive inheritance

There are two temporal orderings in the paper.

The first is external to the recurrent production system: distributive positions occur in different sequences. Because those positions support different activities, reversing their order can change the inherited productive state.

The second is internal to a recurrent organization: the same production phases can themselves be repeated in different non-cyclic orders. With three or more phases, changing their temporal adjacency is not merely changing the date at which we begin observing the same cycle. Different non-cyclic orders can have different reproduction and valuation properties.

The important result is that these two meanings of order need not remain separate. In the exact support-to-order construction, one fixed grand catalogue is used throughout. Conditional on the distributive word, competitive conditions determine the active traverse processes, wages, and wage-net physical maps. Their composition determines the inherited stock. A terminal activity-analysis problem then selects which of two non-cyclic recurrent orders of the same phase catalogue can be supported. Thus:

$$\begin{aligned}\text{order of distributive conditions} &\longrightarrow \text{order of supported activities}\\ &\longrightarrow \text{composition inherited from the traverse}\\ &\longrightarrow \text{order of recurrent production.}\end{aligned}$$

The two histories leave the same aggregate stock but select different recurrent orders, conventionally labelled $ABC$ and $ACB$. Those recurrent orders use exactly the same three elementary phase technologies once each; only their non-cyclic order differs. They have different internal Sraffian Standard properties. The final selection step uses an explicit recurrent-capacity closure rather than a maximum-profit ranking; Section 33A explains why that distinction matters.

The abstract composition lemma is deliberately modest: it states when such links persist once strict supports are in place. The exact rational witness performs the stronger primitive derivation. This distinction prevents the paper from assuming the support separation that it claims to explain.

### P.7 What is Sraffian here, and what is broader than Sraffa

A reader does not need to be a specialist in the Standard commodity to understand the economic argument. The broad claim is about reproduction, inherited productive structure, and the historical effects of a sequence of profitability conditions. Sraffian theory supplies a particularly disciplined environment in which to ask the question because technical coefficients can remain fixed while distribution and competitive support are treated explicitly.

The paper uses that discipline in two ways. First, prices and the distributive variable help determine which activities are competitively supportable during a traverse; they do not by themselves determine the quantity path. Second, once a recurrent organization is specified, Sraffian long-period objects – maximum profit factors, Standard constructions, basicity, and joint-production realizability – can be studied for that historically selected organization.

This division of labour is central. Long-period price equations do not secretly become a complete short-period dynamics. The quantity path requires a stated closure. Conversely, the inherited quantity state does not replace price and distribution theory. It conditions the domain on which long-period comparisons become economically relevant.

For a Marxian reader, the useful proposition is not that the paper supplies a theory of exploitation or accumulation. It does not. The proposition is narrower: even with a fixed technical catalogue, the sequence of profitability conditions can alter the material structure through which reproduction must proceed. The same technical knowledge need not reproduce the same effective productive configuration.

For an institutionalist or evolutionary reader, the mechanism provides a form of path dependence without building the result out of changing preferences, learning curves, institutional mutation, or increasing returns. Institutions or policy regimes may matter by changing the profitability sequence that governs renewal and scrapping; the model then traces the material inheritance of that sequence.

For a Post-Keynesian or structuralist reader, the paper's limitation is equally important. Effective demand, utilization, saving, finance, and investment expectations are not endogenized. The construction identifies a productive channel through which those forces could leave durable compositional effects; it does not replace theories that explain the forces themselves.

### P.8 What the exact results establish

The paper contains several examples because no single calibration should be forced to establish logically different propositions.

The first multistage witness proves a genuine order effect and an exact minimum reconstruction horizon. It also reveals that a particular myopic quantity rule is destructive. That collapse is treated as a diagnosis of the closure, not as evidence that history must permanently trap the economy.

A primitive perturbation removes the pure periodicity of the target graph and separates first passage from persistent restoration. It shows that an economy can temporarily regain a benchmark and then lose it again.

A separate rank-two example places optimal reconstruction on a proper active cone. Both inherited goods carry positive shadow values. The dominant and subdominant recovery modes both matter, and the spectral gap sharply improves the date at which feasible reconstruction can be certified. The same example separates the growth factor of the Sraffian valuation system from the factors governing optimal reconstruction and the slower realized closure.

The integrated support-to-order witness then connects the external and internal parts of the programme. A fixed external activity catalogue has one unique competitive support in one distributive region and another support in the other. The corresponding physical maps do not commute. Reversing the distributive sequence changes terminal composition while leaving aggregate stock unchanged. The terminal capacity problem consequently supports a different non-cyclic recurrent order. The result is not confined to a single numerical point: the inequalities remain strict throughout the regular two-parameter distributive rectangle derived from the catalogue.

Finally, the internal periodic module shows why ordered reproduction has its own structure. Cyclic rotations merely change the calendar origin and preserve the relevant spectrum. Non-cyclic permutations can change the Standard ratio. Under joint production they can also change whether a positive-price Standard construction is physically realizable with non-negative process intensities. The distinction between a formal value solution and an operable production system therefore matters inside the recurrent system itself.

### P.9 Why the mathematics is layered rather than ornamental

Several mathematical languages appear in the paper because they answer different economic questions.

Linear programming and complementarity describe scarcity, support, and reconstruction. Polyhedral geometry describes which inherited compositions make a recurrent organization feasible. Perron–Frobenius theory describes dominant reproductive directions and rates inside positive linear systems. Spectral gaps describe how fast transients die out when an active reconstruction basis repeats. Dated graph and matrix-pencil methods distinguish calendar relabelling, basicity, positive prices, and physical realizability under joint production.

These are not interchangeable metaphors. A price vector is not an inherited stock. A Perron root of a recurrent production system is not automatically the growth factor of a chosen behavioral closure. A feasible reconstruction path is not the same thing as the path generated by a quantity rule. A positive-price joint-production root is not by itself proof that all required processes can operate at non-negative scale.

The value of the formal structure is precisely to block these shortcuts. The paper's broader message becomes more credible, not less, when the different objects are kept distinct.

### P.10 What the paper does not claim

The strongest defensible interpretation is conditional. The distributive word is taken as given. Conditional on that sequence and on the stated traverse and terminal closures, competitive support determines wage-net maps; their non-commuting composition changes inherited productive composition; and a stated terminal recurrent-capacity problem can select a different recurrent order with different internal Sraffian properties. That selector is not identified with classical maximum-profit technique choice.

The paper does not explain why society moves through one distributive sequence rather than another. It does not derive saving, investment demand, utilization, final demand, finance, expectations, or bargaining institutions. It does not claim a unique economy-wide short-period equilibrium. It does not treat every slow recovery as irreversible, and it does not turn a stochastic matrix product into a Sraffian maximum-profit rate by relabelling its Lyapunov exponent.

These are limits, but they also locate the contribution. The model isolates a production-theoretic channel of historical dependence that can be inserted into richer theories of distribution, institutions, accumulation, and demand. It shows what productive history can do before those additional mechanisms are specified.

### P.11 A map for different readers

A reader interested mainly in path dependence, institutions, or structural change can read this Prologue, Part I, the historical-domain and reconstruction results, the integrated support-to-order section, and the concluding synthesis. The mathematical appendices can be treated as certification.

A reader interested in Marxian reproduction can focus on the distinction between technical knowledge and productive inheritance, the role of distributive conditions in renewal and scrapping, the compositional nature of inherited means of production, and the separation between feasible, realized, and durable reconstruction.

A reader interested in Sraffian theory can proceed to the periodic Standard construction, dated basicity, joint-production realizability, and the exact relation between external support selection and internal phase ordering.

A reader interested in mathematical economics can read the paper as a sequence of positive-system, parametric-LP, complementarity, Perron, and matrix-pencil results, with exact certificates separating imported mathematical infrastructure from the genuinely production-theoretic propositions.

The shortest statement of the programme is this: **history need not change the technical coefficients; it can change the productive structure that those coefficients act upon.** Once productive structure has an order, history can affect not only which technique is reachable, but which recurrent organization is economically supportable.

# PART I — RETHINKING THE "GIVEN SYSTEM"

## 1. Sraffa's austerity and the problem of historical reproduction

Sraffa's starting point is deliberately austere. Methods of production are given: definite quantities of commodities and labour are required to produce definite quantities of commodities. From those physical data, together with a distributive variable, one studies prices of production, wage–profit relations, alternative methods, the Standard commodity, and basic and non-basic commodities. The gain from this austerity is fundamental. Distribution is not derived from the marginal productivity of an aggregate quantity of capital, and heterogeneous means of production need not first be valued as a scalar factor endowment.

The apparent cost is that history seems to disappear. If the methods remain given, it is tempting to assume that whenever the same distributive conditions return, the same productive alternatives automatically return with them. That inference is stronger than the fixed-coefficient premise. A catalogue records what is technically known. It does not by itself determine the dated composition of inherited means of production, which activities are competitively worth operating during a transition, or how long reconstruction of a recurrent configuration takes after the capital structure has been transformed.

The programme distinguishes **technical knowledge** from **historically effective reproduction**. The distinction must itself be disciplined. Under constant returns, divisibility, and free suballocation, any strictly positive inherited state contains a sufficiently small scaled copy of every positive recurrent bundle. If "available" means only "seedable at some positive scale", robust interior exclusion collapses. The Interior Seedability Lemma is a conceptual boundary condition: it determines the form that a non-degenerate historical question must take without being presented as a deep mathematical theorem.

That question is a traverse question. A simple economic reading is a productive district whose firms still know the previously dominant method but inherit, after a temporary sequence of profitability conditions, the wrong composition of machines, stages, or complementary inputs to operate it immediately at a relevant scale. The issue is not technological forgetting. It is whether the inherited productive state permits reconstruction, which path can achieve it, and whether the quantity rule actually followed by the economy preserves the reconstructed position.

The comparison requires an explicit counterfactual. Let $x^\circ$ be the state at the beginning of a $T$-period historical comparison and let $\kappa^0$ be a specified no-excursion history leading to the same comparison date. If $\tau^G$ is the target recurrent technique at the final distributive position $r_f$, define the reference state

$$x_T^0 = \Phi^c_{\kappa^0}(x^\circ).$$

Two benchmarks must then be distinguished. The *comparison-date baseline* freezes the target capacity that the no-excursion branch possesses at date $T$. The *moving counterfactual* continues the no-excursion branch beyond $T$ under a stated reference closure and compares the two branches again at each later date. Restoration to a dated baseline and catch-up with a moving counterfactual are different economic questions; neither is treated as a law of nature.

Wage payment is physical. In the central multistage example workers receive the wage in $K_0$ and consume it before the next productive state is inherited. Distribution can alter transition quantities without changing a technical coefficient: a different wage and a different competitively supported renewal process change how much gross output remains as productive stock.

The external chain is

$$\begin{aligned}\text{distribution and continuation} &\longrightarrow \text{dated profitability of fixed activities}\\ &\longrightarrow \text{operated renewal processes}\\ &\longrightarrow \text{net inherited dated-stage state}\\ &\longrightarrow \text{capacity relative to } x_T^0\\ &\longrightarrow \text{baseline restoration or counterfactual catch-up.}\end{aligned}$$

History enters in reproduction, not in the technical coefficients.

A second historical problem is internal. Once a recurrent organization is specified, it may itself be an ordered periodic system rather than a one-period stationary matrix. Dated commodities then become natural objects and one may ask what becomes of Sraffa's Standard commodity, maximum rate, and basicity. Under joint production one must also distinguish a positive dated-price vector from a non-negative vector of Standard process multipliers.

The programme contains two analytically distinct meanings of order. External history concerns the temporal order of distributive positions and the competitive traverse supports they induce. Internal history concerns the non-cyclic order of phases inside a recurrent organization. Section 33A composes them in one block support problem: conditional only on the distributive word, complementarity determines the active traverse processes, wages, physical maps and inherited composition, and a terminal activity-analysis KKT system selects the recurrent phase order. The integrated witness keeps the two notions conceptually distinct while linking them causally through inherited composition.

## 2. Five objects, two historical layers, one discipline

The reconstruction keeps five economic objects distinct.

1. The **technical catalogue** $\mathcal{T}$: fixed input, output, labour, and mandatory carry-over coefficients.
2. The **inherited productive state** $x$: reproducible commodity stocks, including vintages where chronology matters.

3. A **recurrent organization** $\tau \in \Omega$: a complete reproducible arrangement generated by the catalogue.
4. A **traverse closure**: a rule for transition quantities on the process set supported by dated competitive prices. Sraffian price equations do not determine these quantities by themselves.
5. The **internal Sraffian packet** $\mathfrak{I}(\tau)$: wage curve and, where defined, Standard ratio, Historical Standard Cycle, dated basic structure, and joint-production realizability status.

The external module adds a comparison construction, not a technical primitive. A reference history $\kappa^0$ and a comparison date $T$ generate $x^0_T$. From that state one may define either a fixed comparison-date baseline or a moving no-excursion counterfactual. The operational chain is

$$\mathcal{T} \longrightarrow (x^\circ, \kappa^0, h) \longrightarrow (x^0_T, x^h_T) \longrightarrow \Omega^{\text{rest}}_H(x^h_T, r_f \mid x^0_T) \longrightarrow \mathfrak{I}(\tau).$$

The scale is endogenous conditional on the stated counterfactual path; the choice of the counterfactual itself is an economic identification assumption and is reported as such.

Three distinctions discipline the argument. Material self-reproduction is not a wage-ranking criterion. Long-period wage curves are not short-period traverse wages. LP dual variables used to certify restoration bounds are not Sraffian production prices unless a separate decentralization theorem says so.

## 3. The traverse is the missing genealogy of the external module

The closest conceptual ancestor of the external module is the traverse tradition. Hicks and Lowe asked how an economy moves between productive configurations when inherited capital is not already composed as required by the destination. Gehrke and Hagemann made bottlenecks and efficient traverses explicit in structural settings. Later post-Keynesian discussions emphasize the same logical difficulty: a long-period position is economically meaningful only if there is an account of the out-of-equilibrium movement toward it and of whether that movement changes what can subsequently be reproduced. Amendola and Gaffard place sequential restructuring and viability at the centre of an explicitly out-of-equilibrium production process: their constraints–decisions–constraints sequence is a particularly close antecedent for the distinction drawn here between feasible reconstruction and the path generated by a specified closure.

The present framework isolates a fixed-coefficient mechanism within the broader traverse problem; saving, investment demand, utilization, and finance remain outside the model. Mandatory ageing and dated competitive support make inherited dated-stage composition operative. The state transition can depend on the temporal order of distributive regimes even when the catalogue of activities is fixed.

This genealogy also clarifies switching and reswitching. A conventional Sraffian switch changes which wage curve is highest at a given rate of profit. By itself it does not create memory. A traverse creates memory only when the temporary regime changes the inherited productive state or the support of the activities through which a recurrent position must be reconstructed.

Nor is the mechanism identical to David–Arthur lock-in. Increasing returns, network effects, and adoption externalities are not required. In the interior witness below, history acts through stage-specific quantity transformation and finite reconstruction time. In the boundary witness, it acts through destruction of a productive siphon. The mathematical vocabulary of path dependence is useful, but the production-theoretic content is more specific.

## 4. Main results and structure

The repertoire is organized by logical status: elementary boundary results, general regular-cell results, exact witnesses, classical consequences, and open integrations are kept distinct.

**First: interior seedability.** Under divisibility and free suballocation, positive-scale restartability collapses in the strict interior. This is an elementary boundary lemma for the homogeneous theory; it forces any non-degenerate historical criterion to fix an economically meaningful scale.

**Second: two benchmarks and three clocks.** A stated no-excursion history generates a fixed comparison-date baseline and, if continued, a moving counterfactual. For the fixed baseline, $H^{\min}$ measures minimum feasible reconstruction, $D^{\chi}$ first passage under a stated closure, and $\bar{D}^{\chi}$ persistent restoration. When the closure-generated path is admissible,

$$H^{\min} \leq D^{\chi} \leq \bar{D}^{\chi}.$$

**Third: exact external order effects.** The original multistage witness proves a common-metric $H_A^{\min,\phi}(EP) = 3$ by catalogue-complete LP certificates and diagnoses a stock-destructive myopic closure. A primitive period-one perturbation removes the graph-period objection and gives $D_A^{\chi,\phi}(EP) = 2$ but $\bar{D}_A^{\chi,\phi}(EP) = +\infty$.

**Fourth: genericity and comparative statics.** On a connected regular cell, a non-identically-zero valuation-visible commutator makes order-sensitive capacity open dense in primitives and positive inherited states. Strict integer-delay configurations satisfy open persistence. On the certified original cell, increasing $r_E - r_P$ raises $PE$ capacity, lowers $EP$ capacity, and widens the order gap exactly.

**Fifth: common-metric packet selection.** Cross-technique comparisons use one common positive commodity-value functional. The primitive witness supplies a stationary opportunity-level Double Historical Condition: external order changes which distinct recurrent packet reaches the same scale, without identifying that packet with a realized short-period equilibrium.

**Sixth: ordered Standard systems.** For an ordered periodic recurrent organization, the temporal lift yields a Historical Standard Cycle. The genuinely distinctive internal results are the phase-value invariance boundary in Section 29.6 and the exact joint-production result of Section 33, where non-cyclic order changes both the selected positive-price root and physical Standard realizability. Classical block-cyclic, graph, Stiemke, Frobenius, and von Neumann ingredients are identified as such.

**Seventh: Support-to-Order.** Section 33A supplies an exact support-to-order construction. A single fixed grand catalogue contains four traverse processes and three recurrent phase systems. At gross distributive factors $\mu_P = 11/10$ and $\mu_E = 6/5$, the full-catalogue zero-excess-profit complementarity problem has unique regular supports $\{P_1, P_2\}$ and $\{E_1, E_2\}$. Their wage-net maps generate equal aggregate but different terminal compositions under $PE$ and $EP$. A terminal recurrent-capacity LP, with exact KKT certificates, then selects uniquely $ABC$ after $PE$ and $ACB$ after $EP$. The internally selected orders use the same phase technologies once each and have different Historical Standard packets.

**Eighth: three spectral structures and a genuine rank-two recovery cell.** Standard monodromy, physical closure, and active-recovery transfer answer different questions. A proper-cone rank-two construction with partial accumulation has

$$1 + r = \frac{11}{10}, \qquad \rho_R = \frac{43}{40}, \qquad \rho_\chi = \frac{21}{20},$$

and $H^{\min} = 37 < D^\chi = \bar{D}^\chi = 55$. Its transverse amplitude is $K = 72/115$, so the spectral-gap bound does substantive work rather than merely restating an exact modal decomposition.

The construction is conditional on a stated distributive word. External maps and terminal recurrent order are determined under the stated traverse and terminal-capacity closures; the origin of the distributive sequence, richer utilization behavior, and stochastic extensions remain outside the model.

## 5. Scope, neighbouring literatures, and evidential status

This master repertoire is deliberately broader than a journal article. It keeps together historical traverse and productive restoration, ordered Standard systems, and joint-production realizability because the aim is to map their exact relations. The two uses of order are analytically distinct. Section 33A connects them through an exact block complementarity construction in which state maps and terminal recurrent order are solved under stated closures.

The external module belongs explicitly to the traverse tradition. The novelty claim is not that dated productive stages, vintages, or traverses are difficult. It is the exact decomposition between interior seedability, counterfactual benchmark choice, common-metric reconstruction, physical wage consumption, dated support, generic order sensitivity, and the distinction among feasible, first-passage, and persistent recovery time.

The internal module has a different mathematical genealogy. Block-cyclic spectral identities are classical periodic-system mathematics. The Sraffian question is which properties that make the

Standard commodity economically distinctive survive when the given reproductive system is an ordered periodic object. Under joint production, the problem becomes a dated matrix pencil with distinct left-price and right-multiplier conditions.

The mathematical ingredients used in the synthesis—Perron–Frobenius theory, block-cyclic products, linear programming duality, Sherman–Morrison inversion, and dynamic input–output stability arguments—are largely classical. The contribution claimed here lies in their production-theoretic synthesis: the exact connection between Sraffian valuation, wage-net physical reproduction, historically inherited state, and the recurrent Standard packet. In particular, the paper does not claim to discover dynamic Leontief instability; it identifies a specific extremal spectral role for Sraffian prices under the stated physical closure and derives a finite-viability consequence from it.

Several limits are part of the statement. The no-excursion path is an explicit identifying counterfactual, not a general-equilibrium determination of the benchmark. Labour is elastically available at the stated wage; no scarcity rent is inferred from the LP. The wage basket is explicit but final demand, saving, finance, and capitalist consumption are not derived. The full-resource transition rule is only a regular-cell closure; the paper demonstrates one alternative objective over the central comparison window rather than claiming closure invariance. The boundary siphon example remains topology-dependent. Rectangular joint-production support selection is incomplete. The positive-wage three-step factor in the original recovery LP is now an exact active-basis identity on the certified cell; what remains open is eventual periodicity of optimal bases in general. And the paper does not identify the attainable alternative recurrent opportunity with the realized traverse equilibrium.

Those boundaries define the domain of the claims made here.

# PART II — PRODUCTIVE STATE, REPRODUCTIVE CONES, AND HISTORICAL DOMAINS

## 6. Fixed activity technology and productive state

Let there be $n$ reproducible commodities and $m$ elementary activities. Activity $j$ has non-negative commodity input vector $a_j$, non-negative output vector $b_j$, and direct labour coefficient $\ell_j \geq 0$. Collect commodity inputs and outputs in matrices $A$ and $B$. Technical coefficients are fixed throughout the analysis.

At date $t$, the inherited reproducible stock is $x_t \in \mathbb{R}^n_+$. If $y_t \geq 0$ denotes activity scales, current material feasibility requires

$$Ay_t \leq x_t.$$

Unused stocks need not be frozen. A mandatory carry-over operator $D_t \geq 0$ may represent storage, depreciation, ageing, or transformation of the unused remainder. The general physical transition is

$$x_{t+1} = D_t(x_t - Ay_t) + By_t.$$

In fixed-capital applications, chronological ageing must be represented either in $D_t$ or in explicit activities. Omitting it would accidentally create a free technology for preserving a vintage indefinitely.

A recurrent organization $\tau \in \Omega$ is a complete reproducible arrangement generated by the technical catalogue. Depending on the application it may be stationary, periodic, or a support selected from a joint-production system. The catalogue $\Omega$ must be economically complete for the comparison being made. If repeating a phase or combining available methods yields another recurrent system, that system cannot be silently excluded merely because it is inconvenient for an example.

## 7. The self-reproduction factor and reproductive cones

### 7.1 Definition

Fix a recurrent organization $\tau$ of period $T_\tau$. Its phase-specific material relations are written

$$A^\tau_t y_t \leq x_t,$$

$$x_{t+1} = D_t^\tau(x_t - A_t^\tau y_t) + B_t^\tau y_t, \qquad t = 0, \ldots, T_\tau - 1.$$

Labour is not included in the reproduced state vector; it enters the price and distribution problem. For $x \neq 0$, define the **self-reproduction factor**

$$g_\tau(x) = \sup \left\{ g \geq 0 : \begin{array}{l} \exists\, x_1, \ldots, x_{T_\tau}, y_0, \ldots, y_{T_\tau - 1} \geq 0, \\ \text{satisfying the phase constraints above and } x_{T_\tau} \geq gx \end{array} \right\}.$$

The economic interpretation is narrow. $g_\tau(x)$ asks how much of the **same inherited composition** can be reproduced after one complete recurrent cycle. It is not a welfare index, a wage criterion, or an equilibrium objective.

For $\gamma \geq 0$, define

$$\mathcal{V}_\tau(\gamma) = \{x \in \mathbb{R}_+^n : g_\tau(x) \geq \gamma\}.$$

The central object is

$$\mathcal{V}_\tau := \mathcal{V}_\tau(1),$$

the **reproductive cone**: the set of productive states from which $\tau$ can reproduce at least the inherited bundle proportionally.

The value 1 is not a calibrated threshold. It is the simple-reproduction boundary. Scale disappears from the criterion because the technology is homogeneous.

### 7.2 Reproductive-Cone Geometry Theorem

**Theorem 7.1 (Reproductive-Cone Geometry).** Suppose the programme defining $g_\tau(x)$ is feasible and finite on the relevant non-zero state cone. Then, for every $\gamma \geq 0$:

1. $g_\tau(\lambda x) = g_\tau(x)$ for all $\lambda > 0$;
2. $\mathcal{V}_\tau(\gamma)$ is a closed convex polyhedral cone;
3. if $\gamma_2 > \gamma_1$, then $\mathcal{V}_\tau(\gamma_2) \subseteq \mathcal{V}_\tau(\gamma_1)$;
4. on the composition simplex $\Delta^{n-1} = \{x \geq 0 : \mathbf{1}^\top x = 1\}$, the section $\mathcal{V}_\tau(\gamma) \cap \Delta^{n-1}$ is a compact convex polytope, possibly empty;
5. $g_\tau$ is quasi-concave on the composition simplex.

**Proof.** If a trajectory witnesses the feasibility of factor $g$ from $x$, multiplying all state and activity quantities by $\lambda > 0$ witnesses the same factor from $\lambda x$. The reverse scaling gives equality, proving degree-zero homogeneity.

Fix $\gamma$. The set of all tuples $(x, x_1, \ldots, x_T, y_0, \ldots, y_{T-1})$ satisfying the phase constraints and $x_T \geq \gamma x$ is defined by homogeneous linear equalities and inequalities. It is therefore a polyhedral cone. Its projection on the initial-state coordinates is $\mathcal{V}_\tau(\gamma)$. Linear projections of polyhedral cones are polyhedral cones, hence closed and convex. Nestedness follows directly from the terminal inequality. Intersecting a closed polyhedral cone with the compact simplex gives a compact polytope. Since the superlevel sets of $g_\tau$ are convex, $g_\tau$ is quasi-concave. $\square$

The result is stronger than the simple-reproduction statement. Every fixed reproduction level has a polyhedral geometry. It is weaker than concavity of $g_\tau$, which is neither needed nor generally implied.

### 7.3 Projective interpretation

Because $g_\tau$ is homogeneous of degree zero, the natural state space is projective: productive composition rather than absolute scale. A ray $\{\lambda x : \lambda > 0\}$ is either entirely inside or entirely outside $\mathcal{V}_\tau(\gamma)$.

For a finite recurrent catalogue $\Omega$, the finitely many facets of the polytopes

$$P_\tau = \mathcal{V}_\tau \cap \Delta^{n-1}$$

generate a finite polyhedral refinement of the composition simplex. On the relative interior of each cell, the set

$$\Omega^R(x) = \{\tau \in \Omega : x \in \mathcal{V}_\tau\}$$

of immediately whole-bundle-self-reproducing organizations is constant. This statement is geometric. It must not yet be interpreted as a theorem about operational technique availability, because a divisible economy can allocate only a subbundle of its inherited stock to a recurrent system.

