Mathematical Models and Methods in Applied Sciences
Vol. 7, No. 4 (1997) 435–455

WAXY CRUDE OILS: SOME ASPECTS OF THEIR DYNAMICS

A. FARINA
Dipartimento di Ingegneria Aeronautica e Spaziale, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy

Communicated by G. P. Galdi
Received 20 March 1996

Waxy crude oils are highly non-Newtonian fluids known to cause pipelining difficulties because their rheological properties are strongly affected by paraffin crystallization. On the basis of experimental data, a physical model has been developed to describe the behavior of these crudes. The corresponding mathematical problem has been studied in planar geometry proving the existence and uniqueness of a classical solution. A condition on the pressure gradient has been found ensuring that the system do not come to a complete stop in finite time.

1. Introduction

Waxy crude oils contain a great deal of heavy hydrocarbons which, at sufficiently low temperature, may precipitate as a waxy phase. This can cause severe difficulties in pipelining and storage (especially if the mixtures are transported across artic regions or cold oceans).

The compounds of the waxy phase are usually paraffins. They consist of straight and branched-chain of n-alkanes (usually ranging from $C_{18}H_{38}$ to $C_{40}H_{82}$), mixed with organic and inorganic materials.[14]

Paraffin solubility in crude oils depends on the chemical composition of the crude as well as on pressure and temperature. Paraffin will begin to crystallize as soon as the equilibrium temperature and pressure is reached (cloud point). When the product is cooled to a temperature lower than the cloud point, the crystals agglomerate leading to the formation of a gel structure (pour point).[5]

The rheology of waxy crude oils is strongly affected by the presence of crystallized wax. For most crudes at sufficiently high temperatures, the viscosity, at a given temperature, is constant and the oil, although chemically very complex, can be considered a simple Newtonian fluid. Below the crystallization temperature the flow becomes non-Newtonian. A yield-stress (a minimum stress that must be overcome before any fluid movement takes place) can be detected. The flow properties of waxy crude oils are further complicated by their critical dependence upon the mechanical and thermal "history". The viscosity of the non-Newtonian oil can be greatly reduced by mechanical shear. The disintegration of large wax agglomerates appears to be the primary cause of the lower viscosity.[15,16]

This paper is devoted to the study, in planar geometry, of the low temperature dynamic behavior of waxy crude oils. In Sec. 2 we will define a physical model to describe this class of fluids. Starting from this model we will develop the corresponding mathematical-physics problem. Section 3 is devoted to the analytical study of the problem. Existence and uniqueness of a classical solution will be proved. We will also show the existence of a condition preventing the system to come to a complete stop in a finite time.

2. The Physical Problem

2.1. *Physical assumptions*

2.1.1. *Low, uniform and constant temperature*

A first simplifying assumption is that the temperature is uniform, constant and below the so-called pour point, so that the density of crystallized wax is constant in time and space. Therefore the non-Newtonian behavior has to be attributed to the agglomeration of wax crystals only. Relaxing this assumption leads to a much more complicated problem which will be considered in a forthcoming paper.

2.1.2. *Incompressible fluid*

A very reasonable assumption, consistent with the previous one, is to take $\rho =$ constant, ρ being the oil density. Typically $\rho \cong 0.8\ \mathrm{gr/cm}^3$. Thus if $\mathbf{v}(\mathbf{x}, t)$ is the velocity field and V the domain where the oil flows, from the continuity equation, we have

$$\nabla \cdot \mathbf{v}(\mathbf{x}, t) = 0 \quad \mathbf{x} \in \mathrm{V}\ ;\ t \geq 0\,.$$

2.1.3. *Laminar flow*

This assumption is justified by the fact that, for low temperatures, the Reynolds number (evaluated with typical pipeline values) is less than the threshold of turbulent flow.

2.1.4. *Planar geometry*

For the sake of simplicity we consider a one-directional flow between two parallel plates at a distance $2L$. The coordinate along the direction of motion is x and y is the direction perpendicular to the plates. In the xy-plane, the velocity has the form $\mathbf{v} \equiv (v(y, t), 0)$.

2.2. *The rheological model*

Waxy crude oils show, at low temperatures, the presence of an yield-stress.[15] According to this experimental evidence we describe them as Bingham fluids. A Bingham fluid is a non-Newtonian fluid which behaves like a rigid body when the shear stress τ is less than a threshold value τ_0, while for $\tau > \tau_0$ the relationship between the stress τ and shear rate $\gamma(\left|\frac{\partial v}{\partial y}\right|$ in planar geometry) is linear

$$\tau = \tau_0 + \eta\gamma$$

η being the Bingham viscosity.

In order to consider the evolution of the sheared system exhibiting a kind of "thixotropy" we introduce a time-dependent parameter α defined as the ratio between the density of agglomerated paraffin and the density of crystallized paraffin. So α is a nondimensional quantity in the interval $[0, 1]$. The quantity mainly influenced by the agglomeration factor seems to be the yield stress τ_0:

$$\tau_0 = \tau_0(\alpha),$$

where τ_0, as function of α, is C^1 and nondecreasing.

A natural way of writing down an evolution equation for α accounting for both spontaneous aggregation of paraffin crystals and agglomerates fragmentation (explaining "thixotopy") is the following:

$$\dot{\alpha} = k_1(1 - \alpha) - k_2\alpha W, \tag{1}$$

where W is the power dissipated in the flow for unit volume and k_1, k_2 are constant parameters. Unfortunately it has to be noted that no experimental data are available for the time evolution of α, so that the values of k_1, k_2 are not known, although it is quite clear that the validation of (1), or of a similar low, is a necessary step towards a correct description of such systems.

It is important to underline that the assumption $\alpha = \alpha(t)$ for each $\mathbf{x} \in \mathrm{V}$ limits the applicability of the model to implant characterized by the presence of pumps mixing the fluid over time intervals much smaller than the evolution time scale of α. Such a situation is indeed occurring in laboratory experimental loops.

2.3. *The mathematical formulation*

Owing to the presence of a yield stress τ_0, a Bingham flow between two parallel plates has a central rigid core bounded between two planes $y = \pm s(t)$ and a fluid region in which the equation of motion is parabolic

$$\rho v_t - \eta v_{yy} = f_0, \qquad s(t) < |y| < L \tag{2}$$

where $f_0 = -\partial P/\partial x$ is the driving pressure gradient which is assumed to be a known constant.

Considering only the upper part of the layer, the initial and boundary conditions for v and s are

$$v(L,t) = 0\,, \qquad t > 0 \tag{3}$$

$$v_y(s(t),t) = 0\,, \qquad t > 0 \tag{4}$$

$$v(y,0) = v_0(y)\,, \qquad s_0 < y < L \tag{5}$$

$$s(0) = s_0\,, \qquad 0 < s_0 < L\,. \tag{6}$$

Operating as in p. 91 of Ref. 12 we get to the additional free boundary condition

$$v_t(s(t),t) = \frac{1}{\rho}\left(f_0 - \frac{\tau_0(\alpha(t))}{s(t)}\right), \qquad t > 0 \tag{7}$$

which is nothing but the equation of motion of the rigid core.

Taking into account the expression of W, we obtain the following equation for the parameter α

$$\dot{\alpha} = k_1(1-\alpha) - \alpha\frac{k_2}{L}\left[\eta\int_{s(t)}^{L}(v_y)^2 dy + \tau_0(\alpha)\,v(s,t)\right] \tag{8}$$

whose initial condition is

$$\alpha(0) = \alpha_0\,, \qquad 0 \leq \alpha_0 \leq 1\,. \tag{9}$$

We look for a classical solution of the system (2)–(9) in the following referred as problem (P).

We can reduce problem (P) to a problem of Stefan type, assuming the necessary regularity of the solution, by means of the transformation (see Ref. 4 for instance)

$$z(y,t) = v_t(y,t) \tag{10}$$

whose inverse is

$$v(y,t) = v_0(y) + \int_0^t z(y,\theta)d\theta \tag{11}$$

leading to the system (defined problem (P_z))

$$\rho z_t - \eta z_{yy} = 0\,, \qquad t > 0;\ s(t) < y < L \tag{12}$$

$$z(L,t) = 0\,, \qquad t > 0 \tag{13}$$

$$z(s(t),t) = \rho^{-1}\left(f_0 - \frac{\tau_0(\alpha)}{s(t)}\right), \qquad t > 0 \tag{14}$$

$$z(y,0) = z_0(y)\,, \qquad s_0 \le y \le L \tag{15}$$

$$z_y(s(t),t) = (\eta\, s(t))^{-1}\tau_0(\alpha)\,\dot{s}(t)\,, \qquad t > 0 \tag{16}$$

$$s(0) = s_0\,, \qquad 0 < s_0 < L \tag{17}$$

$$\dot{\alpha} = k_1(1-\alpha) - \frac{1}{L}k_2\alpha\left[\eta\int_s^L (v_y)^2 dy + \tau_0(\alpha)\, v(s,t)\right]\,, \qquad t > 0 \tag{18}$$

$$\alpha(0) = \alpha_0\,, \qquad 0 \le \alpha_0 \le 1 \tag{19}$$

where

$$z_0(y) = \frac{\eta}{\rho}v_0''(y) + \frac{f_0}{\rho}\,. \tag{20}$$

3. Existence and Uniqueness Theorem for (P_z)

Let us first list the assumption needed.

A - *Assumptions on* τ_0

(a.1) $\tau_0 : [0.1] \to [\tau_m\,, \tau_M]$ with $0 < \tau_m < \tau_M < \infty$ and $\tau_0(\alpha) \in C^1([0,1])$.
(a.2) $0 \le \frac{d}{d\alpha}\tau_0(\alpha) \le N$ for some positive constant N.

B - *Assumptions on* $\alpha_0\,, s_0$ *and* $v_0(y)$

(b.1) $s_m \le s_0 \le s_M$ where $s_m = \tau_m/f_0; s_M = \tau_M/f_0$.
(b.2) $0 \le \alpha_0 \le 1$.
(b.3) $v_0(y) \in C^3([s_0, L])$.
(b.4) $v_0(y) \ge 0, v_0(L) = 0; v_0'(y) < 0, v_0'(s_0) = 0; v_0''(y) < 0, v_0''(s_0) = -\tau_0(\alpha)/s_0$
(b.5) $v_m'(y) \le v_0'(y) \le v_M'(y)$ for $s_0 \le y \le L$, where $v_m(y)$ and $v_M(y)$ are the stationary velocities calculated respectively with the minimum value τ_m and the maximum value τ_M of the yield-stress $\tau_0(\alpha)$.

The solution of (P_z) in the domain

$$D_{T_0} = \left\{(y,t) \in \mathbb{R}^2 : 0 < t < T_0; s(t) < y < L\right\} \tag{21}$$

is defined as follows:

Definition 3.1. A time T_0 and a triple of functions $(z(y,t), s(t), \alpha(t))$ is a solution of problem (P_z) in the domain D_{T_0} if:

(i) $z(y,t) \in C^{2,1}(D_{T_0}) \cap C(\overline{D}_{T_0})$
(ii) z_y is continuous up to $y = s(t)$ for each $0 \le t \le T_0$
(iii) $s(t) \in C^1((0,T_0)) \cap C([0,T_0])$
(iv) $0 < s(t) < L$ for each $0 \le t \le T_0$
(v) $\alpha(t) \in C^1((0,T_0)) \cap C([0,T_0])$
(vi) $0 \le \alpha \le 1$ for each $0 \le t \le T_0$
(vii) $z(y,t)$, $s(t)$ and $\alpha(t)$ satisfy the equations of (P_z). □

We remark that if (P_z) has a solution then $v(y,t)$, given by (11), is a classical solution of problem (P).

To prove the local existence and uniqueness of solution of (P_z) we will use a technique based on the Schauder fixed point theorem (p. 211 of Ref. 10). Taken α in a suitable set Σ of Lipschitz functions we solve the Stefan problem (12)–(17) for $z(y,t)$ and $s(t)$. After that we consider the function $\widetilde{\alpha}$ obtained by solving the linear Cauchy problem

$$\begin{cases} \dfrac{d\widetilde{\alpha}}{dt} = k_1(1-\widetilde{\alpha}) - \dfrac{1}{L}k_2\,\widetilde{\alpha}\,W(\alpha)\,, \\ \widetilde{\alpha}(0) = \alpha_0\,, \end{cases}$$

where W is the power calculated with the previously selected α. Showing that, for each $\alpha \in \Sigma$, the function $\widetilde{\alpha}$ exists and belongs to Σ, an operator $\mathcal{T}$ from Σ to Σ is defined. If $\mathcal{T}$ is such that the Schauder theorem can be applied, then $\mathcal{T}$ has at least one fixed point. Thus (P_z) has at least one solution. Uniqueness will be proved using the contraction theorem.

3.1. *The set* Σ

Let us consider a time $T_0 > 0$ and the positive constant

$$A = k_1 + \frac{5}{6}k_2\,\frac{f_0^2\,L^2}{\eta}\,. \tag{22}$$

Definition 3.2. Σ is the set of all functions α such that:

(i) $\alpha : [0,T_0] \to [0,1]; \alpha(0) = \alpha_0$
(ii) $\alpha \in C([0,T_0])$
(iii) $|\,\alpha(t_1) - \alpha(t_2)\,| \leq A\,|\,t_1 - t_2\,| \quad \forall \; t_1, t_2 \in [0,T_0]$. □

It is easy to show that Σ is a compact set with respect to the norm of $C([0,T_0])$. Moreover its elements have L^∞ first derivative (see, for example, Ref. 11 or 13).

An important feature of functions α is the possibility to be approximated with functions possessing a bigger regularity.

Definition 3.3. Let Σ_1 be the set of all functions β such that:

(i) $\beta : [0,1] \to \mathbb{R}$,
(ii) $\beta(0) = \alpha_0$,
(iii) $\beta \in C^\infty([0,T_0])$,
(iv) $|\,\frac{d}{dt}\beta(t)\,| \leq A \;\forall\; t \in [0,T_0]$. □

Proposition 3.1. *For each* $\alpha \in \Sigma$ *there exists a sequence* $\{\beta_n\} \subset \Sigma_1$ *uniformly converging to* α.