### 7.4 Free suballocation and the Interior Seedability Lemma

The distinction between reproduction of the *whole inherited bundle* and availability of a technique is decisive. Suppose stocks and activities are divisible and a producer may allocate a subbundle to one recurrent organization while leaving the residual stock to other uses or free disposal. Define the operational upward closure of the reproductive cone by

$$\mathcal{U}_\tau = \{x \geq 0 : \exists z \in \mathcal{V}_\tau \setminus \{0\} \text{ with } z \leq x\}.$$

**Lemma 7.2 (Interior Seedability / Scale-Free Collapse).** If $\mathcal{V}_\tau$ contains any non-zero vector, then

$$\mathbb{R}^n_{++} \subseteq \mathcal{U}_\tau.$$

Hence every strictly positive inherited state contains a positive-scale subbundle from which $\tau$ can reproduce. If the recurrent plan defining that subbundle is competitively supported at rate $r$, constant returns to scale imply that the scaled plan is competitively supported as well.

**Proof.** Choose $v \in \mathcal{V}_\tau \setminus \{0\}$. For $x \gg 0$, let

$$0 < \varepsilon < \min_{i:v_i>0} \frac{x_i}{v_i}.$$

Because $\mathcal{V}_\tau$ is a cone, $\varepsilon v \in \mathcal{V}_\tau$, and by construction $\varepsilon v \leq x$. Divisibility permits the subbundle $\varepsilon v$ to be allocated to $\tau$. Scaling every activity and dated stock in its recurrent plan by $\varepsilon$ preserves all material inequalities. Under constant returns, zero-profit equalities and inactive-process inequalities are homogeneous in activity scale, so competitive support is preserved as well. □

**Corollary 7.3 (No interior scale-free historical wedge).** Suppose the unique global wage maximizer $\tau^G(r)$ has a non-empty reproductive cone and its recurrent plan is competitively supportable at $r$. Under divisible constant-returns production and free suballocation,

$$W^H(x,r) = W^G(r) \qquad \text{for every } x \gg 0$$

for any operational admissibility criterion that requires only a non-zero reproductive seed. A strict historical wedge can then arise only on a support boundary, or after adding a non-homogeneous or non-convex restriction such as a fixed social-output requirement, a fixed employment requirement, fixed costs, indivisibilities, financing constraints, or another scale-fixing closure.

This lemma identifies a limit of the reconstruction rather than a defect that can be repaired by further coefficient search. Adding more vintage classes or making the ageing map non-surjective does not by itself overcome the result: every strictly positive state still dominates a sufficiently small copy of any finite non-zero launch bundle. The whole-bundle cones remain useful geometric objects, but their interior facets cannot by themselves be interpreted as operational technique-exclusion boundaries under free suballocation.

## 8. Engineering entry into a reproductive cone

The Interior Seedability Lemma changes the interpretation of finite-horizon cone entry. A state outside $\mathcal{V}_\tau$ may indeed be transformed into a state inside it, and that reachability problem has a useful polyhedral geometry. But under divisible constant-returns production with free suballocation, entry of the *whole terminal state* into $\mathcal{V}_\tau$ is not the same thing as operational availability of $\tau$: an interior state already contains a sufficiently small reproductive subbundle. The cone-entry object below is therefore retained as a whole-state composition diagnostic and as an exact tool for support-boundary cases. It becomes an economically operative admissibility criterion only when an explicit closure requires the relevant reference stock, output, employment, or another positive scale to be reproduced.

Fix a horizon $H$ and a prospective recurrent organization $\tau$. Let $\mathcal{R}^e_H(x)$ denote the set of terminal states reachable from $x$ by physically feasible use of the full technical catalogue under the stated quantity closure, without imposing profitability restrictions.

A naive condition

$$\mathcal{R}^e_H(x) \cap \mathcal{V}_\tau \neq \varnothing$$

is insufficient because $0 \in \mathcal{V}_\tau$. Let $e \gg 0$ be arbitrary and define the **engineering cone-entry value**

$$\rho^e_{\tau,H}(x) = \max\{e^\top z : z \in \mathcal{R}^e_H(x) \cap \mathcal{V}_\tau\}.$$

Then

$$\rho^e_{\tau,H}(x) > 0$$

if and only if a non-zero point of the reproductive cone is reachable. The sign is independent of the particular $e \gg 0$ because $e^\top z > 0$ exactly when $z \geq 0$ and $z \neq 0$.

With a fixed finite-horizon activity itinerary, the stacked constraints are linear and homogeneous in the inherited state. Standard parametric linear programming therefore yields positive homogeneity, concavity, continuity, piecewise linearity, and a dual bottleneck representation for the cone-entry value. The dual variables are physical shadow valuations of inherited resources. They are **not** Sraffian production prices.

## 9. Dated competitive support

### 9.1 Forward-looking margins

An activity initiated at date $t$ uses current inputs and delivers dated outputs at $t+1$. Let $p_t$ be dated commodity prices, $w_t$ the wage, and $r_t$ the normal rate of profit. The forward-looking margin of activity $j$ is

$$\Pi^F_{j,t} = p^\top_{t+1} b_j - (1 + r_t) p_t^\top a_j - w_t \ell_j.$$

An active competitive activity satisfies equality; an inactive available activity must have non-positive margin. The output price is indexed by continuation. There is generally no unique price “under regime $E$” independently of what follows, because the economic value of a produced machine depends on its dated future use.

This is not an Arrow–Debreu completion of the model. Consumption, saving, financing, and scarcity rents are not derived. The role of the dated price system is narrower: to test whether the activities used by a proposed historical path can be competitively supported at the normal rate.

### 9.2 Regular competitive itineraries

Let

$$\kappa = (\kappa_0, \ldots, \kappa_{H-1})$$

denote a **regular competitive itinerary**. At each date the active set is fixed; all inactive-process inequalities are strict; the price system is locally nonsingular under the chosen numeraire; and the quantity closure selects a unique material transition on the relevant cell.

On such a cell, stack all intermediate state and activity variables into $\xi$ and write the material cone-entry programme as

$$G_{\tau,H,\kappa}\xi + L_{\tau,H,\kappa}z \leq N_{\tau,H,\kappa}x, \qquad z \in \mathcal{V}_\tau.$$

The dated prices associated with the itinerary determine whether the active and inactive margin conditions are satisfied. Define a minimum competitive slack $s_{\tau,H,\kappa}(x, r; \theta)$ that is strictly positive when all relevant positive-price, positive-wage, active equality, inactive strict-inequality, and cell-consistency conditions hold.

The distinction between material and competitive conditions is essential. An engineering route can exist even when every route of that type requires an activity with a strictly negative forward-looking margin.

## 10. The Polyhedral Historical-Domain Theorem

For a fixed regular itinerary define

$$\begin{aligned}\rho^c_{\tau,H,\kappa}(x; r) = \max\{\, e^\top z :\ & \text{the stacked itinerary constraints hold,} \\ & z \in \mathcal{V}_\tau,\ \kappa \text{ is competitively supported}\,\}.\end{aligned}$$

Equivalently, when the supporting price path is fixed by the regular itinerary, this is the engineering LP with activities excluded whenever their dated margin is strictly negative.

Define the itinerary-specific historical domain

$$\mathcal{D}^c_{\tau,H,\kappa}(r) = \{x \geq 0 : \rho^c_{\tau,H,\kappa}(x; r) > 0\}.$$

**Theorem 10.1 (Polyhedral Historical Domain).** Fix $H$, $r$, $\tau$, and a regular competitive itinerary $\kappa$. Assume feasibility and finiteness of the corresponding cone-entry programme. Then:

1. $\rho^c_{\tau,H,\kappa}$ is non-negative, positively homogeneous, concave, continuous, and piecewise linear on its feasible state cone;
2. $\mathcal{D}^c_{\tau,H,\kappa}(r)$ is either empty or a convex relatively open polyhedral cone;
3. its intersection with the projective simplex is a relatively open polytope;
4. if only finitely many regular competitive itineraries are possible over horizon $H$, the complete historical domain

$$\mathcal{D}^c_{\tau,H}(r) = \bigcup_{\kappa} \mathcal{D}^c_{\tau,H,\kappa}(r)$$

is a finite union of relatively open polyhedral cones and need not be convex.

**Proof.** The stacked material programme is an LP whose right-hand side depends linearly on $x$. Homogeneity and convex combinations of feasible paths give positive homogeneity and concavity. The dual feasible set is independent of $x$, so the value can be written locally as the minimum of finitely many linear forms,

$$\rho^c_{\tau,H,\kappa}(x;r) = \min_j a_{\kappa j}^\top x.$$

Therefore strict positive cone entry is equivalent to finitely many strict homogeneous linear inequalities. Intersecting these with the linear sign conditions defining the regular itinerary yields a relatively open polyhedral cone. The projective section is a relatively open polytope. The finite union over itineraries need not be convex. □

The global non-convexity is economically important. Different initial states can lead to different competitively supported itineraries even when they share the same final target cone. There is no reason for the union of those itinerary regions to be convex.

### 10.2 Engineering as an outer bound

By construction,

$$\rho^c_{\tau,H}(x;r) \leq \rho^e_{\tau,H}(x).$$

Hence

$$\rho^e_{\tau,H}(x) = 0 \quad \Longrightarrow \quad \rho^c_{\tau,H}(x;r) = 0.$$

This one-way implication is all that is needed for many impossibility results. It avoids the false converse that physical feasibility establishes competitive restoration.

# PART III — FROM INTERIOR SEEDABILITY TO COUNTERFACTUAL RECONSTRUCTION

## 11. Why interior seedability changes the question

The Interior Seedability result identifies the boundary of the homogeneous theory. If $x \gg 0$, constant returns and divisibility allow a sufficiently small recurrent seed to be carved out of the inherited stock whenever the relevant reproductive cone is non-empty. Positive-scale restartability can therefore be a support statement, but it cannot by itself distinguish robustly between interior states at an economy-relevant scale.

The historical question must contain a positive reference magnitude. One route is to impose a non-homogeneous datum such as a demand, capacity, labour endowment, fixed cost, or minimum viable scale. The route used here is different: the scale is generated by a stated counterfactual history observed at the same date as the historical branch. This prevents the reference path's own dated dynamics from being misattributed to the historical sequence.

## 12. Wage-consistent transitions and metric-indexed reconstruction

Let $c^w \geq 0$ be a physical wage-basket vector. For technique $\tau$ with matrices $(A_\tau, B_\tau)$ and labour vector $\ell_\tau$, a period-$t$ transition operated at wage $w_t$ satisfies

$$A_\tau y_t \leq x_t,$$

$$x_{t+1} = D(x_t - A_\tau y_t) + B_\tau y_t - c^w w_t \ell_\tau^\top y_t \geq 0.$$

The wage term is a physical stock-flow closure: the specified wage basket is removed before the next productive state is inherited.

A terminal recurrent plan $v \geq 0$ for technique $\tau$ at final rate $r_f$ satisfies

$$A_\tau v \leq x_H, \qquad A_\tau v + c^w w_\tau(r_f) \ell_\tau^\top v \leq B_\tau v.$$

Let $m_\tau(v)$ be a stated positive linear terminal metric. Define

$$S^m_{\tau,H}(x; r_f) := \sup m_\tau(v),$$

where the supremum ranges over the admissible $H$-period transition and terminal recurrent plan. The labour-capacity object used in several exact certificates is the special case

$$\Lambda^{c,\mathrm{net}}_{\tau,H}(x; r_f) = S^\ell_{\tau,H}(x; r_f), \qquad m_\tau(v) = \ell_\tau^\top v.$$

For cross-technique comparisons the paper instead fixes a single positive commodity-value covector $\phi \gg 0$ and uses
$$S^{\phi}_{\tau,H}(x;r_f) = \sup \phi^{\top} B_{\tau} v.$$
Because the same $\phi$ is applied to every technique, a common rescaling of the numeraire multiplies all relevant scales by the same factor. Labour remains a quantity flow and a legitimate within-technique metric; it is not treated as a technique-neutral cross-technique scale.

For fixed support, wages, terminal metric, and active basis, these objects are linear programmes. As functions of inherited stocks they are non-negative, positively homogeneous, concave, continuous, and piecewise linear whenever finite.

## 13. Comparison-date baseline and moving counterfactual

Fix an initial state $x^{\circ}$, a comparison horizon $T$, a final distributive parameter $r_f$, a terminal target technique $\tau^G$, and an explicit reference history $\kappa^0$ of length $T$. Let
$$x^0_T := \Phi^{\chi^0}_{\kappa^0}(x^{\circ}).$$
For a chosen terminal metric $m$, define the **comparison-date reference level**
$$L^0_T(m) := S^m_{\tau^G,0}(x^0_T;r_f) > 0.$$
This level is generated by technology, initial state, reference history, comparison date, physical distribution convention, and metric. It is not a freely calibrated threshold. It is held fixed only when the question is restoration to the productive scale possessed by the reference economy at date $T$.

A different question continues the reference branch after $T$. For a stated reference closure $\chi^0$, define the **moving no-excursion counterfactual**
$$L^{0,m}_{T,H} := S^m_{\tau^G,0}\left(\Phi^H_{\chi^0}(x^0_T);r_f\right).$$
A historical branch can therefore restore the comparison-date level while never catching the moving reference, or catch a declining reference without having achieved a persistently viable reconstruction. The two benchmarks must not be conflated.

For a historical state $x^h_T$, the fixed-baseline ratio is
$$\mathcal{R}^m_{\tau,H} := \frac{S^m_{\tau,H}(x^h_T;r_f)}{L^0_T(m)}.$$
By construction the reference branch has ratio one at $H = 0$. A moving-reference ratio instead

divides by $L^{0,m}_{T,H}$ and compares the two branches at the same later date $T+H$.

## 14. Restoration sets and attainable recurrent packets

For a common metric $m$, define the fixed-baseline restoration set

$$\Omega^{\text{rest}}_{H,m}(x^h_T, r_f \mid x^0_T) = \left\{\tau \in \Omega(r_f) : S^m_{\tau,H}(x^h_T; r_f) \geq L^0_T(m)\right\}.$$

If $\mathcal{I}(\tau)$ denotes the recurrent Sraffian packet attached to technique $\tau$, the historically attainable packet set is

$$\mathcal{I}^{\text{att}}_{H,m} = \{\mathcal{I}(\tau) : \tau \in \Omega^{\text{rest}}_{H,m}\}.$$

History changes the economically effective domain; it does not change the internal invariant packet of a fixed recurrent organization.

A moving-reference attainable set is defined by replacing $L^0_T(m)$ with $L^{0,m}_{T,H}$. The fixed set answers a restoration question; the moving set answers a relative catch-up question. The corresponding long-period wage envelope remains an opportunity object, not the wage actually paid during the traverse.

## 15. Three reconstruction clocks: feasibility, first passage, and persistence

The reconstruction problem contains three different clocks.

**Definition 15.1 (Minimum feasible reconstruction horizon).**

$$H^{\min,m}_{\tau}(x \mid x^0_T) := \inf\left\{H : \sup_{\text{admissible } H\text{-step paths}} m_\tau(v) \geq L^0_T(m)\right\}.$$

It asks how soon reconstruction *can* occur.

**Definition 15.2 (First-passage restoration time).** For a specified quantity closure $\chi$,

$$D^{\chi,m}_{\tau}(x \mid x^0_T) := \inf\left\{H : S^m_{\tau,0}(\Phi^H_\chi(x); r_f) \geq L^0_T(m)\right\}.$$

It asks when the stated closure first crosses the comparison-date baseline.

**Definition 15.3 (Persistent restoration time).**

$$\bar{D}^{\chi,m}_{\tau}(x \mid x^0_T) := \inf\left\{H : S^m_{\tau,0}(\Phi^h_\chi(x); r_f) \geq L^0_T(m) \text{ for every } h \geq H\right\}.$$

It asks when the closure has restored the baseline and never loses it again.

**Proposition 15.4 (Three-clock inequality).** If the closure-generated path is admissible for the reconstruction problem, then

$$\boxed{H_\tau^{\min,m} \le D_\tau^{\chi,m} \le \bar{D}_\tau^{\chi,m}.}$$

Either inequality may be strict, and either trajectory time may be $+\infty$.

**Proof.** A closure path that first crosses at $D$ is one admissible path, so feasibility cannot require more time. Persistent restoration implies a first crossing no later than the date from which all later dates remain above the threshold. □

Moving-reference versions are obtained by replacing the fixed level $L_T^0(m)$ at horizon $h$ by $L_{T,h}^{0,m}$. These are *counterfactual catch-up* times rather than restoration times. The distinction is substantive when the reference branch itself is non-monotone or leaves its maintained closure cell.

## 16. Polyhedral structure and common-metric invariance

For a fixed itinerary and final technique, stack transition variables and the terminal plan into $z$. The reconstruction LP has the form

$$\max c_m^\top z \quad \text{s.t.} \quad Gz \le Nx,$$

plus homogeneous equalities and non-negativity. Hence $S_{\tau,H}^m(x; r_f)$ is concave, positively homogeneous, continuous, and piecewise linear on regular cells. For fixed $x_T^0$ the domain

$$\mathcal{D}_{\tau,H}^m(x_T^0) = \{x : S_{\tau,H}^m(x; r_f) \ge L_T^0(m)\}$$

is a closed affine polyhedron on each active cell and a finite union across finitely many cells.

If $m(v) = \phi^\top B_\tau v$ with a common $\phi$, multiplication $\phi \mapsto c\phi$, $c > 0$, leaves every restoration ratio and attainable set unchanged. This is the relevant numeraire invariance for cross-technique reconstruction.

## 17. Generic order sensitivity and open persistence of delays

On a connected regular parameter cell $\Theta$, suppose the two supported state maps and the terminal capacity row are analytic:

$$F_P(\theta), \qquad F_E(\theta), \qquad a(\theta)^\top.$$

Define the valuation-visible commutator row

$$g(\theta)^\top = a(\theta)^\top \left[F_E(\theta)F_P(\theta) - F_P(\theta)F_E(\theta)\right].$$

**Theorem 17.1 (Generic order sensitivity on a regular cell).** Let $\Theta$ be a connected open regular parameter cell. If $g$ is not identically zero on $\Theta$, then

$$\Theta_{\text{ord}} = \{\theta \in \Theta : g(\theta) \neq 0\}$$

is open and dense. For every $\theta \in \Theta_{\text{ord}}$, the positive inherited states satisfying

$$a(\theta)^\top F_E F_P x \neq a(\theta)^\top F_P F_E x$$

form an open dense subset of the positive state cone.

**Proof.** A non-identically-zero real-analytic map has a zero set with empty interior. For fixed $g \neq 0$, the exceptional states belong to the hyperplane $g^\top x = 0$. □

This theorem concerns order-sensitive capacity. A stronger statement that unequal integer delays are open dense is false without threshold-transversality conditions: there are open regions in which both orders are above the benchmark at horizon zero or cross the same threshold at the same date.

**Proposition 17.2 (Open persistence of a strict delay configuration).** Suppose at one regular parameter point all support margins, active-basis inequalities, reference inequalities, and threshold comparisons that identify two finite delays are strict. Then the same two delays persist on an open neighbourhood of primitives.

The original and primitive witnesses below supply strict open regions for, respectively, minimum feasible reconstruction and closure-generated delay.

### 18. Historical bifurcations and comparative statics in distributive distance

Three boundaries must be kept distinct: competitive-support boundaries, active-basis boundaries, and reconstruction-threshold boundaries. A wage switch alone does not generate productive history. History becomes visible when the dated state operators fail to commute on the inherited state and the difference is seen by the chosen terminal metric.

For two regimes write

$$\Delta(r_P, r_E; x) = a^\top \left[F_E(r_E)F_P(r_P) - F_P(r_P)F_E(r_E)\right] x.$$

On a maintained regular cell,

$$\frac{\partial \Delta}{\partial r_E} = a^\top \left[F_E'(r_E)F_P - F_P F_E'(r_E)\right] x,$$

and an analogous expression holds for $r_P$. Thus distributive comparative statics of the order effect are ordinary derivatives of the valuation-visible commutator as long as support and active basis remain unchanged.

Part IV derives an exact signed comparative static for the central calibration: increasing $r_E - r_P$ raises the $PE$ capacity, lowers the $EP$ capacity, and strictly widens their gap on the maintained cell.

## 19. What the external theory now establishes

The external module combines general regular-cell results with exact delay certificates:

1. interior seedability makes arbitrarily small restartability too weak as an interior historical criterion;
2. a stated no-excursion history supplies a non-arbitrary comparison-date baseline, while its continuation defines a distinct moving counterfactual;
3. reconstruction has three clocks: minimum feasible, first passage, and persistent restoration;
4. cross-technique reconstruction requires a common metric;
5. order-sensitive capacity is generic on regular cells whenever the valuation-visible commutator is not an analytic identity;
6. strict integer-delay configurations persist on open parameter sets;
7. the order gap admits local comparative statics with respect to distributive parameters;
8. closure viability is an independent question and must be diagnosed rather than inferred from a threshold crossing.

The exact witnesses perform different jobs. The original witness isolates minimum feasible reconstruction and a scalar active-basis scaling law; the primitive witness separates first passage from persistent restoration; the rank-two viable witness exercises the genuinely multidimensional spectral-delay theorem under a closure that remains in the positive cone indefinitely.

# PART IV — EXACT DATED MULTISTAGE WITNESSES AND CLOSURE DIAGNOSTICS

## 20. What the exact witnesses must establish

The external theory requires more than a non-zero commutator. The witnesses are designed to separate five questions: whether order changes the inherited state; whether the difference is visible in a common reconstruction metric; whether reconstruction is feasible within a stated horizon; whether the stated quantity closure actually realizes that reconstruction; and whether the timing survives once the target technology is primitive rather than a pure cycle.

The central construction is described as *dated multistage production*. Its three reproducible stages constitute dated multistage production rather than a full Sraffian fixed-capital model with used machines appearing as joint products. A separate boundary example carries the genuine fixed-capital interpretation.

## 21. Original three-stage witness: four processes and two recurrent techniques

There are three reproducible stages $(K_0, K_1, K_2)$ and four elementary processes $(\rho, d, a_0, a_1)$. The complete input and gross-output matrices are

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 9/16 & 0 \\ 0 & 0 & 0 & 3/5 \\ 11/16 & 9/20 & 0 & 0 \end{pmatrix}, \qquad \mathcal{B} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and direct labour is

$$\ell = \left(\frac{9}{25}, \frac{3}{5}, \frac{1}{25}, \frac{1}{20}\right)^{\top}.$$

Thus

$$\rho : \frac{11}{16}K_2 + \frac{9}{25}L \to K_0, \qquad d : \frac{9}{20}K_2 + \frac{3}{5}L \to K_0,$$
$$a_0 : \frac{9}{16}K_0 + \frac{1}{25}L \to K_1, \qquad a_1 : \frac{3}{5}K_1 + \frac{1}{20}L \to K_2.$$

Idle stages age according to

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The complete recurrent catalogue contains exactly

$$A = \{\rho, a_0, a_1\}, \qquad B = \{d, a_0, a_1\},$$

with

$$A_A = \begin{pmatrix} 0 & 9/16 & 0 \\ 0 & 0 & 3/5 \\ 11/16 & 0 & 0 \end{pmatrix}, \qquad A_B = \begin{pmatrix} 0 & 9/16 & 0 \\ 0 & 0 & 3/5 \\ 9/20 & 0 & 0 \end{pmatrix},$$

$B_A = B_B = I_3$ and

$$\ell_A = \left(\frac{9}{25}, \frac{1}{25}, \frac{1}{20}\right)^\top, \qquad \ell_B = \left(\frac{3}{5}, \frac{1}{25}, \frac{1}{20}\right)^\top.$$

Technique $A$ uses more old-stage material but less renewal labour; $B$ uses less old-stage material but more renewal labour.

Let $R = 1 + r$, normalize $p_0 = 1$, and pay wages at the end of the period. The wage curves are

$$w_A(R) = \frac{25(1280 - 297R^3)}{4(132R^2 + 275R + 2880)},$$

$$w_B(R) = \frac{25(1600 - 243R^3)}{12(36R^2 + 75R + 2000)}.$$

They cross at the relevant real root of

$$13527R^3 + 912R^2 + 1900R - 38400 = 0, \qquad R^s \simeq 1.361290270.$$

Use

$$P: \quad r_P = \frac{1}{4}, \qquad w_P = w_A(r_P) = \frac{223975}{175616},$$
$$E: \quad r_E = \frac{1}{2}, \qquad w_E = w_B(r_E) = \frac{155975}{210576}.$$

At $P$, $A$ is the unique wage maximizer; at $E$, $B$ is the unique wage maximizer. The exact inactive-renewal margins along $PE$ and $EP$ are

$$-\frac{576833787953}{3432254031360}, \quad -\frac{32415}{280768}, \quad -\frac{12859302599}{117008660160}, \quad -\frac{20941}{175616},$$

so the support inequalities are strictly separated from zero.

## 22. The two histories, the no-excursion path, and the local quantity closure

The wage basket is one unit of $K_0$. If technique $\tau$ is supported at date $t$, the baseline myopic closure chooses

$$\max_{y\geq 0}\ \ell_\tau^\top y$$

subject to

$$A_\tau y \leq x, \qquad D(x - A_\tau y) + B_\tau y - e_0 w_\tau(r_t)\ell_\tau^\top y \geq 0.$$

There is no labour ceiling. On the interior cells used to construct the two-period histories, all three current material constraints bind and the post-wage constraints remain slack. The closure therefore reduces locally to

$$x_{t+1} = F_\tau(r_t)x_t, \qquad F_\tau(r) = \left[I - e_0 w_\tau(r)\ell_\tau^\top\right]A_\tau^{-1}.$$

For the two regimes,

$$F_P = \begin{pmatrix} -8959/98784 & -223975/2107392 & 8635/10976 \\ 16/9 & 0 & 0 \\ 0 & 5/3 & 0 \end{pmatrix},$$

$$F_E = \begin{pmatrix} -6239/118449 & -155975/2526912 & 21665/17548 \\ 16/9 & 0 & 0 \\ 0 & 5/3 & 0 \end{pmatrix}.$$

Their commutator is non-zero.

Starting from

$$x^\circ = \left(\frac{5}{8}, \frac{1}{6}, \frac{1}{2}\right)^\top,$$

the order-reversed terminal states are

$$x^{PE} = \left(\frac{128585365955}{499236950016}, \frac{1344355}{2370816}, \frac{50}{27}\right)^\top \gg 0,$$

$$x^{EP} = \left(\frac{3450161185}{71319564288}, \frac{2901395}{2842776}, \frac{50}{27}\right)^\top \gg 0.$$

The no-excursion reference is

$$x^{PP} = \left(\frac{29775581155}{416353222656}, \frac{1344355}{2370816}, \frac{50}{27}\right)^\top \gg 0.$$

Thus the order effect is strictly interior at the comparison date.

The same $PE$, $EP$, and $PP$ endpoints are selected by a second local objective that maximizes dated next-stock value at the supporting continuation prices. This is an objective-robustness result over the comparison window only; it is not a global closure-invariance theorem.

## 23. Original witness: common-metric minimum feasible reconstruction

For within-technique diagnosis, the original labour capacities remain useful. At the common comparison date,

$$\Lambda_{A,0}(PP) \simeq 0.1266372741, \qquad \Lambda_{A,0}(PE) \simeq 0.4560873862, \qquad \Lambda_{A,0}(EP) \simeq 0.08566313045.$$

The fact that $B$ lies above the $PP$ reference after $EP$ is used only as a labour-absorption comparison, not as a cross-technique attainability criterion.