Proof. Any $\alpha \in \Sigma$ is continuous on the interval $[0, T_0]$. From Weierstrass polynomial approximating theorem we have that, for every $\varepsilon > 0$, there exists an algebraic polynomial

$$\Pi_n(t) = q_0 + q_1 t + q_2 t^2 + \cdots + q_n t^n$$

such that

$$|\,\alpha(t) - \Pi_n(t)\,| < \varepsilon$$

on the interval $[0, T_0]$ (see p. 521 of Ref. 1).

Now if we take a number sequence $\{\varepsilon_k\}$ converging to zero we can construct the corresponding sequence of algebraic polynomials $\{\Pi_k(t)\}$ uniformly convergent to $\alpha(t)$ on $[0, T_0]$.

To construct effectively the polynomials $\Pi_n(t)$ we use the Tonelli's method (p. 31 of Ref. 9). Therefore with $\Pi_n(t)$ we can define the nth Tonelli polynomial for the function $\alpha(t)$.

It is easy, using the definition of the Tonelli polynomials, to show that

$$|\,\Pi_n(t_1) - \Pi_n(t_2)\,| \le A\,|\,t_1 - t_2\,|$$

for each $t_1, t_2 \in [0, T_0]$.

Consequently, if we set

$$\beta_n(t) = \Pi_n(t) - \Pi_n(0) + \alpha_0$$

we have that $\beta_n \in \Sigma_1$. Thus we have found a sequence $\{\beta_n\}$ which converges uniformly to α. □

From the assumption (a.1) τ_0, as a function of α, is defined on $[0, 1]$. Therefore $\tau_0(\beta_n)$ cannot be defined. To avoid this difficulty we extend τ_0 (with the appropriate regularity, i.e. C^1) to $[-\varepsilon, 1+\varepsilon]$ (where $\varepsilon > 0$) taking care that the assumption (a.2) is still valid. Now the values τ_m and τ_M are the new limits of τ_0.

3.2. *The auxiliary problem*

For a given $\alpha \in \Sigma$ we consider the Stefan-like problem: (12)–(17) defined as problem (P_z^α).

A pair of functions $(z(y, t); s(t))$ is said to be solution of (P_z^α) on D_{T_0} (given by (21)) if it has the properties (i), (ii), (iii) and (iv) of Definition 3.1, and functions z, s satisfy the equations of (P_z^α).

Unfortunately, because α is not $C^1([0, T_0])$, it is not possible to apply Theorem 2 of Ref. 6 to problem (P_z^α). Anyway, using Proposition 3.1 α can be approximated by a sequence $\{\beta_n\}$. Since for each β_n the corresponding problem $(\mathrm{P}_z^{\beta_n})$ is solvable we obtain a sequence of approximating solutions (z_n, s_n). We will prove the existence of a time T_0 (which does not depend on the particular function β_n) such that each problem has a unique solution in the time interval $(0, T_0)$. Next we will show that the sequences $\{z_n(y, t)\}$ and $\{s_n(t)\}$ converge to two functions: $z(y, t)$ and $s(t)$. The following sections are devoted to this demonstration.

3.2.1. *Integral formulation of the free boundary equation*

Considering the domain

$$D_t = \{0 < \theta < t \,:\, s(\theta) < y < L\}$$

and applying to it to Green formula

$$\iint_{D_t} (\phi \mathcal{L} \psi - \psi \mathcal{L}^+ \phi)\, dy d\theta = \oint_{\partial D_t} \omega$$

with

- $\phi = y - L;\ \psi = z,$
- $\mathcal{L} = \frac{\eta}{\rho} \frac{\partial^2}{\partial y^2} - \frac{\partial}{\partial t};\ \mathcal{L}^+ = \frac{\eta}{\rho} \frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial t},$
- $\omega = \phi \psi\, dy - \frac{\eta}{\rho} (\psi\, \phi_y - \phi\, \psi_y)\, d\theta,$

we get (as in Ref. 4) this integral equation

$$\frac{\eta}{\rho} \int_0^t \left(\frac{\tau_0\,(\alpha)}{s} - f_0 \right) d\theta = \rho \int_{s(t)}^{L} (L - y)\, z\,(y,t)\, dy - \rho \int_{s_0}^{L} (L - y)\; z_0\,(y)\, dy$$
$$+ \frac{1}{2} f_0 \left[(L - s_0)^2 - (L - s\,(t))^2 \right] \tag{23}$$

where we have taken into account of the equations of (P_z^{α}).

Proposition 3.2. *If* $(z(y,t), s(t))$ *is solution of* (P_z^{α}) *in* $(0, T_0)$ *then* z *and* s *satisfy* (23). *Vice versa if* $s(t) \in C([0, T_0])$ *and is Lipschitz;* $z(y,t)$ *satisfies the Dirichlet problem:* (12)–(15) *on* D_{T_0} *and the pair* $(z(y,t), s(t))$ *satisfies* (23) *in* $(0, T_0)$ *then:*

(i) $s(t) \in C^1([0, T_0])$,
(ii) $z_y(s(t), t)$ *is continuous in* $(0, T_0)$,
(iii) $s(t)$ *and* $z(y,t)$ *satisfy*

$$z_y(s(t), t) = \frac{\tau_0(\alpha)}{\eta\, s(t)} \frac{d}{dt} s(t)\,. \tag{24}$$

Proof. The first part of the proposition has already been proved. Point (ii) follows from Theorem 14.4.1 p. 247 of Ref. 2. To show point (i) we need to prove that for each $t \in (0, T_0)$ the limit

$$\lim_{\delta \to 0} \frac{s(t + \delta) - s(t)}{\delta}$$

exists and defines a continuous function. Considering (23) at times t and $t + \delta$ and taking the limit for $\delta \to 0$ we obtain:

$$\frac{\tau_0(\alpha)}{s(t)} \lim_{\delta \to 0} \frac{s(t + \delta) - s(t)}{\delta} = \eta\, z_y(s(t), t)\,.$$

Since $z_y(s(t),t)$ is a continuous function in $(0,T_0)$, point (i) follows immediately. Point (iii) is trivial. □

This result establishes the equivalence between Eqs. (23) and (24).

3.2.2. *Principal features of the flow*

Assume that (P_z^α) has a unique solution $(z(y,t),s(t))$ on D_{T_0}. Under such hypothesis the function $v(y,t)$, given by (11), satisfies the following problem (denoted by (P_v^α)):

$$(P_v^\alpha)\begin{cases}\rho v_t-\eta v_{yy}=f_0\,,\\ v(L,t)=0\,,\\ v_y(s(t),t)=0\,,\\ v(y,0)=v_0(y)\,,\\ v_t(s(t),t)=\rho^{-1}[f_0-\tau_0(\alpha)\,/\,s(t)]\,,\\ s(0)=s_0\,.\end{cases}$$

Proposition 3.3. *If* $(v(y,t),s(t))$ *is the solution of* (P_v^α) *on* D_{T_0} *then*

(i) $v(y,t)\geq 0\,,$
(ii) $v_y(y,t)\leq 0\,,$
(iii) $v_{yy}(y,t)\leq 0$ *which implies* $v_t(y,t)\leq f_0/\rho\,.$

Proof. Consider, on D_{T_0}, the function w given by

$$w=v_y\,. \tag{25}$$

It satisfies the equation $\rho w_t-\eta w_{yy}=0$ with the conditions (see assumption (b.4)):

$$w(s(t),t)=0\,,$$
$$w(y,0)=v_0'(y)\leq 0\,,\qquad s_0\leq y\leq L$$
$$w_y(L,t)=-f_0\,\eta^{-1}\,.$$

Since $w_y(L,t)<0$, Hopf's lemma implies $w(L,t)<0$ on $(0,T_0)$. Then the maximum principle implies (ii). Furthermore, inverting (25)

$$v(y,t)=-\int_y^L w(\xi,t)\,d\xi$$

we obtain (i).