For reconstruction, fix the common final-$A$ price covector

$$\phi = p_A(r_P) = \left(1, \frac{132439}{175616}, \frac{1727}{2744}\right)^\top$$

and measure the gross recurrent product of every technique by $\phi^\top B_\tau v$. The comparison-date $PP$ baseline is

$$L_2^{0,\phi} = \frac{243155405343060695}{452419404215353344} \simeq 0.53745573925.$$

The immediate common-metric $A$ scales are

$$S^{\phi}_{A,0}(EP) \simeq 0.36355916088, \qquad S^{\phi}_{A,0}(PE) \simeq 1.93566061085.$$

Thus the order reversal is large under a technique-independent metric. The calibration is a stress test rather than an empirical magnitude: $r_P = 25\%$ and $r_E = 50\%$, and the comparative-static cell is stated explicitly below.

Exact catalogue-complete recovery LPs at the actual final-$A$ wage give

$$S^{\phi}_{A,1} = 0.42607864849, \qquad S^{\phi}_{A,2} = 0.49096758530,$$

both below $L_2^{0,\phi}$. The exact horizon-three optimum is

$$S^{\phi}_{A,3} = 0.84746068303 > L_2^{0,\phi}.$$

Therefore

$$\boxed{H_A^{\min,\phi}(EP) = 3.}$$

This is a minimum feasible reconstruction horizon, not a trajectory statement.

The lower robustness margin is the binding one:

$$\frac{L_2^{0,\phi}}{S_{A,2}^{\phi}} - 1 \simeq 0.09469.$$

Thus the threshold can fall by about 9.5% relative to the certified two-period optimum before the integer horizon changes. The much larger distance to the horizon-three optimum is reported but is not described as the binding robustness margin.

The reported common-metric reconstruction values are evaluated at the actual final-$A$ wage; no lower-wage relaxation enters the stated $H^{\min} = 3$ claim.

## 24. Closure diagnostics, the primitive first passage, and the need for persistence

### 24.1 The original myopic rule is a closure pathology diagnostic

Continue the exact myopic $P$ closure of Section 22 from $x^{EP}$. For two periods it coincides with the full-resource path. At the third step the constrained optimum reaches

$$K_{0,3} = 0,$$

after which required stages remain missing and recurrent $A$ capacity is zero. Hence

$$H_A^{\min,\phi}(EP) = 3, \qquad D_A^{\chi,\phi}(EP) = +\infty.$$

This formally separates reachability from the trajectory generated by the specified rule.

The economic interpretation is deliberately narrow. The pathology is not unique to the historical branch. Starting from the comparison-date $PP$ reference state, the same myopic rule also reaches zero recurrent $A$ capacity at the third continuation date. It is therefore a diagnostic of a closure that maximizes current labour absorption subject only to one-period non-negativity and can consume the productive structure required for later reproduction. Section 24.1 is not used as evidence that history alone causes the infinite delay. Its role is to show why a viable traverse closure cannot be inferred from a myopic objective.

### 24.2 Primitive period-one witness: first passage is not persistent restoration

Perturb both recurrent input matrices by the same rational self-requirement:

$$A_A^{\varepsilon} = \begin{pmatrix} 2/25 & 9/16 & 0 \\ 0 & 0 & 3/5 \\ 11/16 & 0 & 0 \end{pmatrix}, \qquad A_B^{\varepsilon} = \begin{pmatrix} 2/25 & 9/16 & 0 \\ 0 & 0 & 3/5 \\ 9/20 & 0 & 0 \end{pmatrix}.$$

Then $(A_A^\varepsilon)^4 \gg 0$, so the target graph has period one. The wage switch survives exactly. For the initial state and complete catalogue listed in Appendix A.7, the common metric is

$$\phi_\varepsilon = \left(1, \frac{13347}{17920}, \frac{171}{280}\right)^\top,$$

and the comparison-date $PP$ baseline is

$$L_{2,\varepsilon}^{0,\phi} \simeq 1.70249897347.$$

From $x^{EP}$ the same myopic closure generates the exact common-metric sequence

$$1.55276241453, \quad 1.49234574701, \quad \mathbf{1.74192849313}, \quad 1.11393727667, \quad 0.77674960515, \quad 0, \ldots$$

Thus the first crossing occurs at date two:

$$\boxed{D_A^{\chi,\phi}(EP) = 2,}$$

but it is a transient peak. The reconstructed scale is immediately lost again, so

$$\boxed{\bar{D}_A^{\chi,\phi}(EP) = +\infty.}$$

This witness removes the graph-period objection while separating first passage from persistent restoration.

The same point also disciplines benchmark language. If the $PP$ reference branch is continued under the same myopic closure, its own target scale is non-monotone and eventually collapses. A comparison with $L_{2,\varepsilon}^{0,\phi}$ is restoration to a *comparison-date baseline*; a horizon-by-horizon comparison with the continued $PP$ branch is a different, moving-counterfactual question.

### 24.3 Common-metric Double Historical Condition

At the comparison date of the primitive witness,

$$S_{A,0}^{\phi}(EP) \simeq 1.5527624 < L_{2,\varepsilon}^{0,\phi} \simeq 1.702499 < S_{B,0}^{\phi}(EP) \simeq 1.7856424.$$

After $PE$, $A$ is above the same comparison-date baseline. The two recurrent techniques also have distinct maximum-profit rates and Perron/Standard compositions. Hence this witness supplies a common-metric **Double Historical Condition at the opportunity-set level**. It is not a theorem that $B$ is the realized short-period equilibrium, and it is not a statement of persistent restoration.

### 24.4 Why a proper-cone rank-two construction is required

The original recovery value rows satisfy $a_H \propto e_0^\top$ on the certified horizons: the shadow values of the other two inherited stages are zero. Its active-basis factor is therefore genuinely useful but scalar. The primitive witness solves the graph-period problem but still does not exercise a non-trivial active-basis spectral gap or persistent restoration. Section 36.7 constructs a third exact witness with a full-rank $2 \times 2$ active-basis transfer, a non-zero subdominant mode, and an indefinitely viable closure.

## 25. Boundary benchmark: genuine fixed-capital extinction

A separate fixed-capital example isolates a different mechanism. There is one circulating commodity, two machine vintages, and labour. Used machines are treated as dated productive objects, mandatory ageing creates a productive siphon, and continuation-indexed prices make renewal the unique supported use of the vulnerable old machine in one sequence and terminal destruction the supported use in another. One order preserves the machine subsystem; the other empties it. With no bootstrap process, the empty face is absorbing.

This boundary construction is exact but topologically different from the interior multistage witnesses. A small bootstrap link can destroy literal absorption. It is retained as a theorem about boundary support extinction, not as evidence for interior finite reconstruction delay.

## 26. What the external module establishes

1. Positive seedability is too weak to identify interior historical exclusion at an economically meaningful scale.
2. A no-excursion history supports two different questions: restoration to a fixed comparison-date baseline and catch-up with a moving counterfactual.
3. Reconstruction has three distinct times: $H^{\min}$, $D^{\chi}$, and $\bar{D}^{\chi}$.
4. The original multistage witness proves the exact catalogue-complete result $H_A^{\min,\phi}(EP) = 3$. Its infinite trajectory delay is used only as a closure-pathology diagnostic, because the same myopic rule also destroys the reference branch.
5. The primitive period-one witness gives $D_A^{\chi,\phi}(EP) = 2 < \bar{D}_A^{\chi,\phi}(EP) = +\infty$, so first passage need not be persistent and the timing is not a three-cycle artefact.
6. A common commodity-value metric is required for cross-technique comparison. Under that ruler the primitive witness supplies an opportunity-level Double Historical Condition.
7. Order-sensitive capacity is generic on regular cells when the valuation-visible commutator is not an analytic identity; strict integer-delay configurations persist on open parameter sets; and the order gap admits exact distributive comparative statics.

8. Boundary extinction in a genuine fixed-capital construction remains a separate topology-dependent historical mechanism.

The two multistage witnesses are complementary rather than exhaustive. They establish order-conditioned reconstruction and distinguish feasibility, first passage, persistence, and closure pathology. The multidimensional spectral recovery construction belongs to Part VI, while Section 33A supplies the stronger bridge in which external order selects internal recurrent order.

# PART V — INTERNAL HISTORY: ORDERED STANDARD SYSTEMS

## 27. From the historical domain to the Sraffian invariant packet

Parts II–IV study one historical order: the order of temporary supported environments that acts on an inherited productive state. Part V studies another: the order of phases inside a recurrent organization whose elementary technical systems are held fixed. These are different objects and should not be conflated.

For a fixed recurrent organization $\tau$, inherited state does not alter its Standard ratio, dated production graph, or joint-production pencil. Those are structural properties of $\tau$. External history can nevertheless change which recurrent organization is attainable. The primitive witness of Part IV establishes this only for distinct stationary techniques.

A stronger integration is developed at the end of this Part. Section 33A places the traverse support problem and terminal recurrent-order choice in one block mixed-complementarity formulation. The distributive word selects external processes and wages; the resulting inherited composition then selects, through a second primal-dual support system, the non-cyclic recurrent order. Thus the two meanings of order are first separated analytically and then joined by an exact support-to-state-to-order mechanism.

The object may itself be ordered and periodic. If a fixed collection of phase systems is repeated in a sequence $\sigma = (A_1, \ldots, A_T)$, a temporal lift treats commodity-date pairs as distinct commodities. The question is then whether the resulting dated system possesses an economically meaningful counterpart of Sraffa's Standard commodity, whether the relevant invariant is unchanged by a mere shift of calendar origin, and whether a genuinely non-cyclic reordering changes that invariant.

The answer is affirmative under stated conditions. In single production the temporal lift yields a Historical Standard Cycle with Standard proportions, a cycle profit identity, a common-rate equivalent, a linear Standard wage–profit relation, and a conditional value-invariance property. In reducible systems, basicity becomes a dated graph property plus a separate spectral-admissibility condition. Under joint production, positive prices and physically realizable Standard multipliers are different sides of a matrix pencil; the distinction requires an inverse-free, support-aware treatment.

## 28. Static benchmark, scope, and notation

### 28.1 Single production

Consider n commodities and a square single-product technique represented, after normalizing each gross output to one unit, by a non-negative input matrix A. With the wage paid at the end of the period, the price equations are

$$p^\top = (1+r)p^\top A + w\ell^\top.$$

On an irreducible productive basic system, let $q \gg 0$ satisfy $Aq = \lambda q$ with $\lambda = \rho(A) < 1$. The Standard ratio and the maximum rate of profits are

$$R = \lambda^{-1} - 1.$$

The Standard net product is $(I - A)q = (1 - \lambda)q$. With labour and the Standard net product normalized in the usual way, the distributive relation becomes $r = R(1 - w)$. This paper takes these familiar facts as a benchmark, not as results requiring reproving.

### 28.2 What "historical" means here

History enters in two distinct senses and they should not be conflated. First, an ordered sequence $\sigma$ can contain the same elementary systems as another sequence $\tau$ while arranging them differently. This is internal historical order. Second, an inherited state can determine which ordered sequences are materially attainable. That is external productive history and belongs to the historical-domain problem developed in Parts II–IV. Throughout Sections 29–33 the elementary matrices are exogenous and fixed.

The main analysis is periodic. A finite one-off sequence has dated price and quantity implications, but it does not by itself define a long-period Standard ratio. Periodic repetition gives a well-defined cycle and permits an exact comparison with Sraffa's reproductive construction. Non-periodic and stochastic sequences are left for later work.

### 28.3 Relation to existing extensions

The term "dynamic Standard commodity" is avoided deliberately. Pasinetti (1993) uses that expression in a theory of structural dynamics in which sectoral productivity changes through time. Kurose and Yoshihara (2019) generalize the Standard commodity to convex production sets so that a distributive change may be associated with a change of technique. The construction below instead fixes the elementary technical systems and studies the consequences of their order. Solow's (1959) dynamic input-output valuation and von Neumann's (1945) growth model provide important precedents for dated multisector reproduction.

The block-cyclic lift and monodromy identity used below are also standard objects in the mathematics of periodic linear systems; Floquet-type representations, lifting, and monodromy are treated systematically by Bittanti and Colaneri (2009). Periodic Leontief systems likewise appear in the dynamic input-output/control literature. No priority is claimed for the identity $\rho(\bar{A})^T = \rho(M)$ itself. The economic contribution is the Sraffian interpretation: identifying exactly which Standard properties survive in dated commodity space, which fail, and how non-cyclic permutation affects those properties.

## 29. The Historical Standard Cycle in single production

### 29.1 Temporal lift

Let $\sigma = (A_1, \ldots, A_T)$ be a periodically repeated ordered sequence of non-negative $n \times n$ input matrices. $A_t$ gives the means of production, available at phase $t-1$, required per unit of the commodities produced at phase $t$. Phase indices are modulo $T$. Define the $nT \times nT$ temporal lift $\bar{A}(\sigma)$ by the explicit block convention

$$\bar{A}(\sigma)_{t-1,t} = A_t, \qquad t = 1, \ldots, T,$$

with phase indices modulo $T$ and all other blocks zero. Thus the $(t-1)$-st block of $\bar{A}Q$ is exactly $A_t q_t$. Its commodities are dated pairs $(i, t)$. With this convention the monodromy matrix attached to the phase-0 block is

$$M(\sigma) = A_1 A_2 \cdots A_T.$$

Using the opposite row/column orientation reverses the written sequence and therefore interchanges labels such as 123 and 132 up to cyclic rotation; it does not change the underlying dated system. The convention above is used throughout all determinant and kernel calculations in this paper.

The block-power and Perron ingredients are standard periodic-system/Floquet mathematics; the Sraffian content lies in the dated Standard interpretation that follows.

**Construction 29.1 (Historical Standard Cycle from the periodic lift).** *Assume that $\bar{A}(\sigma)$ is irreducible and productive, $\rho(\bar{A}(\sigma)) < 1$. Then*

$$\lambda(\sigma) := \rho(M(\sigma))^{1/T} = \rho(\bar{A}(\sigma)) < 1.$$

*There is a unique $Q(\sigma) = (q_0, \ldots, q_{T-1}) \gg 0$, up to scale, such that*

$$A_t q_t = \lambda(\sigma) q_{t-1} \qquad \forall t$$

*(indices modulo $T$). The sequence $Q(\sigma)$ is the Historical Standard Cycle.*

**Proof.** Raising $\bar{A}(\sigma)$ to the $T$-th power produces a block-diagonal matrix whose diagonal blocks are the cyclic products $A_1 A_2 \cdots A_T$, $A_2 \cdots A_T A_1$, and their rotations. Cyclic products have the same non-zero spectrum, hence

$$\rho(\bar{A}(\sigma))^T = \rho(M(\sigma)).$$

Perron–Frobenius gives the unique positive eigenvector up to scale; productivity gives $\lambda(\sigma) < 1$. $\square$

### 29.2 The cycle profit factor

The spectral object is more general than a common period-by-period profit rate. At zero wages, dated prices satisfy

$$p_t^\top = (1 + r_t) p_{t-1}^\top A_t.$$

After one complete cycle, periodicity of prices implies

$$p_0^\top = \left[ \prod_{t=1}^{T} (1 + r_t) \right] p_0^\top M(\sigma).$$

**Identity 29.2 (Cycle profit factor).** *If $p_0 \gg 0$ satisfies the zero-wage periodic price equation displayed immediately above, then the gross profit factors satisfy $\prod_{t=1}^{T}(1 + r_t) = G(\sigma)$, where $G(\sigma) = 1/\rho(M(\sigma))$. If the same rate r is imposed in every phase, its constant-rate Standard equivalent is $R(\sigma) = \rho(M(\sigma))^{-1/T} - 1$.*

**Proof.** The periodic equation gives $p_0^\top M(\sigma) = \alpha p_0^\top$, where $\alpha = [\prod_{t=1}^{T}(1 + r_t)]^{-1}$. Since $M(\sigma)$ is non-negative and $p_0 \gg 0$, the lower and upper Collatz–Wielandt quotients of $M(\sigma)^\top$ evaluated at $p_0$ coincide at $\alpha$; hence $\alpha = \rho(M(\sigma))$. The product of gross profit factors is therefore $1/\rho(M(\sigma))$, and the common-rate formula follows by taking the T-th root. $\square$

The distinction between $G(\sigma)$ and $R(\sigma)$ matters. $G(\sigma)$ is the invariant attached to the entire cycle even when phase-specific profit rates differ. $R(\sigma)$ is the constant-rate equivalent relevant when the Sraffian comparison imposes one uniform rate across all dated processes.

### 29.3 The Standard wage–profit line

Let P collect all dated prices and L all dated direct labour coefficients. Under a common r and a common standardized wage w, the lifted price system is $P^\top = (1 + r) P^\top \bar{A}(\sigma) + wL^\top$. Normalize $L^\top Q(\sigma) = 1$ and use the Standard net product as numeraire: $P^\top [I - \bar{A}(\sigma)] Q(\sigma) = 1$. Since $\bar{A}(\sigma) Q(\sigma) = \lambda(\sigma) Q(\sigma)$, multiplication of the price equations by Q($\sigma$) gives the familiar linear relation.

**Corollary 29.3 (Historical Standard wage–profit relation).** *Under the normalizations just stated and a common profit rate in all phases, r=R($\sigma$)(1−w).*

**Proof.** Multiplication by $Q$ gives

$$P^\top Q = (1 + r) \lambda P^\top Q + w.$$

The numeraire implies $(1-\lambda)P^{\top}Q = 1$. Eliminating $P^{\top}Q$ gives

$$r\lambda = (1-w)(1-\lambda), \qquad r = (\lambda^{-1}-1)(1-w) = R(\sigma)(1-w).$$

### 29.4 Standard proportions and uniqueness

Let $Q = Q(\sigma) \gg 0$ be the Perron vector of the productive irreducible temporal lift $\bar{A}(\sigma)$, with $\bar{A}Q = \lambda Q$ and $R = \lambda^{-1} - 1$. Define the Historical Standard means of production and net product by

$$X_{\sigma} := \bar{A}(\sigma)Q(\sigma), \qquad Y_{\sigma} := [I - \bar{A}(\sigma)]Q(\sigma).$$

Then $X_{\sigma} = \lambda Q$ and $Y_{\sigma} = (1-\lambda)Q$, so
Y$\sigma$ = $R(\sigma)$ X$\sigma$.

This is the Standard-proportion property in dated commodity space. It also characterizes the positive Standard bundle uniquely up to scale: if $\widetilde{Q} \gg 0$ satisfies $[I - \bar{A}]\widetilde{Q} = \widetilde{R}\bar{A}\widetilde{Q}$ for some $\widetilde{R} > 0$, then $\bar{A}\widetilde{Q} = [1/(1+\widetilde{R})]\widetilde{Q}$. Perron–Frobenius therefore implies $\widetilde{Q} \propto Q$ and $\widetilde{R} = R$. The Historical Standard Cycle is consequently not an arbitrary positive eigenbundle to which a Standard label has been attached; it is the unique positive dated composite bundle whose net product stands in Standard proportions to its own means of production.

### 29.5 Historical Standard value invariance

Normalize dated direct labour by $L^{\top}Q = 1$. Consider two price systems for the same fixed ordered cycle, expressed in the same comparison unit,

$$P^{\top} = (1+r)P^{\top}\bar{A} + wL^{\top}, \qquad P'^{\top} = (1+r')P'^{\top}\bar{A} + w'L^{\top},$$

with $0 \le r, r' < R$. Since Q=X+Y, multiplication by Q gives $P^{\top}Y = rP^{\top}X + w$ and the analogous primed identity. Let $\Delta P = P' - P$, $\Delta r = r' - r$, and $\Delta w = w' - w$. Using Y=RX yields

$$(1 - r'/R)\,\Delta P^{\top}Y = \Delta r\, P^{\top}X + \Delta w.$$

**Observation 29.4 (Composite value identity).** For a fixed irreducible productive ordered cycle, under a common profit rate and common end-of-period wage, the value of the Historical Standard net product is unchanged between two interior distributive positions if and only if the change in profits valued on the Standard means of production is exactly offset by the change in wages. This is an algebraic identity inside the fixed ordered system. It is retained because it separates the composite Standard-value relation from the stronger phase-by-phase invariance result established in Section 29.6; it is not counted as a separate novelty claim.

Once Y is chosen as numeraire, $P^\top Y = 1$ and $P^\top X = 1/R$, so the distributive relation becomes $r = R(1-w)$. The linear wage–profit line is therefore downstream of the Standard-proportion and value-invariance structure rather than the sole reason for calling Q a Historical Standard object.

**29.6 Calendar-origin covariance and the boundary of phase-value invariance**

A cyclic rotation $\tau$ of $\sigma$ merely relabels dated commodity coordinates. There is a permutation matrix $\Pi$ such that $\bar{A}(\tau) = \Pi^\top \bar{A}(\sigma)\Pi$. Hence $Q(\tau) = \Pi^\top Q(\sigma)$, $X_\tau = \Pi^\top X_\sigma$ and $Y_\tau = \Pi^\top Y_\sigma$, up to the same normalization. With correspondingly relabelled prices and labour, $P_\tau^\top Y_\tau = P_\sigma^\top Y_\sigma$. Thus a cyclic rotation does not generate a different Historical Standard commodity; it generates the same Historical Standard Cycle represented from a different temporal origin.

The invariance belongs to the composite cycle, not generically to each dated component. Write $Q = (q_0, \ldots, q_{T-1})$, normalize $\sum_t \ell_t^\top q_t = 1$, and let $h_t = \ell_t^\top q_t$. Under the Standard net-product numeraire define $u_t(r) = (1-\lambda)p_t^\top q_t$. The block price equations imply the cyclic affine recursion

$$u_t = au_{t-1} + (1-a)h_t, \qquad a = (1+r)\lambda \in (0,1).$$

Solving the cyclic recursion gives

$$u_t(a) = \frac{1-a}{1-a^T}\sum_{k=0}^{T-1} a^k h_{t-k}.$$

This is a circulant linear map $u(a) = C(a)h$. Let $\omega = e^{2\pi i/T}$. The Fourier mode indexed by $j$ is an eigenvector of $C(a)$, with eigenvalue

$$G_j(a) = \sum_{k=0}^{T-1} \frac{(1-a)a^k}{1-a^T}\omega^{jk} = \frac{1-a}{1-a\omega^j},$$

where $\omega^{jT} = 1$ cancels the finite-geometric-series factor. If $u(a) = u(b)$ for two distinct distributive positions $a \neq b$, then $[C(a) - C(b)]h = 0$. A Fourier mode can belong to this kernel only if $G_j(a) = G_j(b)$. Cross multiplication yields

$$(b-a)(1-\omega^j) = 0.$$

Since $a \neq b$, only $j = 0$ survives. The kernel is therefore spanned by the uniform vector. With $\sum_t h_t = 1$, one obtains

$$h_t = \frac{1}{T} \qquad \text{for every } t.$$

Conversely, uniform labour weights make every phase value identical and therefore invariant across distributive positions. Thus the complete Historical Standard net product is the invariable composite; phase-by-phase values remain distribution-sensitive except in the uniform-labour special case. This argument closes the gap that would arise from inferring functional identity merely from equality at two points.

**29.6A Which Standard properties survive?**

| Property | Historical-cycle status | Conditions / boundary |
|---|---|---|
| Positive composite Standard bundle | Preserved | Irreducible productive temporal lift |
| Uniqueness up to scale | Preserved | Perron uniqueness; equivalently the Standard-proportion characterization |
| Net product proportional to means of production | Preserved | $Y_\sigma = R(\sigma)X_\sigma$ |
| Cycle profit identity | Preserved | Positive periodic zero-wage prices; common-rate equivalent when one $r$ is imposed |
| Price-free linear wage–profit relation | Preserved conditionally | Common $r$, common standardized wage, labour normalization, Standard net-product numeraire |
| Distributional value invariance of the composite net product | Preserved conditionally | Same ordered system; interior distributive positions; common comparison unit |
| Calendar-origin invariance | Preserved | Cyclic rotations act by permutation of dated coordinates |
| Phase-by-phase value invariance | Not generic | Across distinct rates only under uniform Standard labour weights |
| Invariance to non-cyclic permutation | Not preserved | Its failure is the historical-order result rather than a defect of the Standard interpretation |

### 29.7 Cyclic invariance and the minimum depth of order effects

Historical order should not depend on the arbitrary date at which an observer starts the clock. A cyclic rotation of $\sigma$ produces a cyclic product with the same non-zero spectrum, so $G(\sigma)$ and $R(\sigma)$ are unchanged. For square A and B, AB and BA have the same characteristic polynomial. Reversing a two-phase cycle therefore does not generate a new Standard ratio: ABAB… and BABA… are the same periodic cycle viewed from a different origin.

**Lemma 29.5 (Cyclic-order invariance and minimum non-cyclic depth).** *$R(\sigma)$ is invariant under cyclic rotations of an ordered periodic sequence. For T=2, the only reversal is a cyclic rotation and therefore cannot change R. For T$\geq$3, non-cyclic permutations exist, and there are strictly positive productive matrices for which a non-cyclic permutation changes R.*

The last statement is an existence result rather than a new matrix theorem. Its economic content is the boundary between relabelling time and changing historical order.

### 29.8 An order effect inside the positive cone

Consider the three strictly positive productive matrices

$$A = \begin{pmatrix} 1/5 & 3/10 \\ 3/10 & 2/5 \end{pmatrix}, \quad B = \begin{pmatrix} 1/10 & 3/10 \\ 2/5 & 2/5 \end{pmatrix}, \quad C = \begin{pmatrix} 2/5 & 1/5 \\ 2/5 & 1/10 \end{pmatrix}.$$

Every matrix has Perron root below one. Yet

$$ABC = \begin{pmatrix} 16/125 & 23/500 \\ 22/125 & 63/1000 \end{pmatrix}, \qquad ACB = \begin{pmatrix} 6/125 & 11/125 \\ 17/250 & 31/250 \end{pmatrix}.$$

Their Perron roots are approximately 0.1911673926 and 0.1721858457. The corresponding Standard ratios are

$$R(ABC) \simeq 0.7359160868, \qquad R(ACB) \simeq 0.7974942812.$$

The two cycles use the same three fixed systems once each; only the non-cyclic order differs. The spectral-radius gap is strict, and spectral radius is continuous in the matrix entries. There is therefore an open neighbourhood of the displayed primitives in which the order effect persists. The result is not an artefact of zero coefficients or reducibility.

## 30. Reducibility and historical basicity

### 30.1 Dated production graph

The irreducible theorem is not enough for Sraffian purposes because basic and non-basic commodities matter precisely when the system is reducible. In the single-product temporal lift, define a directed graph $G(\sigma)$ whose nodes are commodity-date pairs (i,t). There is an edge (i,t−1)→(j,t) whenever $(A_t)$_{ij}>0. A commodity-date pair is historically basic if it is a direct or indirect input into the production of every commodity-date pair in the periodically repeated system; equivalently, its node reaches every node of $G(\sigma)$.

**Graph Lemma 30.1 (Reducible historical basic set).** *The historical basic set $\mathcal{B}(\sigma)$ is non-empty if and only if the condensation graph of $G(\sigma)$ has a unique source strongly connected component. When it exists, that unique source component is exactly $\mathcal{B}(\sigma)$.*

**Proof.** If a node v reaches every node, its strongly connected component S reaches every component of the condensation graph. S cannot have an incoming edge from another component, because S reaches that component and the two would then be mutually reachable. Nor can there be another source component, because S reaches it. Hence S is the unique source.