(iii) is a trivial application of maximum principle to function z. □

Now we prove a monotone dependence result.

Proposition 3.4. *If* (v_1,s_1) *and* (v_2,s_2) *are solutions of* $(P_v^{\alpha_1})$ *and* $(P_v^{\alpha_2})$ *respectively, such that:*

(i) $s_{o_1} > s_{o_2}$,
(ii) $v'_{o1}(y) > v'_{o2}(y)$,
(iii) $\tau_0(\alpha_1) > \tau_0(\alpha_2)$,
then
(a) $s_1(t) > s_2(t)$,
(b) $v_{1y}(y,t) > v_{2y}(y,t)$ *on* $\{s_1(t) \le y \le L;\ 0 < t < T\}$,
(c) $v_1(y,t) \le v_2(y,t)$ *on* $\{s_1(t) \le y \le L;\ 0 < t < T\}$.

Proof. The proof of this proposition is an easy generalization of that of Proposition 1.2 of Ref. 4. □

This result allows us to compare the solution of (P_v^α) with the steady state solution

$$\begin{cases} \eta v_{yy} = -f_0\,, \\ v(L) = 0\,, \\ v_y(s) = 0\,, \end{cases}$$

where $s = \tau_0\,/\,L$ (with τ_0 constant). The stationary solutions corresponding to $\tau_0 = \tau_m$ and $\tau_0 = \tau_M$ are:

$$v_M(y) = -\frac{f_0}{2\eta}(y^2 - L^2) + \frac{f_0 s_M}{\eta}(y - L)\,,$$

$$v_m(y) = -\frac{f_0}{2\eta}(y^2 - L^2) + \frac{f_0 s_m}{\eta}(y - L)\,,$$

with

$$s_M = \frac{\tau_M}{f_0}\,,$$

$$s_m = \frac{\tau_m}{f_0}\,.$$

Proposition 3.5. *If the assumptions* (b.1), (b.4) *and* (b.5) *are correct then*

(i) $s_m < s(t) < s_M$
(ii) $v'_m(y) \le v_y(y,t) \le v'_M(y)$
(iii) $v_M(y) \le v(y,t) \le v_m(y)$. □

Imposing $s_M < L$, i.e.

$$f_0 > \frac{\tau_M}{L} \tag{26}$$

we deduce $s(t) < s_M < L$. Therefore (26) guarantees a positive velocity field. Moreover Proposition 3.5 allows us to estimate the power density dissipated by viscous forces:

$$\frac{f_0}{\eta L}(L - s_M)^2 \left[\frac{f_0}{3}(L - s_M) + \frac{\tau_m}{2}\right] \le \frac{W}{L} \le \frac{5}{6}\,\frac{f_0^2\,L^2}{\eta}\,. \tag{27}$$

3.2.3. *The problem* (P_z^β)

Consider a function $\beta \in \Sigma_1$. The problem (P_z^β) is obtained from (P_z^α) substituting in it the function α with β.

Problem (P_z^β) was studied in Ref. 6, where existence and uniqueness of a solution have been proved for a time $T_0 > 0$. In Ref. 6 it is required that the functions $(\eta x)^{-1}\tau_0(\beta)$ and $\rho^{-1}(f_0 - \tau_0(\beta)\,x^{-1})$ must be bounded. In view of (i) of Proposition 3.5 this request is satisfied. Moreover, the time T_0 does not depend on the particular function β, but depends only on the "physical" parameters: $\eta, \rho, L, f_0, \tau_m, \tau_M, k_1, k_2$ and v_0.

Using Lemma 1 of Ref. 6 it can be proved that there exists a constant Λ_0 (depending only on the "physical" parameters) that controls the "growth" of s uniformly in $(0, T_0)$.

Theorem 3.1. *Under the assumption* (b.4) *there exist a time* T_0 *and a constant* $\Lambda_0 > 0$ *such that, for each* $\beta \in \Sigma_1$, *the problem* (P_z^β) *has a unique solution on* D_{T_0}. *Moreover*

$$|\,\dot{s}(t)\,| \le \Lambda_0 \tag{28}$$

for each $t \in (0, T_0)$. □

Thus Theorem 3.1 specifies both T_0 and the constants characterizing the sets Σ, Σ_1.

3.2.4. *Local existence and uniqueness of solution of problem* (P_z^α)

Take $\alpha \in \Sigma$ and a sequence $\{\beta_n\} \subset \Sigma_1$ converging uniformly to α (Proposition 3.1). Applying Theorem 3.1 to each problem $(P_z^{\beta_n})$ we get two sequences $\{s_n(t)\}$ and $\{z_n(y,t)\}$ given by the solutions of $(P_z^{\beta_n})$. To show that (P_z^α) has a unique solution we operate in the following way:

(i) we prove that $\{s_n\}$ converges uniformly to a Lipschitz continuous function $s(t)$;
(ii) we show that the Dirichlet problem: (12)–(15) has a unique solution $z(y,t)$ on D_{T_0};
(iii) we prove the uniform convergence of $\{z_n\}$ to z, then we show that $s(t)$ and $z(y,t)$ satisfy (23), finally we conclude by applying Proposition 3.2.

Proposition 3.6. *The sequence* $\{s_n\}$ *converges.*

Proof. Consider two different functions belonging to Σ_1: $\beta_n(t)$ and $\beta_k(t)$. The solution of $(P_z^{\beta_n})$ and $(P_z^{\beta_k})$ in D_{T_0} are $(z_n(y,t), s_n(t))$ and $(z_k(y,t), s_k(t))$. Introducing the function

$$u = \frac{1}{\eta}(\rho z - f_0) \tag{29}$$

it is easy to show that, for this function, the integral relationship (23) becomes

$$\int_0^t \tau_0(\beta)\, s^{-1} d\theta = \rho \int_{s(t)}^L (L-y) u(y,t)\, dy + f_0\, t - \rho \int_{s_0}^L (L-y) v_0''(y)\, dy\,. \tag{30}$$

If we calculate (30) in correspondence of the two solutions (z_n, s_n) and (z_k, s_k) and consider their difference we get

$$\int_0^t \left(\frac{\tau_0(\beta_k)}{s_k(\theta)} - \frac{\tau_0(\beta_n)}{s_n(\theta)} \right) d\theta = \rho \int_{s_k(t)}^L (L-y)\, u_k(y,t)\, dy - \rho \int_{s_n(t)}^L (L-y)\, u_n(y,t)\, dy\,. \tag{31}$$

Now, without loss of generality, we can assume that, at the time t, $s_n(t) \le s_k(t)$. Under this assumption (31) becomes

$$-\int_{s_n(t)}^{s_k(t)} (L-y) u_n(y,t)\, dy = -\rho^{-1} \int_0^t \left[\frac{\tau_0(\beta_n)}{s_n(\theta)} - \frac{\tau_0(\beta_k)}{s_k(\theta)} \right] d\theta + \int_{s_k(t)}^L (L-y)[u_n(y,t) - u_k(y,t)]\, dy\,,$$

which we write as

$$|\mathcal{A}| = |\mathcal{B} + \mathcal{C}| \le |\mathcal{B}| + |\mathcal{C}|\,. \tag{32}$$

Let us pass to estimating each term of (32).