Conversely, in any finite acyclic condensation graph every component has a source among its ancestors. If the source S is unique, S reaches every component. Every node in S therefore reaches every node in the original graph. A node outside S cannot reach S, otherwise S would have an incoming path. Thus exactly the nodes of S are historically basic. □

### 30.2 Frobenius normal form and price autonomy

Order the strongly connected components topologically with the unique source first. Under the input-to-output orientation just used, a simultaneous permutation puts the lift in upper block-triangular Frobenius form

$P^\top \bar{A}(\sigma)P$ = [[$B_1$, $C_{12}$, …, $C_1$m], [0, $B_2$, …, $C_2$m], …, [0,…,0,B_m]], (9)

where $B_1$ is the historical basic block. Partitioning the price vector conformably, the first block equation is $p_1{}^\top$=(1+r)$p_1{}^\top B_1$+w$\ell_1{}^\top$. It contains no downstream non-basic price. The historical basic price subsystem is autonomous in the sense required by Sraffa's basic/non-basic logic.

Autonomy, however, does not imply spectral dominance. A downstream non-basic block may have a Perron root at least as large as that of the source. The next proposition states the missing condition.

**Frobenius Criterion 30.2 (Spectral admissibility of historical basics).** *Let $B_1$ be the unique source block and let $\lambda_1$=ρ($B_1$)>0. There exists a strictly positive left eigenvector $p \gg 0$ of the full temporal lift at $\lambda_1$ if and only if $\lambda_1$>ρ($B_j$) for every downstream diagonal block j>1.*

**Proof.** Necessity and sufficiency follow recursively from the block equations $p_j{}^\top(\lambda_1 I - B_j)=c_j{}^\top$. In a topological ordering, every downstream block has an incoming edge from an earlier block;

once the preceding price blocks are strictly positive, this gives $c_j \geq 0$ and $c_j \neq 0$. If $p_j \gg 0$ exists, multiplication by the positive right Perron vector $v_j$ of $B_j$ gives $(\lambda_1 - \rho(B_j)) p_j^\top v_j = c_j^\top v_j > 0$, which implies strict dominance. Conversely, if $\lambda_1 > \rho(B_j)$, then $\lambda_1 I - B_j$ is a nonsingular M-matrix and $(\lambda_1 I - B_j)^{-1} \geq 0$. Irreducibility of the SCC block and $c_j \neq 0$ imply $p_j \gg 0$. Induction over the topological order completes the construction. □

### 30.3 Exact change in the basic set

The following reducible example separates historical basicity from the positive-cone example of Section 29.8. Let

$$A = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \quad B = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad C = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

For the order ABC, the unique source SCC is

$$\mathcal{B}(ABC) = \{(2,0),(1,1),(2,1),(1,2),(2,2)\}.$$

For ACB it is

$$\mathcal{B}(ACB) = \{(2,0),(2,1),(1,2)\}.$$

The same physical commodity can therefore be basic at a given phase under one non-cyclic ordering and non-basic under another. At the same time, $\rho(ABC)=1/4$ and $\rho(ACB)=1/8$, so the two Standard ratios are $4\hat{}(1/3)-1$ and 1. The downstream SCCs in this example are acyclic, with Perron root zero, and the spectral-admissibility condition is satisfied in both orders.

Equations (11)–(12) should not be read as saying that basicity is a primitive function of calendar time. A cyclic rotation merely relabels the phases and carries $\mathcal{B}(\sigma)$ into the corresponding relabelled set. What matters is a non-cyclic reordering of the reproductive relations.

## 31. Joint production: from a matrix to a dated pencil

### 31.1 Dated input-output pair

Joint production requires separate input and output matrices. At phase t let $A_t \in \mathbb{R}_+\hat{}\{n \times k_t\}$ collect commodity inputs and $B_t \in \mathbb{R}_+\hat{}\{n \times k_t\}$ collect joint outputs of $k_t$ processes. A process of phase t uses commodities dated $t-1$ and delivers commodities dated t. Stacking all dated commodities and processes gives rectangular non-negative matrices $\bar{A}(\sigma)$ and $\bar{B}(\sigma)$ of dimension $nT \times K$, where $K=\sum\_t\ k_t$. The dated price equations are

$$P^\top \bar{B}(\sigma) = (1+r) P^\top \bar{A}(\sigma) + wL^\top.$$

At zero wages, a candidate gross profit factor $\mu$=1+r is therefore a root, in the generalized sense, of the matrix pencil

$$C(\sigma, \mu) = \bar{B}(\sigma) - \mu \bar{A}(\sigma).$$

The distinction between left and right null spaces is now economically substantive. A positive left null vector is a positive dated-price system. A non-negative right null vector is a set of process multipliers that reproduces input and output in Standard proportions. Sraffa's Chapter VIII shows why the latter need not be non-negative in a general joint-production system.

### 31.2 Historical basicity under Sraffa's rank criterion

The direct-or-indirect graph criterion used in Section 4 is not a complete definition of basicity under joint production. Sraffa replaces it with a linear-dependence criterion: a linked group of s commodities is non-basic when the matrix formed by their input and output columns has rank no greater than s. The temporal lift permits the same test to be applied to commodity-date pairs without changing its algebraic content.

**Lemma 31.1 (Dated rank decomposition).** *Let S contain $S_t$ commodity-date pairs at phase t. Form the Sraffa matrix [$\bar{A}$_S($\sigma$)$^\top$ $\bar{B}$_S($\sigma$)$^\top$]. After a permutation of its rows and columns, it is a direct sum across phases of the blocks [$A_t$[S_{t−1},:]$^\top$ $B_t$[$S_t$,:]$^\top$]. Hence its rank equals the sum of the ranks of those phase blocks.*

**Proof.** A phase-t process has non-zero input entries only in the commodity-date block t−1 and non-zero output entries only in block t. Once the selected input and output columns are ordered by the process phase to which they belong, different phase contributions occupy disjoint process-row blocks. Rank is additive across the resulting direct sum. □

This lemma does not claim that every reducible joint-production system possesses a unique global basic partition. It says that Sraffa's own rank test can be applied to the dated lift and decomposed phase by phase. Order can alter the rank test because it changes which output block at one phase is paired with which input requirements at the next.

## 32. Positive-price roots, Standard realizability, and competitive support

### 32.1 A useful sufficient class

Before giving the inverse-free result, it is useful to isolate a sufficient square class in which the familiar "lowest economically significant root" can be selected by an ordinary Perron argument. For a square dated pair, let N($\sigma$)=$\bar{B}$($\sigma$)−$\bar{A}$($\sigma$) be nonsingular and define the price-side operator

$$H(\sigma) = \bar{A}(\sigma) N(\sigma)^{-1}.$$

**Theorem 32.1 (Historical lowest-root theorem).** *Suppose H($\sigma$) is non-negative and irreducible. Then R($\sigma$)=1/ρ(H($\sigma$)) is the smallest positive root of det[$\bar{B}$($\sigma$)−(1+r)$\bar{A}$($\sigma$)]=0 and the only positive root that supports a strictly positive dated-price vector. If $x \gg 0$ is the right Perron vector of H($\sigma$) and z=N($\sigma$)$^{-1}$x, then $\bar{B}$($\sigma$)z=(1+R($\sigma$))$\bar{A}$($\sigma$)z. The multiplier vector z need not be non-negative.*

**Proof.** Write $\bar{B}-(1+r)\bar{A}=N-r\bar{A}=(I-r\bar{A}N^{-1})N$. Hence a positive algebraic root r corresponds to a positive eigenvalue 1/r of H=$\bar{A}N^{-1}$. Since every positive eigenvalue of a non-negative irreducible matrix is at most its Perron root, r≥1/$\rho$(H). At r=1/$\rho$(H), the positive left Perron vector supplies positive prices. Conversely any strictly positive price vector is a positive left eigenvector of H and therefore must correspond to $\rho$(H). Finally Hx=$\rho$(H)x and z=N$^{-1}$x imply $\bar{A}$z=$\rho$(H)N z, hence $\bar{B}$z=(1+R)$\bar{A}$z. □

The theorem selects the price-admissible root without imposing $B_t^{-1}$ or the stronger sign restriction $A_tB_t^{-1}\geq 0$. It is a sufficient class: N must be invertible and H must preserve the positive cone. Section 32.3 instead gives a fixed-$\mu$ characterization that requires neither condition.

### 32.2 Three levels of Standard status

For an arbitrary, possibly singular and rectangular, dated pair ($\bar{A}$,$\bar{B}$), fix $\mu$>0 and write C($\mu$)=$\bar{B}-\mu\bar{A}$. Assume only that every process uses at least one input, so that p$^\top$a_j>0 for every $p \gg 0$. Define:

- $\mu$ is price-admissible if there exists $p \gg 0$ with C($\mu$)$^\top$p=0;
- $\mu$ is Standard-realizable if, in addition, there exists z≥0, z≠0, with C($\mu$)z=0;
- $\mu$ is full-support Standard-realizable if z can be chosen strictly positive.

These definitions deliberately separate the sign of prices from the sign of process multipliers. Negative multipliers are admissible in Sraffa's formal Standard construction, but they do not describe a physically operable vector of process scales.

### 32.3 Inverse-free full-support characterization

Let r_i(C) denote the rows and $c_j$(C) the columns of a matrix C. The next result is a finite-dimensional convex-geometric characterization, equivalent to a Stiemke-type alternative.

The next equivalence is a direct finite-dimensional Stiemke-type alternative, used here as a diagnostic rather than claimed as a new separation theorem.

**Stiemke Characterization 32.2 (Two-sided historical pencil condition).** *For fixed μ>0, there exist $p \gg 0$ and $z \gg 0$ satisfying C(μ)$^\top$p=0 and C(μ)z=0 if and only if 0 lies in the relative interior of conv{r_i(C(μ))} and also in the relative interior of conv{$c_j$(C(μ))}. If the row condition holds while the column condition is weakened to 0∈conv{$c_j$(C(μ))}, then μ is price-admissible and Standard-realizable with a non-zero non-negative multiplier vector. No squareness, determinant, inverse or irreducibility assumption is required.*

**Proof.** For a finite indexed family of vectors, zero belongs to the relative interior of its convex hull if and only if zero has a convex representation with strictly positive weights on every listed vector; Appendix B.6 gives a direct proof. Normalizing ∑_i p_i=1 therefore makes $C^\top p=0$ with $p \gg 0$ equivalent to the row relative-interior condition. Likewise, normalizing ∑_j z_j=1 makes Cz=0 with $z \gg 0$ equivalent to the column relative-interior condition. If z is only non-negative and non-zero, the same normalization yields the ordinary convex-hull condition. □

### 32.4 Collatz–Wielandt optimality and the multiplier test

For $p \gg 0$ define the process-specific gross value ratio $\phi$_j(p)=$p^\top$b_j/($p^\top$a_j), and define the maximin bound

$$\Gamma(\bar{A}, \bar{B}) = \sup_{p \gg 0} \min_j \phi_j(p).$$

This is an orientation of the Collatz–Wielandt quotient for a pair of non-negative operators. Friedland (2020) studies such quotients, including rectangular pairs and a commodity-pricing interpretation. In general, an all-binding generalized eigenvector need not by itself be an optimizer; some optimal solutions can live on reduced supports. For the present Sraffian application, the missing condition is exactly the existence of a non-negative right kernel vector. Throughout this subsection assume that every process has a non-zero input column, so that the ratios are defined and strictly positive denominators are available for every strictly positive price vector.

**Proposition 32.3 (All-binding optimality criterion).** *Suppose $p \gg 0$ satisfies $\bar{B}^\top p$=$\mu\bar{A}^\top p$. Then Γ($\bar{A}$,$\bar{B}$)=$\mu$ if and only if ker[$\bar{B}-\mu\bar{A}$] contains a non-zero non-negative vector. If no such vector exists, Γ($\bar{A}$,$\bar{B}$)>$\mu$.*

**Proof.** If z≥0, z≠0 and $\bar{B}$z=$\mu\bar{A}$z, then for every $q \gg 0$ the identity $q^\top\bar{B}$z=$\mu q^\top\bar{A}$z makes $\mu$ a weighted average of the ratios $\phi$_j(q) over the support of z, with non-negative weights z_j $q^\top$a_j. Hence min_j $\phi$_j(q)≤$\mu$ for every q. The all-binding price vector p attains $\mu$, so Γ=$\mu$.

Conversely, if no non-zero non-negative z lies in the kernel, zero is not in the convex hull of the columns of C=$\bar{B}-\mu\bar{A}$. Strict separation gives a vector h with $h^\top c_j$>0 for every j. For sufficiently small $\varepsilon$>0, q=p+$\varepsilon$h remains strictly positive and satisfies $q^\top$b_j>$\mu q^\top$a_j for every process. Hence min_j$\phi$_j(q)>$\mu$ and Γ>$\mu$. □

Proposition 32.3 gives a precise meaning to negative Standard multipliers. If the positive-price root has a one-dimensional right kernel generated by a mixed-sign vector, then the formal Sraffian root is strictly below the pairwise maximin bound Γ. This is an algebraic statement, not yet a theorem of competitive technique selection: processes outside a binding support can display extra-profit ratios and require an explicit selection closure before they are interpreted as inactive.

### 32.5 A three-level realizability index

For a price-admissible root $\mu(\sigma)$, define the Standard status

$$\mathfrak{S}(\sigma) = \begin{cases} 0, & \ker C \cap \mathbb{R}_+^K = \{0\}, \\ 1, & \exists z \geq 0,\ z \neq 0,\ Cz = 0, \\ 2, & \exists z \gg 0,\ Cz = 0. \end{cases}$$

The index is ordinal only in the logical sense. S=0 means that no non-zero non-negative right-kernel vector exists. In a square simple-root case with a one-dimensional nontrivial right kernel, this means that every generator of that kernel has mixed signs; in a rectangular system, S=0 may instead coincide with a trivial right kernel. S=1 means a physically realizable Standard construction on a reduced support, and S=2 a full-support Standard system. Historical order can change this status while every elementary input and output coefficient remains fixed.

### 32.6 Rectangular systems and solving subpencils

Full support is not generically expected for a one-parameter rectangular pencil. Simultaneous non-zero left and right kernels require

$$\operatorname{rank} C(\mu) \leq \min\{m, n\} - 1.$$

In the ambient space of $m \times n$ matrices the corresponding determinantal variety has codimension $|m - n| + 1$. When $m \neq n$, a generic one-dimensional pencil will therefore not meet it without additional structure. This is why support-reduced solving systems arise naturally in rectangular joint production.

If the supremum $\Gamma$ is attained at $p^* \gg 0$, let $J^* = \{j : \phi_j(p^*) = \Gamma\}$. First-order convex geometry gives

$$0 \in \operatorname{conv}\{c_j(\bar{B} - \Gamma \bar{A}) : j \in J^*\},$$

so there is a non-negative right-kernel certificate supported on $J^*$. Because those active columns lie in the $(m-1)$-dimensional hyperplane $(p^*)^\perp$, Carathéodory's theorem permits a certificate using at most $m$ process-date pairs. This is a useful algebraic solving subpencil, but it must not automatically be identified with a competitively selected Sraffian technique.

### 32.7 Competitive support as a complementarity solution

A solving subpencil is not yet an economy. For a dated non-negative pair $(\bar{A}, \bar{B})$, a zero-wage competitive Standard support is a quadruple $(g, y, p, S)$ with $y \geq 0$, $p \geq 0$, $\mathbf{1}^\top y = \mathbf{1}^\top p = 1$, such

that
$$\begin{aligned} \bar{B}y - g\bar{A}y &\geq 0, \\ \bar{B}^\top p - g\bar{A}^\top p &\leq 0, \\ p^\top(\bar{B}y - g\bar{A}y) &= 0, \\ y_j[\bar{B}^\top p - g\bar{A}^\top p]_j &= 0 \qquad \forall j. \end{aligned}$$

The operated support is $S = \{j : y_j > 0\}$. Inactive columns may have strictly negative margins. Under the standard von Neumann connectivity/productivity hypotheses, a normalized saddle point exists and the maximal primal expansion factor equals the minimal dual factor. Consequently at least one competitive support exists; conversely, an algebraic root of a square subpencil that violates a full-system primal or dual inequality is not a competitive support of the rectangular system.

*Classical existence statement 32.4 (rectangular competitive support)*

Under the standard von Neumann connectivity/productivity assumptions, the full dated rectangular system has at least one normalized primal-dual complementarity solution $(g, y, p)$. Its common factor $g$ is simultaneously the maximum technically supportable uniform expansion factor and the minimum factor compatible with non-negative prices excluding positive excess profit. Every operated support is selected by complementarity. If the solution is unique and strict on inactive processes, its support and normalized activity composition are locally stable under small perturbations.

**32.8 Positive wages and the support correspondence**

With dated labour vector $\bar{\ell}$ and wage $w$ in the chosen numeraire, the dual inequality becomes

$$\bar{B}^\top p - g\bar{A}^\top p - w\bar{\ell} \leq 0,$$

with equality on active processes. For fixed $g$, one may maximize $w$ subject to primal feasibility, the dual inequalities, the normalizations, and complementarity. A unique support requires additional nondegeneracy, such as strict complementarity plus a nonsingular active Jacobian, or an explicit refinement such as maximal wage at fixed $g$. The correct general object is therefore a support correspondence,

$$\mathfrak{I}^c(\sigma) = \left\{ (g, S, \mathcal{B}, Q, \mathfrak{S}) : \begin{array}{l} \exists\, y, p \text{ satisfying the normalized} \\ \text{full-system complementarity conditions} \end{array} \right\}.$$

When the complementarity solution is unique, $\mathfrak{I}^c$ is single-valued and reduces to the ordinary Historical Standard packet. When it is multiple, the correspondence records genuine competitive indeterminacy rather than a failure of the theory.

## 33. Exact order-sensitive joint-production example

### 33.1 Primitives

The example has two purposes. It shows that a non-cyclic permutation can change both the lowest positive-price root and Standard realizability, and it displays multiple positive algebraic roots. It also lies outside the stronger phase-by-phase restriction $\mathrm{A}_t\mathrm{B}^{-1}\geq 0$ mentioned in Section 6.1, even though the two dated orders remain inside the sufficient class of Theorem 32.1.

All three phases have the same strictly positive joint-output matrix

$$B = \begin{pmatrix} 1 & 3/5 \\ 2/5 & 1 \end{pmatrix}.$$

and input matrices

$$A_1 = \begin{pmatrix} 2/5 & 2/5 \\ 3/20 & 2/5 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 7/20 & 1/5 \\ 1/5 & 1/2 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 7/20 & 3/10 \\ 1/2 & 9/20 \end{pmatrix}.$$

This is genuine joint production: every phase has two processes and both outputs enter each process's gross-output vector. The phase-by-phase sign restriction fails explicitly: $A_1B^{-1}$ and $A_2B^{-1}$ each contain the negative entry $-1/76$.

### 33.2 Same systems, different order

Compare $\sigma = 123$ with $\tau = 132$. For both orders $N = \bar{B} - \bar{A}$ is nonsingular and $H = \bar{A}N^{-1}$ is strictly positive. Theorem 32.1 selects the unique positive-price root and identifies it as the lowest positive algebraic root.

For 123 the determinant polynomial, after removal of a positive scalar factor, is

$$405r^6 + 2430r^5 + 6075r^4 - 196036r^3 - 606333r^2 - 609978r + 1552173.$$

Its positive roots are approximately 1.0609301694 and 6.9119822603. The economically selected root is therefore

$$R(123) = 1.0609301694\ldots.$$

For 132 the polynomial is

$$405r^6 + 2430r^5 + 6075r^4 - 172248r^3 - 534969r^2 - 538614r + 1575961,$$

with positive roots approximately 1.1514807657 and 6.5789861568, so

$$R(132) = 1.1514807657\ldots.$$

The same three joint-production systems, each used exactly once, therefore have different lowest positive-price Standard roots solely because their non-cyclic order differs.

### 33.3 Order changes Standard realizability

The right kernel at the selected root changes more sharply. Normalized numerical representatives are

$$z(123) \simeq (-0.11685, 0.65954, 0.06066, 0.57042, 0.18793, 0.43241)^\top,$$
$$z(132) \simeq (0.06137, 0.62169, 0.09933, 0.55712, 0.33545, 0.42065)^\top.$$

Mathematical Appendix C certifies these signs exactly from cofactors on rational isolating intervals. The kernels are one-dimensional at the selected simple roots. Hence

$$\mathfrak{S}(123) = 0, \qquad \mathfrak{S}(132) = 2.$$

Under 123 the Standard construction is formal and necessarily uses a negative process multiplier. Under 132 every dated process can enter with a positive multiplier. The result is not merely a difference in the numerical value of R: order changes whether the Standard system is physically realizable as a full-support reproduction vector.

**Table 33.1. Order-sensitive Standard roots and multiplier status**

| Order | Lowest positive-price $R$ | Other positive root | Right-kernel sign | $S$ |
|---|---|---|---|---|
| 123 | 1.0609301694 | 6.9119822603 | $-+++++$ | 0 |
| 132 | 1.1514807657 | 6.5789861568 | $++++++$ | 2 |

Both selected roots are simple, H is strictly positive, and every displayed kernel component is non-zero. These are strict conditions. By continuity of simple roots, Perron data and one-dimensional kernel representatives, they persist on some open neighbourhood of the primitives. The example establishes local, not merely pointwise, order sensitivity. It does not establish robustness over a large parameter region.

## 33A. Support-to-Order under an explicit recurrent-capacity closure

The previous sections distinguish external historical order from internal recurrent order. The remaining question is whether the two can be joined without inserting either the external state maps or the terminal phase order by hand. This section gives an exact conditional construction. It is intentionally block-triangular rather than a single price-unified equilibrium. An external competitive-complementarity block determines prices, wages, traverse supports, and wage-net state maps; the inherited terminal stock then enters a separate activity-analysis block that selects recurrent capacity. The terminal dual variables are resource scarcity values, not production prices, and there is no feedback from them into the preceding price equations. Conditional on the distributive word and this explicit terminal closure, the two blocks form a causal support-to-state-to-order architecture.

### 33A.1 General support-to-order result

Let a fixed external activity catalogue have commodity inputs $a_j$, gross outputs $b_j$, labour coefficients $\ell_j$, and physical wage basket $c \gg 0$. At a gross distributive factor $\mu = 1 + r$, a traverse support satisfies

$$0 \le y_j \ \perp \ \mu p^\top a_j + w\ell_j - p^\top b_j \ge 0, \qquad p^\top c = 1,$$

together with a stated quantity closure. Under full-resource operation the current material equations are $A^e y = x$ and the inherited wage-net state is

$$x^+ = B^e y - cw\ell^\top y.$$

Let $\Sigma$ be a finite set of internally periodic orders of one fixed phase multiset, all represented at the same calendar-entry phase. For each $\sigma \in \Sigma$, let $q_0^\sigma \gg 0$ be the normalized date-0 block of its Historical Standard Cycle, with $\mathbf{1}^\top q_0^\sigma = 1$. Put these vectors in the matrix $Q$. At the comparison date define the recurrent-capacity problem

$$\max_{z \ge 0} \ \mathbf{1}^\top z \qquad \text{s.t.} \qquad Qz \le x_T,$$

with dual resource values $\pi \ge 0$. Its KKT conditions are

$$0 \le z \ \perp \ Q^\top \pi - \mathbf{1} \ge 0, \qquad 0 \le \pi \ \perp \ x_T - Qz \ge 0.$$

Because every column is normalized by $\mathbf{1}^\top q_0^\sigma = 1$, the objective has the exact physical interpretation

$$\mathbf{1}^\top z = \mathbf{1}^\top Qz.$$

It therefore maximizes the aggregate normalized quantity of inherited productive stock absorbed into recurrent operation. Componentwise slack may remain when composition is mismatched, so this is more precisely a *maximum recurrent-absorption* or *recurrent-capacity* criterion than literal full utilization of every inherited good. It is a stated quantity-side closure. It is not a maximum-profit-rate criterion, and $\pi$ are terminal resource shadow values rather than Sraffian prices of production.

Thus an internal order is selected as an active recurrent activity under an explicit capacity/scarcity criterion, not by an externally fixed launch threshold.

**Lemma 33A.1 (Regular Composition and Persistence).** Consider two distributive words $h = (\mu_1, \ldots, \mu_T)$ and $h'$ built from the same finite set of distributive positions. Suppose: (i) at every dated position used by the words, the full external catalogue has a unique regular complementarity support, with strict inequalities on inactive processes; (ii) the stated quantity closure therefore induces locally unique continuous state maps; (iii) at the two resulting terminal states the recurrent-capacity LP has unique strict complementary supports $\{\sigma\}$ and $\{\tau\}$, where $\sigma$ and $\tau$ are non-cyclic orders of the same phase multiset; and (iv) the relevant internal Perron roots are simple and $\mathfrak{I}(\sigma) \neq \mathfrak{I}(\tau)$. Then, conditional only on the distributive word, the stacked external-support, state-transition, and terminal KKT system selects different internal recurrent orders and hence different Historical Standard packets. If the active Jacobians are nonsingular and all inactive-profit, resource-slack, reduced-cost, and spectral-separation inequalities are strict, the entire support-to-order chain persists on an open neighbourhood of the combined primitives.

**Proof.** On each external support the active zero-profit equations plus normalization have a nonsingular Jacobian, so the implicit-function theorem gives a locally unique price–wage solution; strict inactive margins keep the same support. The stated quantity closure then gives a locally unique continuous state map. At the terminal date, a nondegenerate LP basis with strict reduced costs and complementary resource slacks is locally stable. Simple Perron roots make the normalized Standard vectors and packets continuous in the internal phase primitives. Composing the regular blocks gives the claimed local support-to-order map and its open persistence. □

The lemma is a composition-and-stability result. Assumption (iii) already states terminal support separation, so the lemma does not derive that separation from weaker primitives. The substantive support-selection content is supplied by the exact fixed-catalogue witness below, which verifies assumptions (i)–(iv) from explicit rational matrices and strict KKT margins.

**Proposition (Interior CRS blocks state-dependent selection by a pure maximum-rate ranking).** *Let $\Sigma$ be finite, let every normalized recurrent launch vector satisfy $q_0^\sigma \gg 0$, and let the inherited terminal state satisfy $x_T \gg 0$. Under constant returns and divisibility, every $\sigma \in \Sigma$ is feasible at some strictly positive scale. Hence, if terminal choice ranks feasible orders solely by a state-independent scalar $R_\sigma$ and* $\arg\max_{\sigma \in \Sigma} R_\sigma$ *is unique, the selected order is independent of $x_T$. In particular, a pure maximum-profit-rate ranking cannot transmit interior inherited composition into recurrent-order selection.*

**Proof.** For every $\sigma$, choose $0 < \varepsilon_\sigma \leq \min_i x_{T,i}/q_{0,i}^\sigma$. Then $\varepsilon_\sigma q_0^\sigma \leq x_T$, so every order is seedable. A unique ranking by the state-independent $R_\sigma$ therefore has the same maximizer for every $x_T \gg 0$. □

The proposition explains why the terminal closure must do more than rank positive recurrent orders by a state-independent long-period rate if inherited composition is to matter. The recurrent-capacity LP is one transparent state-sensitive closure; fixed costs, indivisibilities, non-homogeneous launch requirements, demand constraints, or other scarcity criteria would be alternatives. No claim is made here that the capacity objective is uniquely privileged.