Recalling that $u_n < 0$, we have

$$|\mathcal{A}| = \left| \int_{s_n(t)}^{s_k(t)} (L-y) u_n(y,t)\, dy \right| \ge |s_n(t) - s_k(t)|\,(L - s_M) \left| \max_{(y,t)\in D_{T_0}} u_n(y,t) \right|.$$

From the assumption (b.4), using the maximum principle, we obtain

$$\left| \max_{(y,t)\in D_{T_0}} u_n(y,t) \right| \ge |\chi| > 0\,,$$

where $\chi < 0$ is given by $\chi = \max\left\{ -\frac{f_0}{\rho}; \max_{y\in[s_0,L]} v_0''(y); -\eta^{-1}\frac{\tau_m}{s_M} \right\}$. Thus we have

$$0 < |\chi|\,|s_n(t) - s_k(t)|\,(L - s_M) \le |\mathcal{A}|\,. \tag{33}$$

From the assumption (a.2) we get to the following inequality:

$$|\mathcal{B}| \le \frac{N s_M}{\rho s_m^2} T_0\, \| \beta_n - \beta_k \|_{T_0} + \frac{\tau_M}{\rho s_m^2} \int_0^t \| s_n - s_k \|_\theta\, d\theta\,, \tag{34}$$

where $\| f \|_t = \max_{\theta\in[0,t]} |f(\theta)|$ for any continuous function in $[0, T_0]$.

In order to estimate $|\mathcal{C}|$, let us define the curve γ and the functions U, ϕ by setting

$$\gamma(t) = \max\{s_n(t); s_k(t)\}\,, \qquad t > 0$$

$$U(y,t) = u_n(y,t) - u_k(y,t)\,, \qquad \{t > 0; \gamma(t) < y < L\}$$

$$\phi(t) = U(\gamma(t), t)\,, \qquad t > 0\,.$$

To estimate $|\phi(t)|$ we can assume that, at the time t, $\gamma(t) = s_k(t)$. Consequently we have

$$\begin{aligned}
|\phi(t)| &= |u_n(s_k(t),t) - u_k(s_k(t),t)| \\
&\le |u_n(s_k(t),t) - u_n(s_n(t),t)| + |u_n(s_n(t),t) - u_k(s_k(t),t)| \\
&\le \frac{s_M N}{\eta\, s_m^2} |\beta_n(t) - \beta_k(t)| + \frac{\tau_M}{\eta\, s_m^2} |s_n(t) - s_k(t)| \\
&\quad + \left| \frac{\partial u_n}{\partial y}(\xi,t) \right| |s_n(t) - s_k(t)|,
\end{aligned}$$

where $s_n(t) < \xi < s_k(t)$.

Since the term $\partial u_n/\partial y$ satisfies the heat equation on $\{0 < t < T_0; s_n(t) < y < L\}$, it can be estimated by maximum principle (see also appendix A). Thus we get:

$$|\phi(t)| \le C_1 |\beta_n(t) - \beta_k(t)| + C_2 |s_n(t) - s_k(t)|, \tag{35}$$

where C_1 and C_2 do not depend on the functions β's. We remark that we would have obtained the same inequality if we had assumed $\gamma(t) = s_n(t)$. This means that (35) is valid for each $t \in [0, T_0]$.

Consider now the function $\hat{U}$ solution of the problem $(\mathrm{P}_{\hat{U}})$

$$(P_{\hat{U}}) \begin{cases} \rho \hat{U}_t - \eta \hat{U}_{yy} = 0, & \{t > 0; y > \gamma(t)\} \\ \hat{U}(\gamma(t),t) = \|\phi\|_t, & t > 0 \\ \hat{U}(y,0) = 0, & y \ge s_0. \end{cases}$$

It is trivial to check that

$$|U(y,t)| \le \hat{U}(y,t). \tag{36}$$

The problem $(\mathrm{P}_{\hat{U}})$ has a unique solution (Chap. 14 of Ref. 2) given by

$$\hat{U}(y,t) = \int_0^t \mu(\theta)\, \Gamma_y(y,t,\gamma(\theta),\theta)\, d\theta, \tag{37}$$

where Γ is the fundamental solution and μ solves the integral equation

$$\|\phi\|_t = \frac{\mu(t)}{2a^2} + \int_0^t \mu(\theta)\, \Gamma_y(y,t,\gamma(\theta),\theta)\, d\theta$$

with $a^2 = \eta/\rho$. Taking into account of (28) and applying Lemma 7 of Ref. 3 we have

$$|\mu(t)| \le H(T_0) \|\phi\|_t, \tag{38}$$

where

$$H(T_0) = 2a^2 \left[1 + \frac{\Lambda_0}{a\sqrt{\pi}}\right] \exp\left\{\pi \frac{\Lambda_0}{a\sqrt{\pi}} T_0\right\}.$$

Consequently, from (36)–(38) we can deduce the following estimate for the integral $|\mathcal{C}|$

$$|\mathcal{C}| \leq \frac{L\,H(T_0)}{a\sqrt{\pi}} \int_0^t \frac{\|\phi\|_\theta}{\sqrt{t-\theta}}\,d\theta\,. \tag{39}$$

Summing up (33) and (34) we obtain the inequality

$$|\chi|\,|s_n - s_k| \leq |\mathcal{C}| + Q_1 \|\beta_n - \beta_k\|_{T_0} + Q_2 \int_0^t \|s_n - s_k\|_\theta\,d\theta\,. \tag{40}$$

At this point, using (39) and applying a generalization of Lemma 7 of Ref. 6 (see also appendix B), from (40) we have:

$$\|s_n - s_k\|_{T_0} \leq \overline{C}\,\|\beta_n - \beta_k\|_{T_0}\,,$$

where $\overline{C}$ is a constant which does not depend on the particular functions α and β. □

Thus we conclude that, since $s(t)$ exists (and is Lipschitz), the domain D_{T_0} is well defined.

Proposition 3.7. *The Dirichlet problem* (12)–(15), *corresponding to the boundary* $y = s(t)$ *just found, has a unique solution on* D_{T_0}. *Moreover* $z_y(s(t), t) \in C\,([0, T_0])$.

Proof. Existence and uniqueness are classical results. Theorem 14.4.1 of Ref. 2 ensures the continuity of z_y up to the boundary $s(t)$. □

It should be noted that we have not yet demonstrated that the pair $(z(y,t), s(t))$ is the solution of (P_z^α). The proof of this fact will be given in Proposition 3.9.