**33A.2 Exact rational witness: external support is endogenous**

There are two reproducible commodities and four external elementary processes, ordered $P_1, E_1, P_2, E_2$. Their input matrix is

$$A^e = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \qquad c = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix},$$

with labour vector

$$\ell^e = \left( \frac{1}{10}, \frac{3}{10}, \frac{1}{10}, \frac{3}{10} \right)^\top.$$

The two pairs of fixed gross-output columns are

$$B_P = \begin{pmatrix} 93/100 & 3/5 \\ 27/100 & 3/5 \end{pmatrix}, \qquad B_E = \begin{pmatrix} 89/120 & 5/24 \\ 73/120 & 137/120 \end{pmatrix}.$$

The full output catalogue is $B^e = [B_{P1}, B_{E1}, B_{P2}, B_{E2}]$.

At

$$\mu_P = \frac{11}{10}$$

the unique full-catalogue support is $\{P_1, P_2\}$, with

$$p_P = (1, 1)^\top, \qquad w_P = 1,$$

and catalogue profit slacks

$$(0, 1/20, 0, 1/20).$$

The active price–wage Jacobian has determinant $-77/1000$. At

$$\mu_E = \frac{6}{5}$$

the unique support is $\{E_1, E_2\}$, with

$$p_E = (1, 1)^\top, \qquad w_E = \frac{1}{2},$$

profit slacks

$$(1/20, 0, 1/20, 0),$$

and active Jacobian determinant $-1/5$. Appendix C.4 enumerates all four complete supports and shows explicitly why each alternative violates at least one full-catalogue inequality.

The supported wage-net maps therefore emerge from the fixed gross-output catalogue:

$$F_P = \begin{pmatrix} 22/25 & 11/20 \\ 11/50 & 11/20 \end{pmatrix}, \qquad F_E = \begin{pmatrix} 2/3 & 2/15 \\ 8/15 & 16/15 \end{pmatrix}.$$

They satisfy

$$\mathbf{1}^\top F_P = \frac{11}{10}\mathbf{1}^\top, \qquad \mathbf{1}^\top F_E = \frac{6}{5}\mathbf{1}^\top, \qquad F_P F_E \neq F_E F_P.$$

Starting from $x^\circ = (1,1)^\top$,

$$x^{PE} = F_E F_P x^\circ = \left(\frac{132}{125}, \frac{198}{125}\right)^\top,$$

$$x^{EP} = F_P F_E x^\circ = \left(\frac{198}{125}, \frac{132}{125}\right)^\top.$$

Yet

$$\mathbf{1}^\top x^{PE} = \mathbf{1}^\top x^{EP} = \frac{66}{25}.$$

The historical effect at the terminal date is purely compositional.

### 33A.3 The same internal phase catalogue, two endogenous recurrent orders

Let the three internally recurrent phase matrices be

$$A = \frac{1}{10}\begin{pmatrix} 7 & 2 \\ 4 & 5 \end{pmatrix}, \qquad B = \frac{1}{10}\begin{pmatrix} 4 & 1 \\ 4 & 6 \end{pmatrix}, \qquad C = \frac{1}{10}\begin{pmatrix} 1 & 5 \\ 5 & 3 \end{pmatrix}.$$

They are strictly positive, nonsingular, and individually productive. Compare $ABC$ with the non-cyclic order $ACB$ at the common $A$-entry date. Their monodromies are

$$M_{ABC} = \frac{1}{1000}\begin{pmatrix} 131 & 237 \\ 206 & 282 \end{pmatrix}, \qquad M_{ACB} = \frac{1}{1000}\begin{pmatrix} 232 & 263 \\ 256 & 239 \end{pmatrix}.$$

**Proposition (Two-dimensional commutator ordering).** *Let $A, B, C$ be $2 \times 2$ positive matrices and put $M_1 = ABC$, $M_2 = ACB$. Then*

$$\det M_1 = \det M_2, \qquad \operatorname{tr} M_1 - \operatorname{tr} M_2 = \operatorname{tr}\big(A[B,C]\big).$$

*Because the Perron root of a positive $2 \times 2$ matrix is strictly increasing in its trace when the determinant is held fixed, the sign of $\operatorname{tr}(A[B,C])$ determines the ordering of $\rho(M_1)$ and $\rho(M_2)$. For a three-phase*

*Historical Standard system,* $R^{\text{Std}} = \rho(M)^{-1/3} - 1$*, so the ordering of the Standard maximum-profit rates is reversed.*

**Proof.** Multiplicativity of determinants gives the first identity, cyclicity of trace gives the second, and $\lambda_+(t,d) = (t + \sqrt{t^2 - 4d})/2$ has $\partial\lambda_+/\partial t > 0$ for a positive $2 \times 2$ matrix. □

In the present witness,

$$\det M_{ABC} = \det M_{ACB} = -\frac{297}{25000}, \qquad \operatorname{tr}(M_{ABC}) - \operatorname{tr}(M_{ACB}) = \operatorname{tr}\big(A[B,C]\big) = -\frac{29}{500}.$$

Thus the commutator, not the calibration alone, determines the direction of the internal order effect: $\rho_{ABC} < \rho_{ACB}$ and therefore $R^{\text{Std}}_{ABC} > R^{\text{Std}}_{ACB}$.

The Perron roots and normalized date-0 Standard compositions are exactly

$$\rho_{ABC} = \frac{11}{25}, \qquad q_{ABC} = \begin{pmatrix} 79/182 \\ 103/182 \end{pmatrix},$$

$$\rho_{ACB} = \frac{99}{200}, \qquad q_{ACB} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}.$$

Hence

$$R^{\text{Std}}_{ABC} = \left(\frac{25}{11}\right)^{1/3} - 1 \simeq 0.314768,$$

$$R^{\text{Std}}_{ACB} = \left(\frac{200}{99}\right)^{1/3} - 1 \simeq 0.264149.$$

The same three phase technologies, each used once, therefore generate distinct internal Standard packets solely because their non-cyclic recurrent order differs.

Let

$$Q = \begin{pmatrix} 79/182 & 1/2 \\ 103/182 & 1/2 \end{pmatrix}.$$

At $x^{PE}$ the recurrent-capacity LP has the exact primal–dual solution

$$z^{PE} = \begin{pmatrix} 24024/9875 \\ 0 \end{pmatrix}, \qquad \pi^{PE} = \begin{pmatrix} 182/79 \\ 0 \end{pmatrix}.$$

The resource slack and reduced costs are

$$x^{PE} - Qz^{PE} = \begin{pmatrix} 0 \\ 2046/9875 \end{pmatrix}, \qquad Q^\top \pi^{PE} - \mathbf{1} = \begin{pmatrix} 0 \\ 12/79 \end{pmatrix}.$$

Thus $ABC$ is the unique active recurrent order. At $x^{EP}$,

$$z^{EP} = \begin{pmatrix} 0 \\ 264/125 \end{pmatrix}, \qquad \pi^{EP} = \begin{pmatrix} 0 \\ 2 \end{pmatrix},$$

with

$$x^{EP} - Qz^{EP} = \begin{pmatrix} 66/125 \\ 0 \end{pmatrix}, \qquad Q^\top \pi^{EP} - \mathbf{1} = \begin{pmatrix} 12/91 \\ 0 \end{pmatrix}.$$

Thus $ACB$ is uniquely selected by the recurrent-capacity criterion.

This direction is economically informative and also marks the exact scope of the result. After $EP$, the higher-rate order $ABC$ is still feasible at strictly positive scale:

$$\bar{z}_{ABC}(x^{EP}) = \min_i \frac{x_i^{EP}}{q_{ABC,i}} = \frac{24024}{12875} \simeq 1.86594 > 0.$$

Yet $R^{\mathrm{Std}}_{ABC} \simeq 0.314768 > 0.264149 \simeq R^{\mathrm{Std}}_{ACB}$ while the capacity LP selects $ACB$. Hence the witness exhibits a genuine divergence between *maximum recurrent absorption of inherited stock* and *maximum Standard profit rate*. If terminal choice were instead defined as a pure maximum-$R^{\mathrm{Std}}$ ranking over seedable orders, $ABC$ would be selected after both $PE$ and $EP$ and the terminal order effect would disappear. The historical result is therefore substantive but closure-conditional: inherited composition can select the lower-maximum-rate recurrent organization under the stated capacity/scarcity criterion. This is not a claim that the lower-rate order is technologically inferior in every sense, nor that competitive long-period technique choice must follow the capacity objective.

Consequently the two exact solutions of the same block architecture are

$$\boxed{(\mu_P, \mu_E) \to \{P_1, P_2\}, \{E_1, E_2\} \to x^{PE} \to ABC \to \mathfrak{I}(ABC),}$$

and

$$\boxed{(\mu_E, \mu_P) \to \{E_1, E_2\}, \{P_1, P_2\} \to x^{EP} \to ACB \to \mathfrak{I}(ACB).}$$

Under the stated full-resource traverse closure and the distinct terminal recurrent-capacity closure, the grand technical catalogue and the distributive word determine the external maps and both support layers. The architecture is causal and block-triangular, not a single price system spanning both layers.

All relevant margins are strict: the inactive external profit margin is $1/20$; the terminal inactive-order reduced costs are $12/79$ and $12/91$; the positive terminal resource slacks are $2046/9875$ and $66/125$; and the internal Perron-root separation is $11/200$. Together with the nonzero active Jacobian determinants, these numbers certify open persistence of the entire chain.

The remaining exogenous historical datum is the distributive word itself. The construction does not explain why the economy experiences $(\mu_P, \mu_E)$ rather than $(\mu_E, \mu_P)$, nor does it derive full utilization from final demand or saving behavior. Those are higher-level closure questions. Appendix C.4 and the executable support-to-order certificate reproduce the construction exactly.

**33A.4 Exact robustness on the distributive support cells**

The benchmark is not a knife-edge in the two distributive factors. Solving the active price–wage systems as functions of the gross factor gives, on the $P$ support,

$$p = (1,1)^\top, \qquad w_P(\mu) = 12 - 10\mu,$$

while each inactive $E$ process has profit slack

$$\frac{9}{4} - 2\mu.$$

Hence the regular positive-wage $P$ cell contains

$$1 \le \mu_P < \frac{9}{8}.$$

On the $E$ support,

$$p = (1,1)^\top, \qquad w_E(\mu) = \frac{9}{2} - \frac{10}{3}\mu,$$

and each inactive $P$ process has slack

$$\frac{2}{3}\mu - \frac{3}{4}.$$

The corresponding regular positive-wage cell is

$$\frac{9}{8} < \mu_E < \frac{27}{20}.$$

For arbitrary $(\mu_P, \mu_E)$ in these cells, the support-selected wage-net maps yield terminal states

$$x^{PE} = \begin{pmatrix} (125\mu_E\mu_P - 50\mu_P + 22)/125 \\ (125\mu_E\mu_P + 50\mu_P - 22)/125 \end{pmatrix},$$

$$x^{EP} = \begin{pmatrix} (500\mu_E\mu_P + 165\mu_E - 66)/500 \\ (500\mu_E\mu_P - 165\mu_E + 66)/500 \end{pmatrix}.$$

Their aggregate stocks coincide identically. The terminal LP selects $ABC$ after $PE$ whenever

$$79x_2^{PE} - 103x_1^{PE} = -\frac{4}{125}\left(750\mu_E\mu_P - 2275\mu_P + 1001\right) > 0,$$

and it selects $ACB$ after $EP$ whenever

$$x_1^{EP} - x_2^{EP} = \frac{33}{250}(5\mu_E - 2) > 0.$$

Both inequalities hold throughout the full rectangle above. The first is minimized on its closure at $(\mu_P, \mu_E) = (1, 27/20)$, where it equals $1046/125 > 0$; the second has infimum at $\mu_E = 9/8$, where it equals $957/2000 > 0$. Thus the qualitative support-to-order result occupies the entire paper-derived two-parameter regular-support rectangle. This statement is analytic; a million-draw Monte Carlo audit returns a hit rate of 100% only as a redundant numerical check. No analogous high-dimensional volume is reported for all technical coefficients because the paper does not specify bounded admissible ranges for every primitive.

# PART VI — SYNTHESIS: SUPPORT-SELECTED ORDER, ORBIT, VALUE, AND THREE SPECTRAL STRUCTURES

## 34. Spectrum and orbit: what order can change

Section 33A supplies the support-to-order link: conditional on a distributive word, a full external complementarity block generates the active processes, wages, and physical maps; the terminal state then enters a second primal-dual support block that selects the recurrent phase order. The spectral–orbital distinctions below clarify which parts of this endogenous chain are carried by inherited state and which are structural properties of the selected recurrent organization.

For an ordered recurrent organization $\sigma$, collect the internal Sraffian objects

$$\mathcal{I}(\sigma) = \big(R(\sigma), \mathcal{B}(\sigma), Q(\sigma), \mathfrak{S}(\sigma)\big).$$

The internal module associates this packet with a recurrent monodromy product or, under joint production, a dated pencil. External history is generated by products of physical state maps

$$\Phi_h = F_{r_T} \cdots F_{r_1}.$$

The first object is spectral and kernel-theoretic; the second acts on an inherited state. History can therefore enter through both spectrum and orbit, but the two channels need not coincide.

For a fixed recurrent organization, $\mathcal{I}(\sigma)$ is structural. History changes the attainable domain:

$$\mathcal{I}^{\text{att}}_{H,m}(x^h_T \mid x^0_T, r_f) = \{\mathcal{I}(\sigma) : \sigma \in \Omega^{\text{rest}}_{H,m}(x^h_T, r_f \mid x^0_T)\}.$$

**Proposition 34.1 (Two-phase spectral–orbital separation).** For square maps $F_P, F_E$, the products $F_E F_P$ and $F_P F_E$ have the same characteristic polynomial, while their action on a state differs whenever

$$(F_E F_P - F_P F_E)x \neq 0.$$

For a two-phase periodically repeated internal system, reversal is only a cyclic rotation, so the Historical Standard Cycle is unchanged up to calendar relabelling. Hence a genuine two-phase historical effect can be purely orbital even when every internal spectral invariant is order-blind.

**Proof.** The characteristic-polynomial identity is the standard $AB/BA$ determinant identity. The orbital statement follows from the commutator. Cyclic invariance of the internal two-phase system follows from Part V. □

Part III strengthens the orbital statement. On a connected regular parameter cell, the row

$$g(\theta)^\top = a(\theta)^\top [F_E(\theta)F_P(\theta) - F_P(\theta)F_E(\theta)]$$

is analytic. Unless it vanishes identically, order-sensitive capacity is open dense in parameters and in positive inherited states. Thus the exact witnesses are not the source of the order effect; they certify particular threshold and closure consequences of a generic non-commuting mechanism.

## 35. Sraffian value as an extremal spectral coordinate of physical reproduction

### 35.1 The price-covector identity

On a full-resource cell with invertible $A_\tau$, define

$$F_\tau(r) = \left(B_\tau - c^w w_\tau(r)\ell_\tau^\top\right)A_\tau^{-1}.$$

Normalize the physical wage basket by $p_\tau(r)^\top c^w = 1$.

**Lemma 35.1 (Price-covector identity).** If

$$p_\tau^\top B_\tau = (1+r)p_\tau^\top A_\tau + w_\tau(r)\ell_\tau^\top,$$

then

$$\boxed{p_\tau(r)^\top F_\tau(r) = (1+r)p_\tau(r)^\top.}$$

**Proof.** Since $p^\top c^w = 1$,

$$p^\top(B_\tau - c^w w\ell^\top) = (1+r)p^\top A_\tau.$$

Post-multiplication by $A_\tau^{-1}$ proves the identity. □

**Corollary 35.2 (Value law on a fixed cell).** While the same full-resource regime remains valid,

$$p_\tau^\top x_t = (1+r)^t p_\tau^\top x_0.$$

The exact value law does not imply convergence of physical composition.

In the original witness,

$$p_A^\top F_P = \frac{5}{4}p_A^\top, \qquad p_B^\top F_E = \frac{3}{2}p_B^\top,$$

while the commutator remains visible under final-$A$ valuation.

### 35.2 Bibliographic location

Dynamic input–output instability and dual stability belong to the established literature rather than to the paper's novelty claims. The relevant lineage includes Morishima (1958), Sargan

(1958), Solow (1959), Jorgenson (1960), Uzawa (1961), Zaghini (1971, 1991), Aoki (1977), Filippini (1983), and Boggio (1985, 1992), under different quantity and price closures. The classical and Sraffian gravitation literature likewise obtains both attraction and non-attraction depending on the specified adjustment mechanism. The result below is narrower: under the particular physical wage-net closure and single production, Sraffian valuation occupies an exact extremal position in the spectrum of the inverse state map.

### 35.3 Inverse-Perron extremality

Suppress the technique subscript and write $c = c^w$.

**Theorem 35.3 (Inverse-Perron Extremality).** Suppose $B = I$, $A \geq 0$ is square and nonsingular, $c, \ell \geq 0$, and there exist $p \gg 0$, $w \geq 0$, $1 + r > 0$ satisfying

$$p^\top c = 1, \qquad p^\top = (1+r)p^\top A + w\ell^\top, \qquad p^\top Ac > 0.$$

For

$$F = (I - cw\ell^\top)A^{-1},$$

one has

$$\boxed{F^{-1} = A + \frac{w}{1 - w\ell^\top c}(Ac)\ell^\top \geq 0,}$$

$$\boxed{\rho(F^{-1}) = \frac{1}{1+r},} \qquad \boxed{\min_{\mu \in \mathrm{spec}(F)} |\mu| = 1 + r.}$$

Thus the Sraffian eigenvalue is a forward eigenvalue of minimum modulus.

**Proof.** Multiplying the price equation by $c$ gives

$$1 - w\ell^\top c = (1+r)p^\top Ac > 0.$$

Sherman–Morrison gives the displayed inverse. Lemma 35.1 implies

$$p^\top F^{-1} = \frac{1}{1+r}p^\top.$$

Since $F^{-1} \geq 0$ has a strictly positive left eigenvector, Collatz–Wielandt identifies its spectral radius with $1/(1+r)$. Taking reciprocals of eigenvalues gives the forward minimum-modulus statement. □

**Corollary 35.4 (Strict separation under primitivity).** If $F^{-1}$ is primitive, then $1 + r$ is simple and every other forward eigenvalue satisfies

$$|\mu| > 1 + r.$$

A sufficient condition is that $A$ be irreducible, $w > 0$, $\ell \gg 0$, and $c \neq 0$.

### 35.4 Finite viability of full-resource reproduction

**Corollary 35.5 (Finite-viability consequence of inverse-Perron extremality).** Maintain Theorem 35.3 and suppose $F^{-1}$ is primitive. Let $v^* \gg 0$ satisfy

$$Fv^* = (1+r)v^*.$$

For any non-zero $x_0 \geq 0$,

$$F^t x_0 \geq 0 \quad \text{for every } t \in \mathbb{N}_0$$

if and only if

$$x_0 \in \mathbb{R}_+ v^*.$$

Thus every off-ray initial state exits the non-negative full-resource cell in finite time.

**Proof.** Set $G = (1+r)F^{-1}$. Then $G \geq 0$ is primitive and $\rho(G) = 1$. Normalize $p^\top v^* = 1$. If all forward states are non-negative, put $y_t = x_t/(1+r)^t$. Since $p^\top y_t = p^\top x_0$, the $y_t$ lie in the compact section $K_\alpha = \{y \geq 0 : p^\top y = \alpha\}$, while

$$y_0 = G^t y_t.$$

Primitive Perron convergence is uniform on $K_\alpha$, so $G^t y_t \to \alpha v^*$ and therefore $x_0 = \alpha v^*$. The converse is immediate. □

This corollary is a Sraffian wage-net formulation of the familiar knife-edge finite-viability logic of dynamic Leontief systems, not a priority claim over Sargan–Jorgenson–Uzawa instability results. It concerns the linear full-resource extension and does not say what utilization, demand, technique, wage, or distribution must do after the cell boundary is reached.

## 36. Active-basis propagation and spectral bounds on reconstruction time

The physical closure map does not by itself govern the minimum feasible reconstruction problem, because $H^{\min}$ optimizes an entire intertemporal path. The missing spectral object is the transfer operator of the optimal recovery basis.

### 36.1 Bellman form and local active-basis propagation

Let $V_0(x)$ be a terminal reconstruction LP. On a non-degenerate terminal basis cell,

$$V_0(x) = a_0^\top x.$$

Let $\Gamma$ denote the one-period feasible state-transition graph and define recursively

$$V_{H+1}(x) = \sup\{V_H(x') : (x, x') \in \Gamma\}.$$

**Theorem 36.1 (Local Active-Basis Propagation).** Suppose, over a stated horizon range, that $V_H(x) = a_H^\top x$ on a regular cone, the one-step Bellman problem has a unique non-degenerate optimal basis, and on that basis the optimal successor state is

$$x' = K_H x.$$

Then

$$\boxed{a_{H+1} = K_H^\top a_H,} \qquad \boxed{V_{H+1}(x) = a_H^\top K_H x.}$$

**Proof.** On the maintained basis, the optimal successor is linear. Hence

$$V_{H+1}(x) = V_H(K_H x) = a_H^\top K_H x.$$

Uniqueness of the linear representation gives the stated row recursion. □

If the active-basis pattern is periodic in horizon with period $h$, define for residue $s$

$$M_s = K_s K_{s+1} \cdots K_{s+h-1}.$$

Then

$$\boxed{V_{s+qh}(x) = a_s^\top M_s^q x.}$$

The intertemporal LP has therefore become a matrix-power observable on the maintained basis regime.

### 36.2 Exact scaling and graph-index timing

**Corollary 36.2 (Exact Active-Cell Scaling).** Suppose $L > 0$ and $V_s(x) > 0$. If

$$a_s^\top M_s = \gamma_s a_s^\top, \qquad \gamma_s > 1,$$

then

$$V_{s+qh}(x) = \gamma_s^q V_s(x).$$

For threshold $L$, the first crossing in residue class $s$ is

$$q_s^* = \max\left\{0, \left\lceil \frac{\log[L/V_s(x)]}{\log \gamma_s} \right\rceil \right\},$$

and

$$\boxed{D = \min_{0 \le s < h} \{s + hq_s^*\}.}$$

The graph/basis period therefore enters calendar time without implying that every delay is a multiple of $h$.

### 36.3 Cone-Perron delay bounds

Exact eigen-covector scaling is special. More generally write

$$S_q = u^\top T^q b,$$

where $T$ preserves a proper cone $K$, is primitive relative to $K$, has Perron root $\rho > 1$ and right Perron vector $v \in \operatorname{int} K$, and $u \in K^* \setminus \{0\}$, $b \in \operatorname{int} K$.

Define

$$\alpha(b) = \sup\{\alpha : \alpha v \le_K b\}, \qquad \beta(b) = \inf\{\beta : b \le_K \beta v\}.$$

**Theorem 36.3 (Cone-Perron Threshold Bracket).** Let $L > 0$. For every $q \ge 0$,

$$\alpha(b)(u^\top v)\rho^q \le u^\top T^q b \le \beta(b)(u^\top v)\rho^q,$$

where the middle term is $u^\top T^q b$. Equivalently, with

$$C_- := \alpha(b) u^\top v, \qquad C_+ := \beta(b) u^\top v,$$

the first crossing

$$q^* = \min\{q : u^\top T^q b \ge L\}$$

satisfies

$$\boxed{\max\left(0, \left\lceil \frac{\log(L/C_+)}{\log \rho} \right\rceil\right) \le q^* \le \max\left(0, \left\lceil \frac{\log(L/C_-)}{\log \rho} \right\rceil\right).}$$

**Proof.** From $\alpha v \le_K b \le_K \beta v$, cone preservation gives

$$\alpha \rho^q v \le_K T^q b \le_K \beta \rho^q v.$$

Apply the positive dual functional $u$. □

For an $h$-periodic basis, apply the bracket to every residue class and then minimize $s + hq_s$. The basis period enters linearly; Perron growth enters logarithmically through the distance to the threshold.

### 36.4 Spectral-gap refinement

Suppose $T$ is diagonalizable with simple Perron root $\rho$, normalized right and left Perron vectors $v, w$ satisfying $w^\top v = 1$, and

$$\eta = \max_{\lambda \neq \rho} \frac{|\lambda|}{\rho} < 1.$$

Choose $\kappa_T$ such that

$$\|\rho^{-q} T^q - v w^\top\| \leq \kappa_T \eta^q.$$

Set

$$C = (u^\top v)(w^\top b), \qquad K = \frac{\|u\| \, \|b\| \, \kappa_T}{C}.$$

**Theorem 36.4 (Gap-Refined Threshold Bracket).** One has

$$\boxed{C\rho^q(1 - K\eta^q) \leq u^\top T^q b \leq C\rho^q(1 + K\eta^q),}$$

whenever the lower factor is positive. For any $0 < \delta < 1$, when $0 < \eta < 1$ and $K > 0$ define

$$q_\delta = \max\left\{0, \left\lceil \frac{\log(K/\delta)}{\log(1/\eta)} \right\rceil\right\}.$$

Use the convention $q_\delta = 0$ when $K = 0$ or $\eta = 0$. If the threshold has not been crossed in the finite prefix $q < q_\delta$, then $q^* \geq q_\delta$ and the crossing date satisfies

$$\max\left\{q_\delta, \left\lceil \frac{\log[L/(C(1+\delta))]}{\log \rho} \right\rceil\right\} \leq q^*$$

and

$$q^* \leq \max\left\{q_\delta, \left\lceil \frac{\log[L/(C(1-\delta))]}{\log \rho} \right\rceil\right\}.$$

The interpretation is exact. $\rho$ governs block growth; $\eta$ governs the transient time before Perron growth is quantitatively accurate; $K$ captures projection and non-normality; $h$ converts block time into calendar time. A bound depending on $h$ and $\eta$ alone is impossible, because the threshold ratio $L/C$ can be changed arbitrarily without changing the spectrum.

A projective version replaces diagonalizability by a Hilbert-metric contraction coefficient for strictly positive transfer operators. It gives multiplicative bounds of the form

$$C\rho^q e^{-d_0 \tau^q} \leq S_q \leq C\rho^q e^{d_0 \tau^q}.$$

### 36.5 Verifying the active basis itself

The previous bounds are useful only while the candidate basis remains optimal. This condition can itself be reduced to matrix-power sign tests.