Proposition 3.8. *The sequence* $\{z_n\}$ *converges uniformly to* $z(y,t)$ *on* D_{T_0}.

Proof. Define the domain

$$D_{T_0}(\sigma) = \{0 < t < T_0; \sigma(t) < y < L\}\,,$$

where σ is the curve

$$\sigma(t) = \min\{s_n(t); s(t)\}\,.$$

Obviously $D_{T_0}(\sigma) \supseteq D_{T_0}$. We also define $z_n(y,t)$ and $z(y,t)$ on the whole rectangle $R = [0, L] \times [0, T_0]$ by setting:

$$\begin{cases} z_n(y,t) = f_n(s_n(t), t)\,, & 0 \leq y \leq s_n(t) \\ z(y,t) = f(s(t), t)\,, & 0 \leq y \leq s(t) \end{cases}$$

where $f(s,t) = \rho^{-1}(f_0 - \tau_0 \,/\, s)$.

Since z_n and z are continuous functions there exists $(\overline{y}, \overline{t}) \in \overline{D}_{T_0}$ such that

$$\|z_n - z\| = |z_n(\overline{y}, \overline{t}) - z(\overline{y}, \overline{t})|\,.$$

In Appendix C it has been pointed out that there exists a constant L_0 such that $|z_{ny}(y,t)| < L_0$ and $|z_y(y,t)| < L_0$ in their respective definition domains.

Assuming, for instance, $s(\bar{t}) < s_n(\bar{t})$ (i.e. $\sigma(\bar{t}) = s(\bar{t})$), there exist three different possibilities.

(i) $s(\bar{t}) < \bar{y} < s_n(\bar{t})$

Since $z_n(\bar{y},\bar{t}) = f_n(s_n(\bar{t}),\bar{t})$ we have

$$\begin{aligned} |z_n(\bar{y},\bar{t}) - z(\bar{y},\bar{t})| &\leq |z(\bar{y},\bar{t}) - f(s(\bar{t}),\bar{t})| + |f(s(\bar{t}),\bar{t}) - f_n(s_n(\bar{t}),\bar{t})| \\ &\leq L_0\,|s_n(\bar{t}) - s(\bar{t})| + A_1\,|s_n(\bar{t}) - s(\bar{t})| \\ &\quad + A_2\,|\alpha(\bar{t}) - \beta_n(\bar{t})|, \end{aligned}$$

where A_1, A_2 are constants.

(ii) $s_n(\bar{t}) \leq \bar{y} \leq L$

Introduce $\delta(t) = \max\{s_n(t), s(t)\}$ and the function $\hat{z}(y,t) = z(y,t) - z_n(y,t)$, defined in the domain $D_{T_0}(\delta) = \{0 < t < T_0, \delta(t) < y < L\}$. It is easy to verify that $\hat{z}$ satisfies the Dirichlet problem

$$\begin{cases} \rho\hat{z}_t - \eta\hat{z}_{yy} = 0\,, & (y,t) \in D_{T_0}(\delta) \\ \hat{z}(\delta(t),t) = \Phi(t)\,, & 0 < t < T_0 \\ \hat{z}(y,t) = 0\,, & s_0 \leq y \leq L \\ \hat{z}(L,t) = 0\,, & 0 < t < T_0 \end{cases}$$

where Φ is bounded by the following estimate:

$$\begin{aligned} |\Phi(t)| &= |z(\delta(t),t) - z_n(\delta(t),t)| \\ &\leq L_0\,\|s - s_n\|_{T_0} + B_1\,\|s - s_n\|_{T_0} + B_2\,\|\beta_n - \alpha\|_{T_0} \end{aligned}$$

with B_1 and B_2 constants. Thus, from the maximum principle, we obtain

$$|\hat{z}(\bar{y},\bar{t})| \leq \|\Phi\|_{T_0}\,.$$

(iii) $\bar{y} = \sigma(\bar{t})$

Since we have assumed $\sigma(\bar{t}) = s(\bar{t})$, there exist two constants C_1 and C_2 such that

$$\|z_n - z\| \leq C_1\,\|s - s_n\|_{T_0} + C_2\,\|\beta_n - \alpha\|_{T_0}\,.$$

Finally we can conclude that there exist two constants, which can be called A_1 and A_2, such that

$$\|z_n - z\| \leq A_1\,\|s - s_n\|_{T_0} + A_2\,\|\beta_n - \alpha\|_{T_0}\,.$$

Consequently, because of the uniform convergence of $\{s_n\}$ and $\{\beta_n\}$ respectively to s and α, the sequence $\{z_n\}$ converges uniformly to z. □

Proposition 3.9. *The functions $(z(y,t), s(t))$ satisfy Eq. (23) in $(0, T_0)$.*

Proof. If we show that $Q(t)$, given by

$$Q(t) = \Bigg| \eta\rho^{-1} \int_0^t \left(\frac{T_0}{s} - f_0\right) d\theta - \rho \int_s^L (L-y)\, z\, dy + \rho \int_{s_0}^L (L-y)\, z_0(y)\, dy - \frac{1}{2} f_0 \left[(L-s_0)^2 - (L-s(t))^2\right] \Bigg|$$

is less than ε, for each $\varepsilon > 0$ and for each $t \in [0, T_0]$, then the thesis is proved. Since the functions $z_n(y,t)$ and $s_n(t)$ solve $(\mathrm{P}_z^{\beta_n})$, they satisfy (23). Subtracting from $Q(t)$ the Eq. (23) (written for z_n and s_n) we obtain

$$Q(t) \leq A_0 \| s - s_n \|_{T_0} + A_1 \| \beta_n - \alpha \|_{T_0} + A_3 \| z_n - z \| .$$

Consequently the thesis is proved. □

Now, in view of Propositions 3.2 and 3.9, we deduce the existence of at least one solution of problem (P_z^{α}).

Uniqueness of the solution

At this point the following question arises: if, chosen $\alpha \in \Sigma$, we consider another sequence $\{\beta_n'\}$ (different from $\{\beta_n\}$) which converges to α, will we obtain the same functions $s(t)$ and $z(y,t)$? In other words we must study the uniqueness of solution of (P_z^{α}).

Proposition 3.10. *If α_1 and α_2 belong to Σ; (z_1, s_1) and (z_2, s_2) are respectively the solutions of $(\mathrm{P}_z^{\alpha_1})$ and $(\mathrm{P}_z^{\alpha_2})$ in $(0, T_0)$ then there exists a constant C (which does not depend on α_1 and α_2) such that:*

$$\| s_1 - s_2 \|_{T_0} \leq C \| \alpha_1 - \alpha_2 \|_{T_0} .$$

Proof. The proof is identical to that of Proposition 3.6. □

Operating as in Proposition 3.8 we can demonstrate the following proposition:

Proposition 3.11. *Given α_1 and α_2 in the set Σ, if (z_1, s_1) and (z_2, s_2) are the solution of $(\mathrm{P}_z^{\alpha_1})$ and $(\mathrm{P}_z^{\alpha_2})$ then*

$$\| z_1 - z_2 \| \leq C_1 \| s_1 - s_2 \|_{T_0} + C_2 \| \alpha_1 - \alpha_2 \|_{T_0}$$

where C_1 and C_2 are opportune constants. □

Thus we may conclude the section stating the uniqueness theorem for (P_z^{α}).