**Theorem 36.5 (Periodic Active-Basis Reduction and Conditional Finite Certification).** Fix an $h$-periodic candidate basis pattern for a time-homogeneous block LP. Gaussian elimination of the repeated basic KKT system yields finite-dimensional primal and dual interface recursions

$$b_{q+1} = T_P b_q, \qquad d_{q+1} = T_D d_q.$$

Every basic variable, inactive primal slack, dual multiplier, and reduced cost is a fixed linear functional of $b_q$ or $d_q$. Hence basis optimality is equivalent to finitely many sign sequences

$$\sigma_j(q) = r_j^\top T_P^q b_0 \geq 0, \qquad \zeta_k(q) = s_k^\top T_D^q d_0 \geq 0.$$

For any one of these sequences, expand the initial condition and the relevant functional on the spectral modes of its transfer matrix. If the nonzero contributing mode of largest modulus is a simple positive real eigenvalue and is strictly separated in modulus from every other contributing mode, then the sign of that sequence is eventually fixed, and an explicit spectral-gap bound gives a finite verification horizon. In particular, when $T$ is primitive with Perron vectors $v, w$ and the Perron projection $(r^\top v)(w^\top b_0)$ is nonzero, the Perron mode supplies such a certificate. If that projection vanishes, or if the leading contributing modes are negative, complex, or tied in modulus, eventual sign stability does not follow and may fail.

**Proof.** A fixed basis turns the KKT conditions into a repeated linear block system. Eliminating within-block basic variables leaves a Schur-complement recursion on interface variables; all eliminated primal and dual quantities are linear functions of those interfaces. Primal and dual feasibility are necessary and sufficient for optimality of the candidate basis, giving the finite family of sign sequences above. For one sequence, spectral decomposition writes it as a finite sum of modal terms. Under the stated simple positive dominant-contribution condition, division by that modal term leaves a remainder that decays geometrically, so its sign is constant beyond an explicitly bounded horizon. Without that condition there is no such conclusion: a negative dominant mode alternates and a dominant complex pair may oscillate. □

The theorem does not assert that every recovery LP has an eventually periodic optimal basis, nor that every proposed periodic basis is finitely certifiable by Perron separation alone. It gives a finite algebraic reduction and states a sufficient spectral condition under which the infinite sign check collapses to a finite one.

### 36.6 The original scalar active-basis factor and its limitation

In the original witness,

$$A_A^3 = \frac{297}{1280} I, \qquad 1 - w_P \ell_{A0} = \frac{94985}{175616},$$

so

$$\boxed{\gamma = \frac{1 - w_P \ell_{A0}}{297/1280} = \frac{43175}{18522} \simeq 2.33101177.}$$

An exact horizon-three primal-dual certificate proves

$$a_3^\top = \gamma a_0^\top.$$

The same three-step factor persists on the verified active basis through later blocks.

But the worked value row is

$$a_H^\top = (\alpha_H, 0, 0)$$

on the certified original horizons. Thus the economically visible active-recovery transfer in this example is scalar: it proves exact block scaling but does *not* instantiate the non-trivial second eigenvalue, gap ratio, or non-normal transient of Theorem 36.4. The general gap theorem therefore requires a separate multidimensional witness.

### 36.7 Genuine rank-two active cell: viable recovery, persistent restoration, and spectral separation

The preceding scalar witness motivates a genuinely multidimensional construction in which the optimal LP basis lives on a proper cone, the subdominant mode is quantitatively material, and the Standard, active-recovery, and realized-closure dominant factors are distinct.

Let

$$A = \begin{pmatrix} 1 & 1/20 \\ 1/100 & 1 \end{pmatrix}, \qquad p = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad c = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix},$$

$$1 + r = \frac{11}{10}, \qquad w = 2, \qquad h = \begin{pmatrix} 1 \\ 10 \end{pmatrix}, \qquad \ell = A^\top h = \begin{pmatrix} 11/10 \\ 201/20 \end{pmatrix}.$$

Let

$$N = \begin{pmatrix} 3/100 & 8/100 \\ 7/100 & 2/100 \end{pmatrix},$$

and define gross output by

$$B = (I + N)A + cw\ell^\top = \begin{pmatrix} 5327/2500 & 20363/2000 \\ 5901/5000 & 22147/2000 \end{pmatrix} \gg 0.$$

Then $p^\top c = 1$ and exactly

$$p^\top B = \frac{11}{10} p^\top A + 2\ell^\top.$$

The full-resource activity vector is $y = A^{-1}x$. Hence its natural active region is the proper cone

$$C = A\mathbb{R}^2_+ = \{x : A^{-1}x \geq 0\},$$

not the whole orthant. Moreover

$$A^{-1}NA = \begin{pmatrix} 2729/99950 & 3213/39980 \\ 17473/249875 & 4537/199900 \end{pmatrix} \gg 0,$$

so every map $I + sN$, $s > 0$, sends $C \setminus \{0\}$ into the interior of $C$. Basis invariance is therefore substantive.

The terminal recurrent-output LP is

$$V_0(x) = \max\{p^\top Bv : Av \leq x,\ v \geq 0\}.$$

On $\operatorname{int} C$ the unique optimum is $v = A^{-1}x$, with value row

$$a_0^\top = p^\top BA^{-1} = \left(\frac{31}{10}, \frac{211}{10}\right) \gg 0.$$

Both inherited goods have positive shadow value.

For feasible reconstruction, suppose at most the fraction $\bar{s} = 3/4$ of wage-net surplus can be accumulated. Storage of unused inputs is permitted, and the Bellman transition has

$$x' = x + z, \qquad 0 \leq z \leq \bar{s}\, NAy.$$

All continuation rows are positive, so on $C$ the unique optimum is $y = A^{-1}x$ and $z = \bar{s}Nx$. The active-recovery transfer is

$$G_R = I + \frac{3}{4}N.$$

Let the realized closure reinvest only $s_\chi = 1/2$ of wage-net surplus:

$$G_\chi = I + \frac{1}{2}N.$$

Because $s_\chi < \bar{s}$, the realized path is admissible for the reconstruction problem, and both maps preserve $C$ indefinitely.

The three dominant factors are now distinct:

$$1+r=\frac{11}{10}, \qquad \operatorname{spec}(G_R)=\left\{\frac{43}{40},\frac{77}{80}\right\}, \qquad \operatorname{spec}(G_\chi)=\left\{\frac{21}{20},\frac{39}{40}\right\}.$$

Hence

$$\boxed{\frac{11}{10}>\rho(G_R)=\frac{43}{40}>\rho(G_\chi)=\frac{21}{20}.}$$

This separation has a general explanation in the non-negative surplus class. If $p \gg 0$, $N \geq 0$, and $p^\top N = rp^\top$ with $r>0$, then Collatz–Wielandt gives $\rho(N)=r$ and

$$p^\top(I+sN)=(1+sr)p^\top, \qquad \rho(I+sN)=1+sr \quad (s\geq 0).$$

Thus complete reinvestment $s=1$ forces the quantity-growth Perron factor to equal $1+r$, while partial accumulation $0<s<1$ separates it from the Sraffian gross-profit factor. If optimal recovery and realized accumulation use different fractions, their dominant factors separate as well.

Take

$$x^h=\begin{pmatrix}7/50\\1/100\end{pmatrix}, \qquad x^0=\begin{pmatrix}1\\1\end{pmatrix}, \qquad L^0=V_0(x^0)=\frac{121}{5}.$$

For $N$, right eigenvectors associated with $1/10$ and $-1/20$ are $v_1=(8,7)^\top$ and $v_2=(1,-1)^\top$, and

$$x^h=\frac{1}{100}v_1+\frac{6}{100}v_2.$$

The optimal-reconstruction value is

$$V_H^R(x^h)=\frac{69}{40}\left(\frac{43}{40}\right)^H-\frac{27}{25}\left(\frac{77}{80}\right)^H,$$

while the realized-closure value is

$$V_H^\chi(x^h)=\frac{69}{40}\left(\frac{21}{20}\right)^H-\frac{27}{25}\left(\frac{39}{40}\right)^H.$$

The transverse amplitude relative to the Perron coefficient is

$$\boxed{K=\frac{72}{115}\simeq 0.626087,}$$

so the second mode is not negligible.

Exact comparison gives

$$V_{36}^R<L^0<V_{37}^R, \qquad V_{54}^\chi<L^0<V_{55}^\chi.$$

Both sequences are strictly increasing: the dominant term grows and the negative subdominant term shrinks in modulus. Therefore

$$\boxed{H^{\min} = 37 < D^{\chi} = \bar{D}^{\chi} = 55.}$$

The construction simultaneously supplies a viable closure, persistent restoration, and strict separation between feasible and realized recovery.

The spectral-gap theorem now does quantitative work. For $G_R$,

$$\eta_R = \frac{77}{86}.$$

The crude cone–Perron bracket is

$$29 \leq H^{\min} \leq 64.$$

With $\delta = 1/50$, the mixing horizon is $q_\delta = 32 < 37$, and the gap-refined bracket collapses exactly to $H^{\min} = 37$. For the realized closure, $\eta_\chi = 13/14$: the crude bracket $43 \leq D^{\chi} \leq 95$ narrows at the same tolerance to $54 \leq D^{\chi} \leq 55$, after which one exact comparison identifies $D^{\chi} = 55$.

This is the worked case in which Theorems 36.1–36.4 operate on a proper active cell with full rank two. Appendix A.9 and the executable certificate reproduce every identity in exact arithmetic.

## 37. From packet selection to endogenous historical order

The primitive period-one witness supplies the first cross-technique bridge. At the common final rate and under one commodity-value covector $\phi_\varepsilon = p_A^{\varepsilon}(r_P)$,

$$S^{\phi}_{A,0}(EP) < L^{0,\phi}_{2,\varepsilon} < S^{\phi}_{B,0}(EP),$$

whereas after $PE$ technique $A$ exceeds the same baseline. The two stationary recurrent techniques have distinct Perron/Standard packets. External history therefore changes the attainable packet set under one common ruler. This remains an opportunity-set result rather than a complete short-period selection theorem.

Section 33A goes further. Its alternatives are non-cyclic recurrent orders of the same internal phase catalogue, and the external state maps are themselves generated by competitive support. The distributive positions $\mu_P$ and $\mu_E$ select different external process pairs with strict inactive-profit margins. Reversing their temporal order produces

$$x^{PE} = \left(\frac{132}{125}, \frac{198}{125}\right)^{\top}, \qquad x^{EP} = \left(\frac{198}{125}, \frac{132}{125}\right)^{\top},$$

with identical aggregate stock. The terminal capacity KKT system then selects

$$\operatorname{supp} z(x^{PE}) = \{ABC\}, \qquad \operatorname{supp} z(x^{EP}) = \{ACB\}.$$

Because $\mathfrak{I}(ABC) \neq \mathfrak{I}(ACB)$, the integrated causal chain is

$$\boxed{\begin{aligned}\text{distributive order} &\to \text{competitive traverse support} \to \text{wage-net state map}\\ &\to \text{inherited composition} \to \text{recurrent-order support}\\ &\to \text{Historical Standard packet.}\end{aligned}}$$

This is the strongest version of the historical-order bridge in the repertoire: the first order causally generates the second under one fixed grand catalogue and stated closures.

## 38. What is now integrated and what remains open

The repertoire contains four distinct maps whose roles should remain separate:

1. a distributive-support block maps a gross distributive factor into active processes, prices, wages, and a local wage-net state map;
2. products of those state maps generate the inherited historical orbit;
3. reconstruction LPs and realized closures generate feasible, first-passage, and persistent recovery times;
4. internally ordered recurrent organizations map phase order into Historical Standard packets.

Section 33A composes the first, second, and fourth maps in a block-triangular two-stage support architecture. It is not a price-unified equilibrium: the full external catalogue selects $F_P, F_E$ through zero-excess-profit complementarity, while the terminal inherited state enters a separate primal-dual recurrent-capacity problem whose dual variables are resource shadow values. Conditional on the distributive word and this terminal criterion, neither the external maps nor the final recurrent order is inserted by hand. Strict margins and nonsingular active Jacobians make the complete conditional chain locally persistent.

The spectral structure remains stratified. Standard monodromy concerns the selected recurrent organization; physical-closure spectrum concerns a realized quantity rule; active-recovery spectrum concerns optimal feasible reconstruction on a maintained basis. Equality of dominant factors can be structural under complete reinvestment, but it is not general: Section 36.7 gives an exact partial-accumulation case in which all three dominant factors differ.

The distributive word itself remains historical data: the model does not derive why $(\mu_P, \mu_E)$ occurs rather than $(\mu_E, \mu_P)$. The full-resource traverse closure also does not derive utilization from final demand, saving, finance, or capitalist consumption. Open problems include eventual periodicity of optimal recovery bases, cone-specific inverse-Perron theory under joint production, and compatible price–quantity–support cocycles for stochastic or non-periodic order.

# PART VII — CLOSURE, METRICS, LIMITS, AND OPEN PROBLEMS

## 39. A hierarchy of closures, benchmarks, and time objects

The framework distinguishes five closure layers: scale-free material feasibility; traverse quantity closure; intertemporal reconstruction optimization; physical distribution closure; and reference-path closure. It also distinguishes two benchmarks and three clocks.

The fixed comparison-date baseline $L_T^0$ asks whether a productive scale once possessed by the no-excursion branch at date $T$ has been reconstructed. The moving counterfactual $L_{T,H}^0$ asks whether the historical branch has caught the no-excursion branch at the later date $T+H$. For the fixed baseline, $H^{\min}$ measures reachability, $D^{\chi}$ first passage, and $\bar{D}^{\chi}$ persistent restoration.

The moving benchmark is informative only when its continuation is economically discriminating. A simple monotonicity result identifies a degenerate case.

**Proposition 39.1 (Common-map non-catch-up).** If two states satisfy $x^0 \geq x^h$ componentwise, $x^0 \neq x^h$, and both branches are continued by the same non-negative linear map $G$, then

$$G^H x^0 \geq G^H x^h \qquad \forall H \geq 0.$$

If $G^H(x^0 - x^h) \gg 0$ and the comparison functional is positive, strict catch-up is impossible at every horizon.

The proposition is elementary but interpretively important. A moving counterfactual is not a universally stronger restoration criterion. Under a common monotone closure and initial dominance it is mechanically unattainable. It becomes informative when branch support, techniques, closures, or other state transformations differ over the continuation, as they can in the order-sensitive multistage witnesses.

## 40. Common metrics, process normalization, and alternative objectives

A quantity objective must be invariant to arbitrary rescaling of activity units. Labour absorption has that property for a fixed technique, while dated stock value at a fixed covector is another normalized objective. Cross-technique reconstruction requires the same ruler on every technique; the paper therefore fixes one positive covector $\phi$ and compares $\phi^\top B_\tau v$.

In the original witness, the labour and common-value recovery rows happen to be proportional on the certified $A$ basis because only $K_0$ has positive local shadow value. The common metric is nevertheless essential for the $A/B$ comparison: it removes the mechanical advantage of evaluating each technique with its own labour coefficients. The primitive witness is where this correction changes the cross-technique conclusion.

### 41. Labour is a quantity flow, but not a universal scale metric

There is no exogenous labour ceiling in the central examples. Labour is elastically supplied at the regime-specific wage, and the quantity programme records how much labour inherited productive stocks can employ. No dual multiplier on a full-employment constraint is identified with the wage.

The original recovery calculations retain labour normalization because several exact active-basis identities are simplest in that metric. Cross-technique opportunity comparisons use the common metric $\phi$. Technique-specific changes in labour requirements remain genuine technological changes and can alter both distributive ranking and reconstruction.

### 42. Closure viability is a separate economic requirement

A one-period objective can be mathematically well defined while being economically destructive over time. The myopic labour-absorption closure in the first two multistage witnesses may consume the stock configuration required for future recurrence. Its boundary failures are therefore diagnostics of the closure, not automatically historical effects.

A viable closure must keep the relevant productive state inside an admissible set and, if persistent restoration is the object, must preserve the reconstructed condition after it is reached. Section 36.7 supplies an exact positive construction on the proper cone $C = A\mathbb{R}^2_+$: both optimal-recovery and realized partial-accumulation maps leave $C$ invariant indefinitely, and the realized closure has $D^\chi = \bar{D}^\chi = 55$.

The distinction between full and partial accumulation also clarifies a spectral issue. Under complete reinvestment of a non-negative wage-net surplus, the physical Perron growth factor can coincide structurally with $1 + r$. Partial accumulation breaks that identity. A richer economic closure would derive the accumulation fraction, utilization, final demand, saving, and capitalist consumption rather than stipulating them; that remains an open extension.

### 43. Claims explicitly excluded

The repertoire does not claim that every positive state excludes some technique, that regime $E$ is intrinsically adverse, or that an arbitrary epsilon-scale seed is economic restoration. It does not call the fixed comparison-date baseline a moving counterfactual. It does not identify first passage with persistent restoration. It does not treat the myopic closure's boundary collapse as a historical effect unique to $EP$. It does not use technique-specific labour as a neutral cross-technique metric. It does not identify an attainable recurrent packet with a realized short-period equilibrium without separate support evidence. It does not equate LP dual variables with production prices. It does not identify the terminal recurrent-capacity objective with classical maximum-profit technique choice: in the exact 33A witness a pure maximum-$R^{\mathrm{Std}}$ ranking selects $ABC$ in both branches. It does not assume that the active-recovery, physical-closure, and Standard dominant factors are always distinct: complete reinvestment can force a coincidence, while partial accumulation can separate them. And it does not identify a generic matrix-product Lyapunov exponent with a Sraffian maximum rate of profits.

## 44. What remains open

The first open problem is an economically richer viable traverse closure with endogenous utilization, final demand, accumulation, and distribution. The proper-cone rank-two construction proves that viability, multidimensional spectral recovery, and persistent restoration are compatible, but its accumulation fractions are deliberately imposed rather than behaviorally derived.

The second is an eventual-basis problem. Theorem 36.5 reduces a candidate periodic basis to finitely many matrix-power sign tests and gives a finite certification horizon only when the first nonzero contributing spectral mode satisfies the stated separation condition; it does not prove eventual periodicity for every recovery LP.

The third is joint-production generalization of inverse-Perron results. For $B \neq I$, the inverse physical map need not be non-negative even though a recovery transfer can preserve a positive cone, as the proper-cone rank-two construction illustrates.

The fourth is the economic closure behind recurrent-order selection. Section 33A supplies an exact primal-dual recurrent-capacity selector for two internally ordered cycles under a stated activity-analysis criterion, and proves why a pure state-independent maximum-rate ranking cannot make interior inherited composition select among positive CRS packets. What remains open is a richer theory that derives the terminal objective itself from utilization, effective demand, accumulation, saving, finance, or other institutions of competitive selection rather than stipulating the capacity/scarcity criterion.

### 44.1 Arbitrary deterministic order

Periodic repetition converts an ordered block into a recurrent object with a cycle invariant. A one-off non-periodic block does not. Finite products can still be summarized by spectral, primal, or dual factors, but these are not automatically Sraffian long-period invariants.

### 44.2 Stochastic order and dual-compatible valuation

A stochastic extension requires a filtration, adapted dated prices, adapted process supports, and compatible primal and dual cocycles. Stochastic Perron–Frobenius theory provides one mathematical point of contact (Evstigneev and Pirogov 2010). A quantity-side Lyapunov exponent is only a growth statistic until linked to positive valuation, non-negative operation, and the distributive convention. Random active-basis dynamics would add a further cocycle.

### 44.3 Empirical reconstruction

The empirical distinction is threefold. One may estimate how soon a target configuration becomes technologically reconstructible, when observed investment first restores a stated baseline, and when the restored position becomes persistent. A separate exercise asks whether the historical branch catches a moving no-excursion counterfactual. Machine cohorts, infrastructure age, sectoral capacity, and district-level capital composition are natural state variables for such tests.

## 45. Why the restrictions strengthen the programme

The restrictions are part of the result. Interior seedability prevents an arbitrary epsilon-scale definition of availability. Benchmark separation prevents a fixed dated baseline from being confused with a moving counterfactual. Three clocks prevent feasibility, first passage, and persistence from being treated as synonyms. A common metric prevents cross-technique comparisons from being driven by technique-specific labour intensity. Closure diagnostics prevent a destructive behavioral rule from being mistaken for an historical law.

The same discipline governs the spectral side. The original witness is explicitly scalar at the active-value level; the proper-cone rank-two construction supplies the non-trivial spectral-gap application. Standard, physical-closure, and active-recovery spectra are conceptually distinct even when special closure structure makes some dominant factors coincide. Endogenous Support-to-Order then composes competitive traverse selection with internal recurrent ordering through inherited composition and a second support problem. The repertoire is intentionally broader than a journal paper but internally stratified: exact witnesses, local theorems, classical corollaries, negative results, and open extensions occupy different logical levels.

# PART VIII — INTELLECTUAL GENEALOGY AND COMPARATIVE POSITIONS

### 46. Why the reconstruction needs a genealogy

The programme developed here should not be presented as a theory without ancestors. Nearly every mathematical ingredient has a well-established origin. Activity analysis and linear optimization provide the polyhedral backbone (Koopmans 1951; Bertsimas and Tsitsiklis 1997), while the matrix theory of non-negative systems provides the spectral language (Berman and Plemmons 1994). Fixed-capital dating, positive-system reachability, Perron–Frobenius theory, Frobenius normal form, convex separation, Collatz–Wielandt quotients, matrix pencils, and von Neumann complementarity enter at more specific points below. The candidate contribution lies in the architecture that makes those tools answer a specifically Sraffian historical question.

The comparison is conjunctive rather than triumphalist. The paper does not claim priority for dated commodities, path dependence, joint-production complications, or the Standard commodity's invariance properties. It asks what follows when several established strands are connected by one distinction: **the fixed catalogue of technical knowledge is not identical to the historically admissible recurrent system**.

### 47. Sraffa: reconstruction from inside the surplus approach

Sraffa's (1960) book supplies the discipline that the paper refuses to abandon. Prices and distribution are studied from given methods of production without invoking an aggregate capital endowment whose marginal product determines the rate of profit. The Standard commodity isolates an internal distributive invariant of the given system. Joint production and fixed capital already reveal that the meaning of "system" can be considerably richer than a square circulating-capital matrix.

The present proposal is not to make Sraffa "dynamic" by appending an adjustment equation to his price system. It is to reconsider what must be specified before one can legitimately say that a system of production is given. The answer proposed here has two layers. The technical coefficients may be given while the historically attainable recurrent domain is not; and the recurrent system to which the Standard construction is applied may itself be an ordered dated system.

This interpretation retains Sraffa's analytical asymmetry between prices/distribution and quantities. The price equations characterize a recurrent organization. They do not, by themselves, determine the historical quantity path by which the economy arrives there.

## 48. Technique choice and traverse: Garegnani, Hicks, Lowe, Gehrke–Hagemann, and later debates

The external-history module sits at the intersection of two traditions that should not be conflated.

The Sraffian choice-of-technique literature, including Garegnani (1970), shows that the wage-profit relation can select different methods as distribution changes and that techniques may switch or reswitch. The related gravitation debate includes both convergence and non-convergence results (Boggio 1985, 1992; Dupertuis and Sinha 2009; Bellino and Serrano 2018). Those results concern the ranking of long-period positions. That literature leaves open the separate traverse question of how the economy moves from one capital composition to another.

Traverse theory asks exactly that second question. Hicks (1973) and Lowe (1976) make the inherited structure of production central to transition. Gehrke and Hagemann's (1996) analysis of efficient traverses and bottlenecks is especially close to the present problem: transition is constrained by the composition of productive means, and the path to a new configuration can be limited by specific bottlenecks. The broader collection edited by Landesmann and Scazzieri (1996) places these issues within production and economic dynamics; Hagemann and Scazzieri (2009) revisit capital, time, and transitional dynamics from adjacent perspectives. Halevi, Hart, and Kriesler (2013) restate the traverse as the problem of movement outside equilibrium and emphasize that the path can matter for the relevance of the equilibrium concept itself. Amendola and Gaffard's (1998) *Out of Equilibrium* is an especially close modern antecedent: restructuring is sequential, current constraints condition decisions that generate new constraints, and viability of the transition is a central analytical object. The present framework is narrower in behavioral content but sharper in separating minimum feasible reconstruction from the trajectory generated by a stated quantity closure.

The present paper adds a specific Sraffian mechanism to that traverse problem. The long-period target is still ranked by the wage curve at the final rate of profit. But mandatory ageing and dated competitive support determine whether the inherited stage vector can be transformed into the target recurrent position at the specified reference scale and within a given horizon. The main three-stage witness is described as dated multistage production rather than as a literal Sraffian fixed-capital model. Genuine fixed capital is retained in the separate joint-production/boundary constructions, where used machines are dated productive objects. This terminology follows the distinction emphasized by the fixed-capital literature itself.

Reswitching is not, by itself, productive memory. If the feasible set is fixed, returning to a rate of profit returns to the same wage ranking. History appears only when the temporary regime changes the inherited productive state or the competitive transition through which the final technique must be reconstructed.

## 49. Schefold: the nearest antecedent on the internal Sraffian side

Schefold (1989) is the closest reference for the internal side of the programme. His work places joint production, fixed capital, the Standard commodity, negative Standard multipliers, balanced growth, and the age and efficiency of machines inside Sraffian production theory. Those are exactly the issues that constrain any serious Historical Standard construction; see also Baldone (1980), Varri (1980), Bidard (2004), and Salvadori and Steedman (1988) on fixed capital and joint production.

The paper does not claim to discover the complications created by joint production, the possibility of negative Standard multipliers, or the need for a refined definition of basic commodities. Those are inherited problems. The departure is narrower: the object evaluated is an **ordered periodic reproduction system** assembled from a fixed collection of elementary phases. A cyclic rotation is treated as a change of calendar origin; a genuinely non-cyclic permutation can alter the Standard ratio, dated basicity, and the sign status of the Standard process vector.

The external historical frontier adds a second distinction. Schefold's fixed-capital analysis makes machine age and efficiency legitimate production-theoretic categories. The present paper uses inherited age composition to determine which recurrent systems can be brought back into self-reproduction after a distributive history.

## 50. Kurz and Salvadori: the immediate long-period framework

Kurz and Salvadori (1995) provide the most systematic framework in which the present reconstruction belongs. Their treatment of joint production and fixed capital, especially the chapters devoted to produced means of production, depreciation, and fixed-capital systems, is the immediate technical background for the dated and fixed-capital constructions used here. Their long-period theory integrates choice of technique, joint production, produced means of production, fixed capital, and the Sraffa–von Neumann tradition. The current programme is not a replacement of long-period production theory by an external control model.

The additional object is a **historically conditioned recurrent domain**. A catalogue may contain many complete recurrent organizations, but inherited stocks and dated competitive support can make only a subset attainable from the current state. The Sraffian wage comparison then operates on that subset. This is a change in the domain of technique choice, not a change in the definition of wage maximization itself.

The distinction also disciplines examples. A recurrent catalogue must be enumerated completely at the level relevant to the claim. One cannot compare two preferred cycles while ignoring a third recurrent organization generated by the same activity set. That requirement is a direct consequence of treating the problem as long-period technique choice rather than as a comparison of two preselected paths.

## 51. Pasinetti: the nearest programme in ambition, not in mechanism

Pasinetti (1981, 1993) is perhaps the closest comparison in ambition. Structural economic dynamics asks how a classical theory of production can address persistent structural transformation, differentiated productivity growth, and learning. The affinity is methodological: production is treated as an economy-wide structural process rather than as a movement of one aggregate quantity of capital.

The mechanism is different by design. Pasinetti allows productive coefficients and sectoral productivity to change. The present paper holds elementary commodity and labour coefficients fixed in its central historical results. History acts on inherited state, on dated competitive support, and on the order of recurrent systems. The question is: **how much historical structure is already available before technical progress is admitted?**

This difference matters for the interpretation of the Historical Standard Cycle. The term "dynamic Standard commodity" is deliberately avoided. Pasinetti's dynamic Standard commodity belongs to a theory of structural change. Here the Standard object belongs to a fixed ordered periodic system. Its historical character derives from dated order, not from changing productivity coefficients.

## 52. von Neumann, Solow, and Lager: dated reproduction as classical infrastructure

von Neumann's (1945) growth model makes production processes, joint inputs and outputs, and system-wide reproducibility central. Solow's dynamic input-output valuation shows that competitive prices can be attached to dated multisectoral interdependence. Lager (1997) explicitly compares the treatment of fixed capital in Sraffian and dynamic input-output frameworks and emphasizes the legitimacy of representing used fixed-capital items as dated joint products within a Sraffa–von Neumann setting.

These contributions matter because they show that dated production is not foreign to the classical-Sraffian tradition. The present temporal lift should not be read as an importation of "dynamics" from an unrelated theory. It is a reorganization of dated commodity relations inside an activity framework.

What is new in the present use is the question asked of that dating. The lift is used to reconstruct specifically Sraffian objects: Standard proportions, the maximum profit invariant, basicity, and the left/right null-space structure of joint production. The finite-horizon historical domain then connects dated competitive valuation to inherited productive states.

## 53. Bellino, Baldone, and Kurose–Yoshihara: what makes a Standard object Standard

A positive eigenvector of a temporal lift does not earn the name "Standard commodity" merely because it exists. Bellino's (2004) analysis of Sraffa's Standard commodity and Baldone's (2006) discussion of its invariance with respect to income distribution provide the appropriate test. The economically relevant property is not the tautology that a chosen numeraire has price one; it is a distributional invariance property derived from the structure of the Standard system.

This is why Part V does more than identify a Perron vector. The Historical Standard Cycle is characterized by Standard proportions, a cycle profit-factor identity, a common-rate Standard equivalent, the linear wage–profit relation under the Standard normalization, and a value-change identity derived before the numeraire is imposed. The composite cycle has the robust invariance property; phase-by-phase value invariance is shown to require the special case of uniform Standard labour weights.

Kurose and Yoshihara (2019) pursue a different and complementary generalization. Their convex-economy construction allows changes in distribution to be associated with changes in the cost-minimizing technique and studies when an invariable measure and the linear distributive relation survive. The present route keeps the elementary phase systems fixed and enlarges the object by making periodic order explicit. One generalization works through a broader production set; the other through a richer temporal organization of fixed systems.

### 54. Hicks and Georgescu-Roegen: time as an organization of production

Hicks's (1973) *Capital and Time* and Georgescu-Roegen's (1970) analysis of production processes are not Sraffian in the narrow sense, but they are important conceptual neighbours. Hicks treats production over time as a process capable of generating sequences of outputs rather than as an instantaneous mapping. Georgescu-Roegen's flow-fund perspective insists on duration and on the internal temporal organization of productive operations.

The affinity lies in refusing to reduce time to an index attached after the fact. A machine's age, a construction period, or the order in which complementary processes occur can matter even if engineering knowledge is unchanged.

The difference is that the present paper connects temporal organization to the specific internal structure of Sraffian theory. The object is not merely dated production; it is dated production with a wage–profit frontier, Standard ratios, basicity, and joint-production realizability.

### 55. Path dependence: David, Arthur, and the difference between lock-in and productive inheritance

The historical vocabulary naturally invites comparison with David (1985) and Arthur (1989). Their classic contributions show how historical events, increasing returns, and adoption dynamics can generate lock-in and path dependence. The present mechanism is different.

No increasing return to adoption is required. A history can matter because fixed productive means have finite lives and because competitive dated prices determine whether those means are preserved, transformed, or exhausted. The resulting state dependence is material. A technique can remain globally superior in the wage comparison while becoming unreachable from the inherited stock.

The distinction also changes the appropriate robustness concept. Lock-in models often emphasize multiple attractors or self-reinforcing shares. The fixed-capital example here instead emphasizes support topology and dated profitability. Other examples can generate interior projective-domain separation without absorbing faces. The historical-domain theory is designed to cover both.

### 56. Positive systems, viability theory, and control

The engineering side of the reconstruction is mathematically adjacent to positive-system reachability, switched systems, and viability theory. Coxson and Shapiro (1987), Valcher (1996), Farina and Rinaldi (2000), and Fornasini and Valcher (2011) provide important reachability results for non-negative systems. Aubin (1991)'s viability theory supplies a broader language for state constraints and viable trajectories.

The paper does not claim novelty for those mathematical facts. Its engineering cone-entry problem differs in economic organization. Controls are activity scales constrained by contemporaneous commodity inputs; outputs and unused stocks jointly determine the next productive state; the target is not an arbitrary point but the self-reproduction cone of a Sraffian recurrent organization; and the physical reachability domain is subsequently intersected with dated normal-profit support.

The dual of the engineering LP is interpreted as a bottleneck certificate for inherited productive resources. It is deliberately not called a price system. Sraffian prices appear on a different layer, through competitive support and the wage frontier.

### 57. Joint production and the danger of excessive Perronization

Joint production is where analogy with ordinary input-output matrices becomes most dangerous. If a process produces several commodities, there is no general reason why the economic problem can be reduced to a non-negative square matrix whose Perron vector simultaneously gives prices and process intensities.

The dated pencil

$$C(\mu) = \bar{B} - \mu \bar{A}$$

makes the distinction explicit. Positive prices require a positive vector in the left kernel. A physically realizable Standard system requires a non-negative vector in the right kernel. The inverse-free convex-hull theorem shows why neither implication is automatic. The Collatz–Wielandt proposition shows further that an all-binding positive-price root can lie strictly below the pairwise maximin bound when no non-negative right-kernel certificate exists.

This is not a technical footnote. It prevents the theory from equating formal Standard solvability with an actually operable process vector. It also explains why support-reduced rectangular systems require complementarity rather than a purely algebraic eigenvalue selection.

## 58. Comparative map

| Tradition | What it already provides | Additional object in the present programme |
|---|---|---|
| Sraffa | Prices, distribution, Standard commodity, basics, joint production | Historically conditioned domain of the "given" recurrent system |
| Garegnani / technique-choice literature | Wage–profit comparison, switching and reswitching | Productive state as a constraint on the domain of comparison |
| Schefold | Joint production, fixed capital, negative multipliers, Standard systems | Non-cyclic order as an argument of dated Standard invariants |
| Kurz–Salvadori | Systematic long-period activity theory | Historical admissibility of complete recurrent organizations |
| Pasinetti | Structural dynamics and changing productivity | Historical reproduction with fixed elementary coefficients |
| von Neumann | Process reproduction and primal-dual growth | Sraffian historical-domain and Standard interpretation |
| Solow | Competitive dynamic input-output valuation | Continuation-indexed pricing inside historical technique reachability |
| Lager | Fixed capital across Sraffian and dynamic IO frameworks | Dated-stage reconstruction joined to an attainable long-period opportunity envelope |
| Bellino / Baldone | Invariance meaning of the Standard commodity | Conditional value invariance of a dated Standard cycle |
| Kurose–Yoshihara | Invariable measure with technique change in convex economies | Ordered fixed-system route to a generalized Standard object |

| Tradition | What it already provides | Additional object in the present programme |
|---|---|---|
| David / Arthur | Path dependence and historical lock-in | Material productive inheritance without increasing-return lock-in |
| Positive-systems / viability literature | Reachability and state constraints | Reproductive cones as Sraffian targets plus dated competitive support |

### 59. The point of intersection

The programme lies at the intersection of four lines.

The first is the Sraffian theory of prices, distribution, choice of technique, fixed capital, basics, and the Standard system. The second is dated process reproduction and competitive valuation. The third is the theory of productive time, vintages, and historical state dependence. The fourth is the mathematics of positive reachability, polyhedral optimization, and matrix pencils.

The reconstruction joins them through a single sequence:

$$
\begin{aligned}
\text{fixed activity catalogue} &\longrightarrow \text{historically inherited productive state} \\
&\longrightarrow \text{competitive entry into recurrent reproductive domains} \\
&\longrightarrow \text{Sraffian wage selection} \\
&\longrightarrow \text{internal Standard packet of the selected system.}
\end{aligned}
$$

This is the sense in which the paper proposes a **relaunch rather than a replacement of Sraffa**. It keeps the fixed-coefficient discipline and the surplus approach, but rejects the tacit identification of technical knowledge with historically effective reproduction.

# PART IX — CONCLUSION: WHAT A HISTORICAL SRAFFA WOULD MEAN

### 60. The first result is a boundary lemma for the homogeneous theory

Under divisible constant returns, every strictly positive state contains a sufficiently small copy of any finite positive reproductive bundle. The Interior Seedability Lemma tells us what a historical theory cannot use as its main criterion: mere positive-scale restartability. Its importance is conceptual rather than technical.

### 61. Historical comparison has two benchmarks and three clocks

A stated no-excursion history generates the reference state at the comparison date. From it the repertoire distinguishes a fixed comparison-date baseline from a moving counterfactual continued beyond that date. The first asks whether a dated productive scale has been restored; the second asks whether the historical branch has caught the no-excursion branch.

For the fixed baseline,

$$\boxed{H^{\min,m} \le D^{\chi,m} \le \bar{D}^{\chi,m}.}$$

These are, respectively, minimum feasible reconstruction, first passage under the stated closure, and persistent restoration. The exact constructions show that the inequalities can collapse to equality or separate sharply.

### 62. Order effects are generic on regular cells and have distributive comparative statics

The external historical mechanism is the non-commuting action of dated state maps on inherited productive states. On a connected regular cell, unless the valuation-visible commutator vanishes identically, order-sensitive capacity is open dense in technology parameters and in positive inherited states. Integer delay differences require threshold crossing and therefore satisfy open persistence rather than an open-dense theorem.

On the original certified cell, increasing $r_E - r_P$ raises $PE$ capacity and lowers $EP$ capacity. The calibration uses high profit rates and is a stress test, not an empirical estimate of magnitudes; the signed comparative static, rather than the factor 5.3, is the theoretically portable result.

### 63. The exact constructions have different jobs

The original multistage witness proves a catalogue-complete $H_A^{\min,\phi} = 3$ and a large order effect under a common metric; its myopic continuation is retained only as a closure-pathology diagnostic.

The primitive period-one witness gives $D_A^{\chi,\phi} = 2 < \bar{D}_A^{\chi,\phi} = +\infty$, so first passage is neither a graph-period artefact nor persistent restoration.

The proper-cone rank-two construction serves a different purpose. It is not an order witness. It exercises the active-basis theory in two dimensions, makes the spectral gap quantitatively relevant, separates optimal from realized accumulation, and gives

$$H^{\min} = 37 < D^{\chi} = \bar{D}^{\chi} = 55.$$

The support-to-order construction performs the integration task: the same fixed grand catalogue, under two distributive words and stated closures, selects different external process supports and then different non-cyclic recurrent orders while leaving aggregate terminal stock unchanged.

## 64. Conditional external support can generate internal recurrent order

The central synthesis is stronger than state-mediated launchability under the stated closures. Section 33A gives the chain

$$\boxed{\text{distributive word} \to \text{competitive process support} \to x_T \to \text{recurrent-order support} \to \mathfrak{I}(\sigma).}$$

At $\mu_P = 11/10$ only $P_1, P_2$ survive the full-catalogue profit inequalities; at $\mu_E = 6/5$ only $E_1, E_2$ do. Starting from the same state, $PE$ and $EP$ leave the same aggregate stock $66/25$ but reverse its composition. Under the explicit recurrent-capacity criterion, the terminal LP selects only $ABC$ after $PE$ and only $ACB$ after $EP$. The internally selected cycles use the same three phase technologies once each and have distinct Standard maximum-profit rates. The direction matters: $ABC$ has the higher $R^{\mathrm{Std}}$, yet after $EP$ it remains seedable while $ACB$ is the unique capacity-maximizing recurrent activity.

Thus external order does not merely constrain a pre-labelled internal option. Under a state-sensitive scarcity closure, it generates the inherited composition that makes a different recurrent order capacity-maximizing. This is stronger than a launchability statement but narrower than a theory of classical profit-rate technique choice. Indeed, the interior-CRS proposition shows that a unique state-independent maximum-rate ranking would select the same order in both branches. The remaining exogenous objects are therefore the distributive word and the economic closure governing terminal recurrent selection; the external maps and the recurrent order are endogenous only conditional on those objects.

## 65. Relaunching Sraffa by distinguishing objects rather than multiplying assumptions

The programme keeps the fixed-coefficient discipline and the asymmetry between valuation and quantities. It adds history not by making technical coefficients fluid, but by distinguishing technical knowledge, inherited productive state, feasible reconstruction, the quantity closure actually followed, persistence after first passage, the counterfactual benchmark, and the recurrent Sraffian packet.

This is not a complete general dynamics. It is a repertoire of exact results and carefully separated open problems showing how a historically organized "given system" can be studied without asking long-period price equations to determine the transition path by themselves.

# MATHEMATICAL APPENDIX A — HISTORICAL-DOMAIN PROOFS AND CERTIFICATES

### A.1 Projection proof for reproductive cones

Fix $\gamma \geq 0$ and a recurrent organization $\tau$ with period $T$. Let

$$\mathscr{K}_\tau(\gamma)$$

be the set of tuples

$$(x, x_1, \ldots, x_T, y_0, \ldots, y_{T-1})$$

satisfying non-negativity, phasewise feasibility,

$$A_t^\tau y_t \leq x_t,$$

state transitions

$$x_{t+1} = D_t^\tau (x_t - A_t^\tau y_t) + B_t^\tau y_t,$$

and

$$x_T \geq \gamma x.$$

All constraints are homogeneous linear equalities or inequalities. Hence $\mathscr{K}_\tau(\gamma)$ is a polyhedral cone. The projection of this cone on the initial-state coordinates is exactly $\mathcal{V}_\tau(\gamma)$. Fourier–Motzkin elimination or the general projection theorem for polyhedra therefore gives a finite homogeneous half-space representation

$$\mathcal{V}_\tau(\gamma) = \{x : H_{\tau,\gamma} x \geq 0\}.$$

Convexity, closedness, and conicity follow immediately. This representation also makes clear that the boundaries relevant to immediate reproductive bifurcations are contained in finitely many hyperplanes.

### A.2 Dual representation of itinerary-specific cone entry

On a fixed competitive itinerary, collect the material variables in $u$ and the terminal reproductive-cone state in $z$. After replacing $z \in \mathcal{V}_\tau$ by the half-space form (A.1), the cone-entry LP can be written

$$\max_{u,z}\ e^\top z$$

subject to

$$Gu + Lz \leq Nx, \qquad H_{\tau,1}z \geq 0, \qquad u = (u, z) \geq 0.$$

After stacking signs into a standard inequality system $Cv \leq Nx$, its dual is

$$\min_{q \in \mathcal{Q}} q^\top Nx,$$

where

$$\mathcal{Q} = \{q \geq 0 : C^\top q \geq c\}$$

is independent of $x$. Under feasibility and finiteness, strong duality yields

$$\rho_{\tau,H,\kappa}(x) = \min_{q \in \mathcal{Q}} q^\top Nx.$$

A polyhedron has finitely many relevant extreme dual solutions for a finite-valued parametric LP on a fixed cone, so locally

$$\rho_{\tau,H,\kappa}(x) = \min_{j=1,\ldots,J} a_j^\top x.$$

This gives the concave piecewise-linear representation used in Theorem 10.1. Each active $a_j$ is a bottleneck certificate for inherited productive resources; it is not a production-price vector.

### A.3 Why the cone-entry positivity test is numeraire-free

Let $e, e' \gg 0$. Define

$$\rho_e(x) = \max\{e^\top z : z \in \mathcal{R}_H(x) \cap \mathcal{V}_\tau\},$$

and analogously $\rho_{e'}(x)$. If the intersection contains only $z = 0$, both values are zero. If it contains any $z \neq 0$, then $e^\top z > 0$ and $e'^\top z > 0$. Hence

$$\rho_e(x) > 0 \iff \rho_{e'}(x) > 0.$$

The positive weight vector is therefore only a detector of non-null cone entry. It is not an economic threshold and carries no distributive meaning.

### A.4 Strict-margin proof of Proposition 17.2

Fix a regular parameter point at which two finite integer delays are identified. For every dated external support used in the comparison, collect the strictly positive inactive-profit margins and a nonsingularity margin for the active price–wage Jacobian. For every maintained recovery basis, collect the positive basic variables, inactive primal slacks, dual multipliers, and reduced costs that certify that basis. Finally collect the strict reference and threshold comparisons that distinguish the dates immediately before and at each finite delay. Let $\eta > 0$ be the minimum of this finite family after continuous local normalizations.

On a support-preserving regular parameter cell, the implicit-function theorem makes the active price–wage solutions and induced state maps continuous in the primitives. Parametric linear-programming stability makes the maintained primal and dual bases continuous while their strict slacks and reduced costs retain sign. Reference levels and the finitely many terminal comparisons that identify the two delays are continuous as well. Hence there is a neighbourhood in which every recorded quantity changes by less than $\eta/2$. The support sequence, maintained bases, and all strict pre-crossing/crossing comparisons are unchanged, so the same two finite delays are obtained. This proves Proposition 17.2.

If an exclusion is defined by an exact zero in the production graph, the relevant neighbourhood is taken in the support-preserving stratum. Allowing a new positive coefficient in a previously zero entry changes the graph and is not covered by the proposition.

### A.5 Exact certificate for the complete four-technique example

At $r_0 = 1/10$, the four stationary price systems under normalization $p_Y = 1$ are obtained from

$$p_i = (1 + r_0)a_i^\top p + w\ell_i$$

for the two active methods in each technique. Exact solution gives

| Technique | $p_X$ | $w$ | $g(3,2)$ | $g(2,3)$ |
|---|---:|---:|---:|---:|
| $\tau_{00}$ | 1139/1673 | 10552/25095 | 60/47 | 5/3 |
| $\tau_{01}$ | 237/166 | 3467/1660 | 20/21 | 5/4 |
| $\tau_{10}$ | 2468/2377 | 12554/35655 | 30/37 | 15/11 |

| Technique | $p_X$ | $w$ | $g(3,2)$ | $g(2,3)$ |
|---|---|---|---|---|
| $\tau_{11}$ | 1152/673 | 3874/3365 | 2/3 | 1 |

The wage ordering is

$$w_{01} > w_{11} > w_{00} > w_{10}.$$

At $(3,2)$ only $\tau_{00}$ satisfies $g \geq 1$. At $(2,3)$ all four do. Thus the *whole-state* diagnostic at horizon zero is strictly positive at the first composition and zero at the second. This is not an operational historical wedge under free suballocation. From $x^A = (3,2)$, operate the two methods of $\tau_{01}$ at scales $y = (2,2)$. Their commodity input is

$$\left(\frac{9}{10}, \frac{9}{5}\right) \leq (3,2),$$

and the gross output is $(2,2)$. At that state $g_{01}(2,2) = 10/9 > 1$. At the $\tau_{01}$ price system the two operated methods have zero margins and the two unused methods have strictly negative margins. Thus the apparent horizon-zero wedge disappears after one competitively supportable reallocation. The calculation is retained because it is an exact finite-dimensional illustration of Corollary 7.3, not because it supplies the historical witness.

Every number in the table is rational and follows from two-by-two linear systems and the closed-form self-reproduction formula displayed in the diagnostic calculation.

## A.6 Exact certificates for the original three-stage reconstruction witness

### A.6.1 Dated support and inactive-process margins

The exact dated-price construction supporting the $PE$ and $EP$ histories is unchanged. In temporal order along $PE$ and then $EP$, the inactive-renewal margins are

$$-\frac{576833787953}{3432254031360}, \quad -\frac{32415}{280768}, \quad -\frac{12859302599}{117008660160}, \quad -\frac{20941}{175616}.$$

All are strictly negative. The historical state transitions used in the two-period comparison are therefore supported on open regular price cells.

### A.6.2 Exact history and local objective robustness

Starting from

$$x^\circ = \left(\frac{5}{8}, \frac{1}{6}, \frac{1}{2}\right)^\top,$$

the exact terminal states are

$$x^{PE} = \left(\frac{128585365955}{499236950016}, \frac{1344355}{2370816}, \frac{50}{27}\right)^{\top},$$

$$x^{EP} = \left(\frac{3450161185}{71319564288}, \frac{2901395}{2842776}, \frac{50}{27}\right)^{\top},$$

$$x^{PP} = \left(\frac{29775581155}{416353222656}, \frac{1344355}{2370816}, \frac{50}{27}\right)^{\top}.$$

The exact continuation-value coefficients used in the alternative two-period objective are strictly positive, so maximizing dated next-stock value and maximizing current labour both exhaust the three current material constraints on the realized $PE$, $EP$, and $PP$ cells. This certifies equality of the two-period endpoints under those two local objectives.

### A.6.3 Common-metric comparison-date benchmark

Fix

$$\phi = p_A(r_P) = \left(1, \frac{132439}{175616}, \frac{1727}{2744}\right)^{\top}.$$

For every technique, the common gross-product metric is $\phi^{\top} B_{\tau} v$. The exact $PP$ reference is

$$L_2^{0,\phi} = \frac{243155405343060695}{452419404215353344}.$$

The immediate target-$A$ values are

$$S_{A,0}^{\phi}(EP) = \frac{28174944330068765}{77497550224883712},$$

$$S_{A,0}^{\phi}(PE) = \frac{1050062687851971895}{542482851574185984}.$$

Thus

$$S_{A,0}^{\phi}(EP) < L_2^{0,\phi} < S_{A,0}^{\phi}(PE).$$

The alternative $B$ technique has

$$S_{B,0}^{\phi}(EP) = \frac{90087893585867405}{190221259642896384} < L_2^{0,\phi}.$$

Hence the original witness is not used for the cross-technique Double Historical Condition.

#### A.6.4 Catalogue-complete one-period common-value certificate

The one-period recovery LP allows all four elementary processes. The displayed dual certificate may be read as a lower-wage outer relaxation; exact recomputation at the actual final-$A$ wage gives the same optimum, so the stated result does not depend on that relaxation. Exact vertex/KKT elimination yields

$$\overline{S}^{\phi}_{A,1} = \frac{4269215285532172970952 5}{10019782264655574938419 2}.$$

An exact dual certificate has inequality multipliers

$$q = \left(\frac{12373958944565}{3211230984192}, 0, 0, 0, \frac{12373958944565}{2497624098816}, 0, \frac{132439}{107743}, 0, \frac{31501113103}{13008458848}\right)^{\top}$$

and equality multipliers

$$\lambda = \left(0, \frac{12373958944565}{2497624098816}, 0\right)^{\top}.$$

All reduced costs are exactly zero and primal and dual objectives coincide. Therefore the value is an exact upper bound, not a numerical relaxation artifact.

#### A.6.5 Catalogue-complete two-period common-value certificate

The exact two-period optimum is

$$\overline{S}^{\phi}_{A,2} = \frac{270543092249203224582 5}{551040639647690994278 4}.$$

A dual certificate is

$$q = \left(\frac{784146240545}{176602517784}, 0, 0, 0, \frac{156829248109}{68678756916}, 0, 0, 0, \frac{156829248109}{45785837944}, \frac{3799828}{1516891}, \frac{208509341055}{266390329856}, 0\right)^{\top},$$

$$\lambda = \left(0, \frac{784146240545}{137357513832}, 0, 0, 0, \frac{156829248109}{45785837944}\right)^{\top}.$$

Again every reduced cost is exactly zero and the dual objective equals the displayed primal value.

Since

$$\overline{S}^{\phi}_{A,1} < L^{0,\phi}_{2}, \qquad \overline{S}^{\phi}_{A,2} < L^{0,\phi}_{2},$$

no admissible recovery can reach the common reference in one or two periods. The lower-wage outer relaxation produces the same optimum at these horizons as the actual-wage programme.

### A.6.6 Horizon-three reconstruction and exact labour optimality

The existing three-period $A$ path has terminal plan

$$v_3 = \left( \frac{67709413255625}{139322211652512}, \frac{148960709162375}{743051795480064}, \frac{744803545811875}{2229155386440192} \right)^\top .$$

Its common-value scale is

$$S^{\phi}_{A,3} = \frac{110586656495519902625}{130491784115026919424} > L^{0,\phi}_2 .$$

At the actual final-$A$ wage this value is also the exact *catalogue-complete* horizon-three optimum. One exact dual certificate has inequality multipliers

$$q = \left( \frac{32052605825}{4182119424}, 0, 0, 0, \frac{6410521165}{1626379776}, 0, 0, 0, \frac{6410521165}{1084253184}, \frac{8166269}{1086624}, 0, 0, 0, \frac{104835}{30184}, \frac{16}{11} \right)^\top$$

and equality multipliers

$$\lambda = \left( 0, \frac{32052605825}{3252759552}, 0, 0, 0, \frac{6410521165}{1084253184}, \frac{8166269}{1086624}, 0, 0 \right)^\top .$$

All reduced costs are non-negative and the dual objective equals the displayed primal value exactly. Thus the catalogue-complete optimum first crosses the comparison-date baseline at horizon three:

$$H^{\min,\phi}_A(EP) = 3.$$

Under the labour-capacity normalization, the same horizon-three plan is not merely feasible but exactly optimal. Its value is

$$\Lambda_{A,3} = \frac{1780486730969915}{8916621545760768}.$$

The exact dual certificate and active-basis sensitivity row are given in Appendix A.8.4.

### A.6.7 The stated myopic closure reaches an absorbing boundary cycle

When the myopic $P$ closure of Section 22 is continued from $x^{EP}$, its first two steps remain on the full-resource cell. At the third step the constrained optimum satisfies

$$K_{0,3} = 0.$$

Subsequent states rotate through faces with one required stage exactly zero. Exact enumeration verifies zero recurrent $A$ capacity at every subsequent state. Hence

$$D_A^{\chi,\phi}(EP) = +\infty.$$

This is the exact closure counterpart to the finite feasible horizon in A.6.6.

#### A.6.8 Executable verification

The executable certificates `Historical_Reproduction_CORE_CERTIFICATE.py`, `Historical_Reproduction_NEW_THEORY_CERTIFICATE.py`, `Historical_Reproduction_SPECTRAL_DELAY_CERTIFICATE.py`, `Historical_Reproduction_GENUINE_RANK2_ACTIVE_CELL_CERTIFICATE.py`, and `Historical_Reproduction_ENDOGENOUS_SUPPORT_TO_ORDER_CERTIFICATE.py` jointly verify the rational primitives, wage curves, $PE/EP/PP$ states, inactive margins, common-value benchmark, catalogue-complete one- and two-period certificates, horizon-three feasible and optimal objects, boundary absorption under the myopic closure, primitive second witness, comparative statics, and active-basis scaling identities.