Theorem 3.2. *For each $\alpha \in \Sigma$, under the assumptions listed at the beginning of Sec. 3, problem (P_z^{α}) has a unique solution on D_{T_0} for each $\alpha \in \Sigma$.* □

3.3. *The operator $\mathcal{T}$*

We have shown that problem (P_z^α) has a unique solution in $(0, T_0)$. So for each $\alpha \in \Sigma$ the power $W(t)$ dissipated by the viscous forces is well defined. In this section, to emphasize that $W(t)$ depends on α, we denote it with W_α.

Given $\alpha \in \Sigma$, consider the linear Cauchy problem

$$\frac{d\widetilde{\alpha}}{dt} = k_1(1-\widetilde{\alpha}) - k_2\widetilde{\alpha}\,\frac{W_\alpha(t)}{L}\,, \tag{41}$$

$$\widetilde{\alpha}(0) = \alpha_0\,. \tag{42}$$

It is easy to show that (41), (42) has a unique solution and that it belongs to Σ. In fact, from results of Sec. 3.3.2 (in particular from (27)) we have:

$$\sup_{t\in(0,T_0)} \left|\frac{d\widetilde{\alpha}}{dt}\right| \le k_1 + \frac{5}{6}k_2\,\frac{f_0^2\,L}{\eta}\,.$$

This explains why the constant A, characterizing the set Σ, has the value given by (22).

Problem (41), (42) can be solved in $(0, T_0)$ and the solution is

$$\widetilde{\alpha} = \left[\alpha_0 + k_1\int_0^t \exp\left\{k_1 + \frac{1}{L}k_2\int_0^{t'} W_\alpha(t'')\,dt''\right\}dt'\right]$$
$$\times \exp\left\{-k_1 t - \frac{1}{L}k_2\int_0^t W_\alpha(t')\,dt'\right\}. \tag{43}$$

Thus the operator $\mathcal{T}$ from Σ to Σ is defined by setting

$$\mathcal{T}\alpha = \widetilde{\alpha}\,.$$

Proposition 3.12. *The operator $\mathcal{T}$ is continuous in Σ.*

Proof. Consider the function

$$\phi_\alpha(t) = \frac{k_2}{L}\int_0^t W_\alpha(t')\,dt'\,.$$

From (43) we have

$$\|\,\mathcal{T}\alpha_1 - \mathcal{T}\alpha_2\,\|_{T_0} \le b\,\|\,e^{-\phi_{\alpha_1}} - e^{-\phi_{\alpha_2}}\,\|_{T_0} \le b\,\|\,\phi_{\alpha_1} - \phi_{\alpha_2}\,\|_{T_0}\,,$$

where $b = \alpha_0 + 2(1 + e^{k_1 T_0})\,e^{2JT_0}$ with $J = L^{-1}\eta^{-1}\,f_0^2\,L^2$. Recalling now (11) and the results about (P_z^α) we have

$$\|\,\mathcal{T}\alpha_1 - \mathcal{T}\alpha_2\,\|_{T_0} \le ET_0\,\|\,\alpha_1 - \alpha_2\,\|_{T_0}\,, \tag{44}$$

where E is a constant depending on T_0 (bounded for T_0 bounded) and on the other parameters characterizing the initial and boundary data. Equation (44) proves the continuity of $\mathcal{T}$ on Σ. □

From Schauder's theorem and from the compactness of Σ we deduce that $\mathcal{T}$ has at least one fixed point in Σ.

3.4. *Existence and uniqueness of solution of* (P_z)

Consider the function $\alpha \in \Sigma$ which is the fixed point of $\mathcal{T}$. In correspondence of this function problem (P_z^α) has a unique solution. Therefore we have found a time $T_0 > 0$ and three functions $(z(y,t); s(t); \alpha(t))$ which have the required regularity and satisfy all the equations of (P_z). Then we can conclude that (P_z) has at least one solution in $(0, T_0)$. To show uniqueness we prove that $\mathcal{T}$ is a contraction (see Chap. 16 of Ref. 8) in a sufficiently small time interval.

Proposition 3.13. *There exists a time* $\overline{T}_o$, $0 < \overline{T}_o \le T_0$ *such that the operator* $\mathcal{T}$ *is a contraction in* $(0, \overline{T}_o)$.

Proof. From (44) we have (for $0 < t \le T_0$)

$$\| \mathcal{T}\alpha_1 - \mathcal{T}\alpha_2 \|_t \le t E_M \| \alpha_1 - \alpha_2 \|_t ,$$

where $E_M = \max_{t\in[0,T_0]} E(t)$. If $T_0 < 1 / E_M$ then $\mathcal{T}$ is a contraction in $[0, \overline{T}_o]$. □

The previous procedure can be used to show that, starting from any time $T_1 \in (\overline{T}_0, T_0)$, $\mathcal{T}$ is a contraction in a suitable time interval (T_1, T_2), so the uniqueness is proved in the whole of $(0, T_0)$.

Theorem 3.3. *The problem* (P_z) *has a unique solution in* $(0, T_0)$. □

Acknowledgments

The author would like to express his sincere gratitude to A. Fasano (Universitá degli studi di Firenze). It would not be possible to complete this work without his useful suggestions and assistance. He would also like to thank L. Preziosi (Politecnico di Torino) for his continued advice and encouragement. He is grateful to S. Meli (Eniricerche, Milano) and E. Carniani (Snamprogetti, Fano) for providing the relevant literature on the experimental background.

This work was partially supported by the Italian CNR Strategic Projects and by EEC-HCM contract CHRX-CT940618.

Appendices

A. Estimate for $u_y(L, t)$

Consider $\beta \in \Sigma_1$ and u given by (29). The function u_y satisfies the following problem:

$$\begin{cases} \rho \dfrac{\partial}{\partial t} u_y - \eta \dfrac{\partial^2}{\partial y^2} u_y = 0\,, & (y,t) \in D_{T_0} \\ u_y(s(t),t) = \dfrac{\rho}{\eta^2} \dfrac{\tau_0(\beta)}{s(t)} \dot{s}\,, & 0 < t < T_0 \\ u_y(y,0) = v_0^{'''}(y)\,, & s_0 \le y \le L \\ u_y(L,t) = \psi(t)\,, & 0 < t < T_0\,. \end{cases}$$

To estimate $|\psi|$ we can use the representation formula for the function u

$$u(y,t) = \frac{f_0}{\rho} \int_0^t G_\xi(y,t,L,\theta)\, d\theta + \int_{s_0}^L u(\xi,0) G(y,t,\xi,0)\, d\xi$$

$$- \frac{1}{\rho} \int_0^t \frac{\tau_0}{s(\theta)} G_\xi(y,t,s(\theta),\theta)\, d\theta\,, \tag{A.1}$$

where

$$G(y,t,\xi,\theta) = \frac{1}{2}[\Gamma(y,t,\xi - L,\theta) - \Gamma(-y,t,\xi - L,\theta)]$$

with

$$\Gamma(y,t,\xi,\theta) = \frac{1}{\sqrt{4\pi a^2 (t-\theta)}} \exp\left\{ - \frac{(y-\xi)^2}{4a^2(t-\theta)} \right\}$$

$a^2 = \eta/\rho$.