### A.7 Primitive closure-consistent witness

This appendix is self-contained. The four elementary processes are $(\rho, d, a_0, a_1)$ with complete input catalogue

$$\mathcal{A}^{\varepsilon} = \begin{pmatrix} 2/25 & 2/25 & 9/16 & 0 \\ 0 & 0 & 0 & 3/5 \\ 11/16 & 9/20 & 0 & 0 \end{pmatrix},$$

gross-output catalogue

$$\mathcal{B} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and direct-labour vector

$$\ell = \left( \frac{9}{25}, \frac{3}{5}, \frac{1}{25}, \frac{1}{20} \right)^{\top}.$$

The two complete recurrent techniques select $(\rho, a_0, a_1)$ and $(d, a_0, a_1)$, giving the perturbed recurrent input matrices

$$A_A^{\varepsilon} = \begin{pmatrix} 2/25 & 9/16 & 0 \\ 0 & 0 & 3/5 \\ 11/16 & 0 & 0 \end{pmatrix}, \qquad A_B^{\varepsilon} = \begin{pmatrix} 2/25 & 9/16 & 0 \\ 0 & 0 & 3/5 \\ 9/20 & 0 & 0 \end{pmatrix}.$$

Direct calculation gives $(A_A^{\varepsilon})^4 \gg 0$. With $R_P = 5/4$ and $R_E = 3/2$, the exact wage comparisons are

$$\frac{3735}{3584} > \frac{20595}{22016}, \qquad \frac{117575}{210576} > \frac{24805}{114864}.$$

For

$$x_{\varepsilon}^{\circ} = \left( \frac{23}{50}, \frac{27}{100}, \frac{77}{100} \right)^{\top},$$

exact vertex enumeration of the myopic LP shows that the three current material constraints form the unique active basis on $P$, $E$, $PE$, $EP$, $PP$, and on the first two subsequent $P$ recovery steps.

The three comparison states are

$$x_{\varepsilon}^{PP} = \left( \frac{2037217302793}{6623232000000}, \frac{391974181}{369600000}, \frac{3704}{3375} \right)^{\top},$$

$$x_{\varepsilon}^{PE} = \left( \frac{38955937166183}{63678182400000}, \frac{101949613}{100800000}, \frac{3704}{3375} \right)^{\top},$$

$$x_\varepsilon^{EP} = \left(\frac{23580343622351}{84055200768000}, \frac{79561004363}{42215223600}, \frac{5816}{6075}\right)^\top.$$

For

$$\phi_\varepsilon = \left(1, \frac{13347}{17920}, \frac{171}{280}\right)^\top,$$

the exact common-metric values are

$$L_{2,\varepsilon}^{0,\phi} = \frac{5044090962413687003}{2962757124096000000},$$

$$S_{A,0}^{\phi}(EP) = \frac{5838424697897602782 1}{3760024484914790400 0},$$

$$S_{B,0}^{\phi}(EP) = \frac{6236644824923141999 9}{3492661702311936000 0},$$

$$S_{A,0}^{\phi}(PE) = \frac{666093815532478279}{213958692864000000}.$$

Continuation under the same myopic $P$ closure begins

$$1.55276241453, \quad 1.49234574701, \quad 1.74192849313, \quad 1.11393727667, \quad 0.77674960515, \quad 0, \ldots$$

relative to the fixed comparison-date baseline 1.70249897347. Hence

$$D_A^{\chi,\phi}(EP) = 2, \qquad \bar{D}_A^{\chi,\phi}(EP) = +\infty.$$

The continued $PP$ reference branch is itself non-monotone under this myopic closure; its first values are approximately

$$1.702499,\ 1.587184,\ 1.852627,\ 1.493385,\ 0.292698,\ 0.341649,\ 0, \ldots$$

so a moving-counterfactual comparison is conceptually distinct from restoration to the fixed date-2 baseline.

All assertions in this section are recomputed by `Historical_Reproduction_NEW_THEORY_CERTIFICATE.py` using exact rational arithmetic.

## A.8 Active-basis transfer and spectral-delay details

### A.8.1 Bellman reduction on a fixed basis

Write the finite-horizon recovery problem recursively as

$$V_{H+1}(x) = \max_{y,x'} \{V_H(x') : (x, y, x') \text{ satisfies the one-period constraints}\}.$$

Suppose $V_H(x') = a_H^\top x'$ on a regular continuation cone and a unique non-degenerate basis is optimal for the one-period problem. After partitioning the basic and non-basic variables, the basic equations have the form

$$B_H z_B = R_H x,$$

with nonsingular $B_H$. Thus

$$z_B = B_H^{-1} R_H x$$

and every basic variable, including $x'$, is linear in $x$. If the successor block is $x' = K_H x$, substitution gives

$$V_{H+1}(x) = a_H^\top K_H x,$$

so $a_{H+1} = K_H^\top a_H$.

If the basis sequence repeats with period $h$, composition over one block produces the monodromy $M_s$ for each residue class and therefore

$$V_{s+qh}(x) = a_s^\top M_s^q x.$$

#### A.8.2 Perron threshold brackets

Let $S_q = u^\top T^q b$ with $T$ primitive on a proper cone $K$, Perron root $\rho > 1$, and Perron vector $v \in \operatorname{int} K$. If

$$\alpha v \le_K b \le_K \beta v,$$

then

$$\alpha(u^\top v)\rho^q \le S_q \le \beta(u^\top v)\rho^q.$$

This proves Theorem 36.3 and the stated logarithmic bracket for the first threshold crossing.

When $T$ is diagonalizable, write

$$T^q = \rho^q(vw^\top + E_q), \qquad \|E_q\| \le \kappa_T \eta^q, \qquad \eta = \max_{\lambda \neq \rho} |\lambda|/\rho.$$

The estimate

$$|u^\top E_q b| \le \|u\|\,\|b\|\,\kappa_T \eta^q$$

gives Theorem 36.4 directly after division by

$$C = (u^\top v)(w^\top b) > 0.$$

#### A.8.3 Periodic-basis sign reduction and the missing projection condition

For an $h$-periodic candidate basis, order all primal KKT equations block by block and eliminate the within-block basic variables. Because the coefficient pattern repeats, the remaining interface variables satisfy a fixed recursion

$$b_{q+1} = T_P b_q.$$

Applying the same elimination to the transposed dual KKT equations gives

$$d_{q+1} = T_D d_q.$$

Every eliminated basic variable and inactive primal slack is a fixed linear functional of $b_q$; every dual multiplier and reduced cost is a fixed linear functional of $d_q$. Feasibility and optimality are therefore equivalent to finitely many sign sequences

$$r_j^\top T_P^q b_0 \geq 0, \qquad s_k^\top T_D^q d_0 \geq 0.$$

Consider one sequence $\sigma(q) = r^\top T^q b_0$. If $T$ is diagonalizable, then

$$\sigma(q) = \sum_i c_i \lambda_i^q, \qquad c_i = (r^\top v_i)(w_i^\top b_0)$$

under a biorthogonal eigenbasis. Let $\lambda_*$ be the eigenvalue of largest modulus among modes with $c_i \neq 0$. If $\lambda_* > 0$ is real and simple and $|\lambda_*| > |\lambda_i|$ for every other contributing mode, then

$$\sigma(q) = c_* \lambda_*^q \big(1 + O(\eta^q)\big), \qquad \eta < 1,$$

so the sign equals $\operatorname{sgn}(c_*)$ after a finite computable horizon. For a primitive non-negative $T$, the Perron mode gives this conclusion only when $(r^\top v)(w^\top b_0) \neq 0$.

That projection condition cannot be omitted. Take

$$T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \qquad v = \begin{pmatrix} \varphi \\ 1 \end{pmatrix}, \qquad r = \begin{pmatrix} 1 \\ -\varphi \end{pmatrix}, \qquad b_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

where $\varphi = (1 + \sqrt{5})/2$. The matrix is primitive and its Perron root $\varphi$ is strictly separated from the other eigenvalue $-1/\varphi$, but $r^\top v = 0$. Consequently $r^\top T^q b_0$ is governed by the negative subdominant mode and alternates in sign forever. This counterexample is why Theorem 36.5 states a condition on the first nonzero contributing spectral mode rather than on the transfer matrix's dominant eigenvalue alone.

#### A.8.4 Exact horizon-three dual certificate

For the original labour-capacity recovery problem at $H = 3$, the exact primal plan used in the repertoire attains

$$\Lambda_{A,3} = \frac{1780486730969915}{8916621545760768}.$$

An exact dual feasible vector $q \geq 0$ is

$$q = \left(\frac{516059}{285768}, 0, 0, 0, \frac{516059}{555660}, 0, 0, 0, \frac{516059}{370440}, \frac{13148}{7425}, 0, 0, 0, \frac{631}{660}, \frac{144}{275}\right)^{\top},$$

with equality multipliers

$$\lambda = \left(0, \frac{516059}{222264}, 0, 0, 0, \frac{516059}{370440}, \frac{13148}{7425}, 0, 0\right)^{\top}.$$

Substitution gives non-negative reduced costs and exact equality of primal and dual objectives. The inherited-state sensitivity row is

$$a_3^{\top} = \left(\frac{1032118}{250047}, 0, 0\right).$$

Since the horizon-zero row is

$$a_0^{\top} = \left(\frac{13148}{7425}, 0, 0\right),$$

one has

$$a_3^{\top} = \frac{43175}{18522} a_0^{\top}.$$

Together with

$$A_A^3 = \frac{297}{1280} I, \qquad 1 - w_P \ell_{A0} = \frac{94985}{175616},$$

this yields

$$\frac{43175}{18522} = \frac{1 - w_P \ell_{A0}}{297/1280}.$$

The executable file `Historical_Reproduction_SPECTRAL_DELAY_CERTIFICATE.py` verifies all identities in exact arithmetic.

### A.9 Genuine rank-two proper-cell active-recovery witness

This appendix certifies Section 36.7. Let

$$A = \begin{pmatrix} 1 & 1/20 \\ 1/100 & 1 \end{pmatrix}, \quad p = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad c = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}, \quad 1 + r = \frac{11}{10}, \quad w = 2,$$

$$\ell = \begin{pmatrix} 11/10 \\ 201/20 \end{pmatrix}, \qquad N = \begin{pmatrix} 3/100 & 8/100 \\ 7/100 & 2/100 \end{pmatrix}.$$

Set

$$B = (I+N)A + cw\ell^\top = \begin{pmatrix} 5327/2500 & 20363/2000 \\ 5901/5000 & 22147/2000 \end{pmatrix}.$$

Then $p^\top c = 1$ and

$$p^\top B = \frac{11}{10} p^\top A + 2\ell^\top.$$

The full-resource basis is defined on $C = A\mathbb{R}^2_+$, and

$$A^{-1}NA = \begin{pmatrix} 2729/99950 & 3213/39980 \\ 17473/249875 & 4537/199900 \end{pmatrix} \gg 0,$$

which proves forward invariance of the proper active cone under $I + sN$, $s > 0$.

The terminal LP value row is

$$a_0^\top = p^\top B A^{-1} = \left( \frac{31}{10}, \frac{211}{10} \right).$$

The feasible recovery fraction is $\bar{s} = 3/4$ and the realized accumulation fraction is $s_\chi = 1/2$, so

$$G_R = I + \frac{3}{4}N, \qquad G_\chi = I + \frac{1}{2}N.$$

Their spectra are

$$\operatorname{spec}(G_R) = \left\{ \frac{43}{40}, \frac{77}{80} \right\}, \qquad \operatorname{spec}(G_\chi) = \left\{ \frac{21}{20}, \frac{39}{40} \right\}.$$

Thus

$$\frac{11}{10} > \frac{43}{40} > \frac{21}{20}.$$

For

$$x^h = \begin{pmatrix} 7/50 \\ 1/100 \end{pmatrix}, \qquad x^0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad L^0 = \frac{121}{5},$$

the exact modal laws are

$$V_H^R = \frac{69}{40} \left( \frac{43}{40} \right)^H - \frac{27}{25} \left( \frac{77}{80} \right)^H,$$

$$V_H^\chi = \frac{69}{40} \left( \frac{21}{20} \right)^H - \frac{27}{25} \left( \frac{39}{40} \right)^H.$$

The relative transverse coefficient is $K = 72/115$. Exact comparison gives

$$V_{36}^R < L^0 < V_{37}^R, \qquad V_{54}^\chi < L^0 < V_{55}^\chi,$$

and monotonicity yields

$$H^{\min} = 37 < D^{\chi} = \bar{D}^{\chi} = 55.$$

For optimal recovery, the crude cone–Perron bracket is $29 \le H^{\min} \le 64$. At $\delta = 1/50$, $q_\delta = 32 < 37$ and the gap-refined bracket identifies $H^{\min} = 37$ exactly. For the realized closure the crude bracket $43 \le D^{\chi} \le 95$ narrows to $54 \le D^{\chi} \le 55$.

The executable file `Historical_Reproduction_GENUINE_RANK2_ACTIVE_CELL_CERTIFICATE.py` verifies the price equation, proper-cone invariance, Bellman basis, modal laws, three distinct dominant factors, three clocks, and spectral bounds in exact arithmetic.

# MATHEMATICAL APPENDIX B — INTERNAL STANDARD PROOFS

### B.1 Block-power identity for the Historical Standard Cycle

Let $E_t$ be the $n$-dimensional phase subspace associated with date $t$. The temporal lift $\bar{A}(\sigma)$ maps $E_t$ into $E_{t-1}$, with phase indices modulo $T$, so $\bar{A}(\sigma)^T$ leaves every $E_t$ invariant. Restricted to $E_0$ it equals $A_1 A_2 \cdots A_T$; restricted to any other phase it equals the corresponding cyclic rotation. Hence

$$\rho(\bar{A}(\sigma))^T = \rho(M(\sigma)).$$

The spectral identity does not require irreducibility or productivity. Irreducibility is used to obtain a strictly positive Perron bundle unique up to scale; productivity supplies $\rho(\bar{A}) < 1$.

### B.2 Cyclic rotations and minimum order-sensitive depth

For square matrices $X, Y$,

$$\det(\lambda I - XY) = \det(\lambda I - YX).$$

Repeated application shows that cyclic rotations of $A_1 A_2 \cdots A_T$ have the same characteristic polynomial and therefore preserve $\rho(M)$, $G$, and $R$. For $T = 2$, reversal is merely a cyclic rotation. At $T = 3$ two distinct cyclic classes exist, so genuine non-cyclic order effects become possible. The strictly positive example in Section 29.8 gives such an effect. For $T > 3$, append $T - 3$ copies of $cI$, $0 < c < 1$, to both three-phase words and then perturb them to strictly positive productive matrices $D_\varepsilon = cI + \varepsilon \mathbf{1}\mathbf{1}^\top$; continuity preserves the strict spectral-radius gap.

### B.3 Frobenius proof of historical basic spectral admissibility

Let the dated lift have Frobenius form with irreducible diagonal blocks $B_1, \ldots, B_m$, where $B_1$ is the unique source block. Put $\lambda_1 = \rho(B_1)$. The left eigenvector equations are

$$p_1^\top B_1 = \lambda_1 p_1^\top,$$

and recursively

$$p_j^\top (\lambda_1 I - B_j) = c_j^\top, \qquad c_j^\top = \sum_{i<j} p_i^\top C_{ij}.$$

Every downstream block has an incoming edge from an earlier block, so after the preceding prices are strictly positive, $c_j \geq 0$ and $c_j \neq 0$. If $v_j \gg 0$ is the right Perron vector of $B_j$, multiplication gives

$$(\lambda_1 - \rho(B_j))p_j^\top v_j = c_j^\top v_j > 0.$$

Thus strict positivity requires $\lambda_1 > \rho(B_j)$. Conversely, under this strict dominance, $\lambda_1 I - B_j$ is a nonsingular $M$-matrix with non-negative inverse; irreducibility and $c_j \neq 0$ imply $p_j \gg 0$. Induction proves Frobenius Criterion 30.2.

### B.4 Dated joint-production rank decomposition

Let $S_t$ select commodity-date pairs at phase $t$. A phase-$t$ process has input support only in block $t-1$ and output support only in block $t$. After grouping process rows by phase and selected commodity columns by adjacent dates, the Sraffa matrix is block diagonal with phase blocks

$$\left[A_t[S_{t-1},:]^\top \; B_t[S_t,:]^\top\right].$$

Therefore

$$\operatorname{rank}[\bar{A}_S^\top \; \bar{B}_S^\top] = \sum_t \operatorname{rank}\left[A_t[S_{t-1},:]^\top \; B_t[S_t,:]^\top\right].$$

This proves the dated rank decomposition used in Section 31.

### B.5 Lowest-root proof in the sufficient joint-production class

Let $N = \bar{B} - \bar{A}$ and $H = \bar{A}N^{-1}$. Then

$$\bar{B} - (1+r)\bar{A} = (I - rH)N.$$

Since $N$ is nonsingular, positive roots $r$ correspond to positive eigenvalues $1/r$ of $H$. If $H$ is non-negative and irreducible, its Perron root is the largest positive eigenvalue and the unique eigenvalue admitting a strictly positive left eigenvector. Hence

$$R = \frac{1}{\rho(H)}$$

is both the smallest positive algebraic root and the unique positive-price root. If $Hx = \rho(H)x$ and $z = N^{-1}x$, then

$$\bar{B}z = (1+R)\bar{A}z.$$

The vector $z$ need not be non-negative.

### B.6 Relative-interior characterization

Let $v_1, \ldots, v_m$ be a finite indexed family. Zero belongs to $\operatorname{relint}\operatorname{conv}\{v_i\}$ if and only if

$$0 = \sum_i \alpha_i v_i, \qquad \alpha_i > 0, \qquad \sum_i \alpha_i = 1.$$

The forward implication follows because a strictly positive convex representation cannot place zero in a proper face. Conversely, if zero is in the relative interior, for each non-zero $v_i$ some $-\delta_i v_i$ remains in the convex hull; combining its convex representation with $v_i$ gives a representation of zero assigning positive weight to $v_i$. Averaging over $i$ gives strictly positive weight to every listed vector. Applied to the rows of $C$, this is equivalent to $C^\top p = 0$ with $p \gg 0$; applied to the columns it is equivalent to $Cz = 0$ with $z \gg 0$. Ordinary convex-hull membership on the column side permits $z \geq 0$, $z \neq 0$.

### B.7 Strict Collatz–Wielandt gap

Assume $p \gg 0$ and $C^\top p = 0$, where $C = \bar{B} - \mu\bar{A}$. If there is no non-zero $z \geq 0$ with $Cz = 0$, then

$$0 \notin \operatorname{conv}\{c_j(C)\}.$$

Strict separation gives $h$ and $\eta > 0$ such that $h^\top c_j(C) \geq \eta$ for every $j$. For sufficiently small $\varepsilon > 0$, $q = p + \varepsilon h \gg 0$, and

$$q^\top b_j - \mu q^\top a_j = \varepsilon h^\top c_j(C) > 0 \qquad \forall j.$$

Therefore $\min_j \phi_j(q) > \mu$ and $\Gamma > \mu$. This proves the strict gap in Proposition 32.3.

### B.8 Support-reduced certificates

Suppose $\Gamma$ is attained at $p^* \gg 0$ and let

$$J^* = \{j : \phi_j(p^*) = \Gamma\}.$$

If zero were not in

$$\operatorname{conv}\{c_j(\bar{B} - \Gamma\bar{A}) : j \in J^*\},$$

a separating direction would improve every binding inequality while inactive inequalities retain positive slack, contradicting optimality. Hence a non-negative right-kernel certificate exists on $J^*$. Since the active columns lie in the $(m-1)$-dimensional hyperplane $(p^*)^\perp$, Caratheodory's theorem gives a certificate using at most $m$ active process-date columns.

# MATHEMATICAL APPENDIX C — EXACT INTERNAL EXAMPLES

### C.1 Strictly positive single-production example

For the matrices in Section 29.8, direct multiplication gives the two ordered monodromy products displayed there. Their Perron roots are

$$\rho(ABC) = 0.191167392564\ldots, \qquad \rho(ACB) = 0.172185845706\ldots,$$

which yield

$$R(ABC) = 0.7359160868\ldots, \qquad R(ACB) = 0.7974942812\ldots.$$

The strict inequality and continuity of spectral radius imply local robustness.

### C.2 Reducible basicity example

For $ABC$, the dated production graph contains the cycles

$$(2,0) \to (1,1) \to (2,2) \to (2,0)$$

and

$$(2,0) \to (2,1) \to (1,2) \to (2,0),$$

which share $(2,0)$. Their union is the unique source strongly connected component and reaches the remaining node $(1,0)$. For $ACB$, the unique source SCC is

$$(2,0) \to (2,1) \to (1,2) \to (2,0).$$

This directly certifies the two historical basic sets reported in Section 30.3.

### C.3 Root isolation and multiplier signs in the joint-production example

For order 123, let $N_{123} = \bar{B}_{123} - \bar{A}_{123}$ and $H_{123} = \bar{A}_{123} N_{123}^{-1}$. Exact arithmetic gives

$$\det N_{123} = \frac{1552173}{4000000},$$

and every entry of $H_{123}$ is positive; its smallest entry is $432/73913$. The selected positive root is isolated by

$$1.06 < R(123) < 1.07.$$

For order 132,

$$\det N_{132} = \frac{1575961}{4000000},$$

$H_{132}$ is strictly positive, its smallest entry is 1520/1575961, and

$$1.15 < R(132) < 1.16.$$

At a simple root, any non-zero column of $\operatorname{adj} C(r)$ spans the one-dimensional right kernel. The first adjugate column suffices for both orders. Sturm root counts show that none of its six cofactor numerators vanishes on the relevant rational isolating interval, so every sign is constant there. The exact sign pattern is

| Order | Cofactor 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 123 | $-$ | $+$ | $+$ | $+$ | $+$ | $+$ |
| 132 | $+$ | $+$ | $+$ | $+$ | $+$ | $+$ |

For transparency, the first cofactor numerators are

$$-(r+1)^2(81r^3 + 243r^2 + 243r + 2893)$$

under 123, and

$$-27(r+1)^2(3r^3 + 9r^2 + 9r - 73)$$

under 132. On the isolating interval for the second order the cubic factor is negative, so the complete cofactor is positive. Since both determinant roots are simple, the kernels are one-dimensional. Consequently

$$\mathfrak{S}(123) = 0, \qquad \mathfrak{S}(132) = 2.$$

This is an exact sign certificate, not a floating-point inference.

### C.4 Exact Endogenous Support-to-Order witness

The external fixed activity catalogue has

$$A^e = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad c = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}, \quad \ell^e = \left(\frac{1}{10}, \frac{3}{10}, \frac{1}{10}, \frac{3}{10}\right)^\top,$$

and

$$B^e = \begin{pmatrix} 93/100 & 89/120 & 3/5 & 5/24 \\ 27/100 & 73/120 & 3/5 & 137/120 \end{pmatrix}.$$

Process order is $P_1, E_1, P_2, E_2$.

At $\mu_P = 11/10$, solving the active zero-profit equations and $p^\top c = 1$ for every complete two-process support gives:

| support | $(p_1, p_2, w)$ | catalogue slack vector |
|---|---|---|
| $P_1P_2$ | $(1, 1, 1)$ | $(0, 1/20, 0, 1/20)$ |
| $P_1E_2$ | $(461/446, 431/446, 190/223)$ | $(0, 17/446, -231/4460, 0)$ |
| $E_1P_2$ | $(253/256, 259/256, 223/256)$ | $(-231/12800, 0, 0, 17/1280)$ |
| $E_1E_2$ | $(1, 1, 5/6)$ | $(-1/60, 0, -1/60, 0)$. |

Hence only $P_1P_2$ satisfies every full-catalogue inequality. Its active Jacobian determinant is $-77/1000$.

At $\mu_E = 6/5$ the corresponding table is

| support | $(p_1, p_2, w)$ | catalogue slack vector |
|---|---|---|
| $P_1P_2$ | $(1, 1, 0)$ | $(0, -3/20, 0, -3/20)$ |
| $P_1E_2$ | $(521/566, 611/566, 243/566)$ | $(0, -30/283, 783/5660, 0)$ |
| $E_1P_2$ | $(289/280, 271/280, 27/70)$ | $(783/14000, 0, 0, -3/70)$ |
| $E_1E_2$ | $(1, 1, 1/2)$ | $(1/20, 0, 1/20, 0)$. |

Thus only $E_1E_2$ is supported; its active Jacobian determinant is $-1/5$.

Subtracting the physical wage from gross output gives

$$F_P = \begin{pmatrix} 22/25 & 11/20 \\ 11/50 & 11/20 \end{pmatrix}, \qquad F_E = \begin{pmatrix} 2/3 & 2/15 \\ 8/15 & 16/15 \end{pmatrix}.$$

For $x^\circ = (1, 1)^\top$,

$$x^{PE} = \left(\frac{132}{125}, \frac{198}{125}\right)^\top, \qquad x^{EP} = \left(\frac{198}{125}, \frac{132}{125}\right)^\top,$$

and both totals equal $66/25$.

The internal phase matrices are

$$A = \frac{1}{10}\begin{pmatrix} 7 & 2 \\ 4 & 5 \end{pmatrix}, \quad B = \frac{1}{10}\begin{pmatrix} 4 & 1 \\ 4 & 6 \end{pmatrix}, \quad C = \frac{1}{10}\begin{pmatrix} 1 & 5 \\ 5 & 3 \end{pmatrix}.$$

Direct multiplication gives

$$M_{ABC} = \frac{1}{1000}\begin{pmatrix} 131 & 237 \\ 206 & 282 \end{pmatrix}, \qquad M_{ACB} = \frac{1}{1000}\begin{pmatrix} 232 & 263 \\ 256 & 239 \end{pmatrix}.$$

Their Perron pairs at the common $A$-entry date are

$$\rho_{ABC} = \frac{11}{25}, \qquad q_{ABC} = \begin{pmatrix} 79/182 \\ 103/182 \end{pmatrix},$$

$$\rho_{ACB} = \frac{99}{200}, \qquad q_{ACB} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}.$$

With $Q = [q_{ABC}\ q_{ACB}]$, the terminal LP is

$$\max\{\mathbf{1}^\top z : Qz \le x,\ z \ge 0\}.$$

At $x^{PE}$ a primal–dual certificate is

$$z = \begin{pmatrix} 24024/9875 \\ 0 \end{pmatrix}, \quad \pi = \begin{pmatrix} 182/79 \\ 0 \end{pmatrix},$$

with

$$x - Qz = \begin{pmatrix} 0 \\ 2046/9875 \end{pmatrix}, \qquad Q^\top \pi - \mathbf{1} = \begin{pmatrix} 0 \\ 12/79 \end{pmatrix}.$$

At $x^{EP}$,

$$z = \begin{pmatrix} 0 \\ 264/125 \end{pmatrix}, \quad \pi = \begin{pmatrix} 0 \\ 2 \end{pmatrix},$$

with

$$x - Qz = \begin{pmatrix} 66/125 \\ 0 \end{pmatrix}, \qquad Q^\top \pi - \mathbf{1} = \begin{pmatrix} 12/91 \\ 0 \end{pmatrix}.$$

Strong duality holds in both cases. The executable file `Historical_Reproduction_ENDOGENOUS_SUPPORT_TO_ORDER_CERTIFICATE.py` checks the support enumeration, active Jacobians, wage-net maps, non-commutation, equal aggregate terminal stock, internal Perron equations, and both terminal KKT certificates in exact arithmetic.

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