Differentiating (A.1) with respect to y, since u_y is continuous up to $y = L$, we obtain

$$u_y(L,t) = \frac{f_0}{\rho} \int_0^t G_{\xi\,y}(L,t,L,\theta)\, d\theta + \int_{s_0}^L u(\xi,0) G_y(L,t,\xi,0)\, d\xi$$

$$-\frac{1}{\rho} \int_0^t \frac{\tau_0}{s(\theta)} G_{\xi\,y}(L,t,s(\theta),\theta)\, d\theta$$

from which it is possible to deduce an estimate for $u_y(L,t)$.

B. Generalization of Gronwall's Inequality

Proposition B.1. *If $f : [0,T] \to \mathbb{R}$ is a function absolutely continuous and*

$$f(\tau) \le k + c \int_0^t f(\eta) \left(1 + \frac{1}{\sqrt{\tau - \eta}}\right) d\eta\,, \tag{B.1}$$

where k and c are constants, then there exist two constants h_1 and h_2 (depending on T) such that:

$$f(t) \le h_1 \exp\{h_2 T\}\,.$$

Proof. Multiplying both sides of (B.1) by $\left(\sqrt{\theta - \tau}\right)^{-1}$ and integrating in τ between 0 and t we obtain from the R.H.S. of (B.1) the following integral:

$$\iint_D \psi(\eta,\tau)\, d\eta d\tau\,,$$

where $D = \{0 < \tau < t; 0 < \eta < \tau\}$. Consequently we can write

$$\iint_D \psi(\eta,\tau)\, d\eta d\tau = \int_0^t \left(\int_\eta^t \psi(\eta,\tau)\, d\tau \right) d\eta$$

obtaining in the present case

$$\int_0^t \frac{f(\tau)}{\sqrt{t-\tau}}\, d\tau \le 2k\sqrt{t} + 2c \int_0^t f(\eta)\sqrt{t-\eta}\, d\eta + \pi c \int_0^t f(\eta)\, d\eta .$$

Thus the following inequality is reached:

$$\int_0^t \frac{f(\tau)}{\sqrt{t-\tau}}\, d\tau \le 2k\sqrt{T} + \left(2c\sqrt{T} + \pi c\right) \int_0^t f(\eta)\, d\eta .$$

Replacing in (B.1) and using Gronwall's inequality the proposition is proved. □

C. Estimate for $z_y(y,t)$

We would like to give an *a priori* estimate of $z_y(y,t)$ in the domain

$$D_{T_0} = \{s(t) < y < L; 0 < t < T_0\}$$

knowing that z satisfies the problem

$$(\mathrm{P}) \begin{cases} \rho z_t - \eta z_{yy} = 0\,, & (y,t) \in D_{T_0} \\ z(L,t) = 0\,, & 0 < t < T_0 \\ z(y,0) = h(y)\,, & s_0 \le y \le L \\ z(s(t),t) = g(t)\,, & 0 < t < T_0 . \end{cases}$$

The functions s, h and g have these properties:

(i) $s(t) \in C([0,T_0])$, $s(t)$ is Lipschitz $|\, s(t_1) - s(t_2)\,| \le \Lambda_0\, |\, t_1 - t_2\,|$
(ii) $s_m < s(t) < s_M$ $\forall t \in [0,T_0]$ with $0 < s_m < s_M < L$
(iii) $h(y) \in C^1([s_0,L])$
(iv) $g(t) \in C([0,T_0])$ and is Lipschitz $|\, g(t_1) - g(t_2)\,| \le \Lambda_1\, |\, t_1 - t_2\,|$.

Under these hypotheses (P) has unique solution z on D_{T_0}, moreover z_y is continuous up to $y = s(t)$ (see Theorems 14.3.1 and 14.4.1 of Ref. 2).

Proposition C.1. *If hypotheses* (i), (ii), (iii) *and* (iv) *are valid and the functions s and g are C^1 then z_y is uniformly bounded in D_{T_0} by a constant A*

$$|\, z_y(y,t)\,| \le A \quad \forall (y,t) \in D_{T_0} .$$

Proof. The proof is an easy generalization of Lemma 3.1 and Corollary 3.1 of Ref. 7. □

The previous result can be generalized assuming only the hypotheses (i)–(iv).

Proposition C.2. *If* (i)–(iv) *are true then there exists a constant A such that*

$$|z_y(y,t)| \leq A \quad \forall (y,t) \in D_{T_0}\,.$$

Proof. The proof is also a generalization of the Theorem 3.1 of Ref. 7. □

References

1. B. M. Budak and S. V. Fomin, **Multiple Integrals, Field Theory and Series** (MIR Publisher, 1973).
2. J. R. Cannon, *The one-dimensional heat equation*, **Encyclopedia of Mathematics and its Applications** (Addison-Wesley, 1984), Vol. 23.
3. J. R. Cannon and C. D. Hill, *Existence, uniqueness, stability and monotone dependence in a Stefan problem with boundary temperature specification*, *J. Math. Mech.* **17** (1968) 1–19.
4. E. Comparini, *A one-dimensional Bingham flow*, *J. Math. Anal. Appl.* **169** (1992) 127–139.
5. J. Denis and J. P. Durand, *Modification of wax crystallization in petroleum products*, *Rev. l'Inst. Française Pétrole* **46** (1991) 637–649.
6. A. Fasano and M. Primicerio, *General free-boundary problems for the heat equations*, *J. Math. Anal. Appl.* **57** (1977) 694–732.
7. A. Fasano and M. Primicerio, *La diffusione del calore in uno strato di spessore variabile in presenza di scambi termici non lineari con l'ambiente. (I) Deduzione di limitazioni a priori sulla temperatura e le sue derivate*, *Rend. Sem. Mat. Univ. Padova* **50** (1973) 269–330.
8. L. V. Kantorovich and G. P. Akilov, **Functional Analysis in Normed Spaces** (Pergamon Press, 1964).
9. S. Lojasiewicz, **An Introduction to the Theory of Real Functions** (Wiley, 1988).
10. L. Lusternik and V. Sobolev, **Précis d'analyse Fonctionelle** (Editions MIR, 1989).
11. M. Reed and B. Simon, **Functional Analysis**, *Methods of Modern Mathematical Physics*, Vol. 1 (Academic Press, 1980).
12. L. Rubinstein, **The Stefan Problem**, American Mathematical Society Translation (AMS, 1971), Vol. 27.
13. W. Rudin, **Real and Complex Analysis** (McGraw-Hill, 1987), 3rd edition.
14. R. N. Tuttle, *High-pour point and Asphaltic crude oils and condensates*, *J. Petroleum Tech.* (1983) 1192–1196.
15. L. T. Wardhaugh and D. V. Boger, *Measurement of the unique flow properties of waxy crude oils*, *Chem. Engrg. Res. Des.* **65** (1987) 74–83.
16. L. T. Wardhaugh and D. V. Boger, *Flow characteristics of waxy crude oils: application to pipeline design*, *AIChE J.* **37** (1991) 871–885.