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Real logarithms of semi-simple matrices

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Abstract
We study the differential and topological structures of the set of real logarithms of any semi-simple non-singular matrix and of the set of real skew-symmetric logarithms of any special orthogonal matrix.

Keywords Logarithm matrix · Semi-simple matrix · Orthogonal matrix · Skew-symmetric matrix · Unitary matrix · Homogeneous space · Homotopy sequence of a bundle · Generalized principal real logarithm

Mathematics Subject Classification 15A24 · 53C30 · 15B10

Contents

Introduction

The aim of this work is to study the differential and topological properties of the set of real logarithms of any semi-simple matrix and of the set of real skew-symmetric logarithms of any special orthogonal matrix. As far as we know, such studies have never been done before. More generally, there are not many papers studying the real logarithms of a matrix from a theoretical point of view. The best known is an old paper by Culver ([2]), in which, among other things, the author proves that a non-singular square matrix M has a real logarithm if

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and only if each of its Jordan blocks corresponding to a negative eigenvalue occurs an even number of times. Furthermore, Culver provides the necessary and sufficient conditions, in terms of Jordan blocks of M, for the real logarithm of M to be unique and the set of real logarithms of M to be countable. A simpler exposition of some of these results can be found in [13].

In the first preliminary Section we fix the notations and, in particular, we define the main homogeneous spaces involved in the structure theorems of the sets of real logarithms and skew-symmetric logarithms.

In Sect. 2 we study the set $\mathcal{L}og(M)$ of the real logarithms of a given semi-simple matrix. The main result of this Section is Theorem 2.5, which states that $\mathcal{L}og(M)$ is a countable disjoint union of differentiable submanifolds of $M(n, \mathbb{R})$; these are all diffeomorphic to suitable homogeneous spaces, which depend on the eigenvalues of the matrices constituting each of these submanifolds. In Sect. 2, we also define and study the set of *generalized principal real logarithms* of a given semi-simple matrix M (see Theorem 2.8). In general, we say that a logarithm X of a matrix M is *generalized principal*, if every eigenvalue of X has imaginary part in $[-\pi, \pi]$. Our definition of generalized principal logarithm is more general than the usual definition of principal logarithm (see [8, Thm. 1.31 p. 20]). Indeed, generalized principal logarithms are also defined for matrices with negative eigenvalues, even if, in this case, they are not unique.

In Sect. 3, we prove analogous Theorems for the set $\mathcal{L}og_{\mathfrak{so}(n)}(Q)$ of *skew-symmetric logarithms* (Theorem 3.4) and for the set of *generalized principal skew-symmetric logarithms* of any special orthogonal matrix Q (Theorem 3.7). For a study of the set of generalized principal skew-symmetric logarithms of a given special orthogonal matrix, we also refer to [4, Sect. 3].

In Sect. 4 we determine, in the simplest cases, some of the homotopy groups of homogeneous spaces involved in the Theorems of Sects. 2 and 3.

Finally, in Sect. 5 we prove that all components of the sets $\mathcal{L}og(M)$ and $\mathcal{L}og_{\mathfrak{so}(n)}(Q)$ are simply connected and that their second homotopy group is a free abelian group, whose rank depends on the eigenvalues of the matrices constituting these components (see Theorems 5.1 and 5.2).

We point out that the techniques used in this paper are similar to those used in [5] to study the set of real square roots of suitable non-singular matrices.

1 Preliminary facts

Notations 1.1 *a) In this paper, for any integer $n \geq 1$, we denote*

- $M(n, \mathbb{R})$*: the $\mathbb{R}$-vector space of real square matrices of order n;*
- $GL(n, \mathbb{R})$ *(and $GL^+(n, \mathbb{R})$): the multiplicative group of non-singular real matrices of order n (with positive determinant);*
- $\mathcal{O}(n)$ *(and $S\mathcal{O}(n)$): the multiplicative group of real orthogonal matrices of order n (with determinant 1);*
- $\mathfrak{so}(n)$*: the Lie algebra of real skew-symmetric matrices of order n;*
- $M(n, \mathbb{C})$*: the $\mathbb{C}$-vector space of complex square matrices of order n;*
- $GL(n, \mathbb{C})$*: the multiplicative group of non-singular complex matrices of order n;*
- $U(n)$*: the multiplicative group of complex unitary matrices of order n;*

We denote by I_n *the identity matrix of order n, by* O_n *the null matrix of order n and by* $\mathbf{i}$ *the imaginary unit. We write* $\bigsqcup_j X_j$ *to emphasize the union of mutually disjoint sets* X_j*;*

furthermore we denote by $|S|$ *the cardinality of any given finite set* S *and by* $\delta_{(i,j)}$ *the usual* Kronecker delta *defined by* $\delta_{(i,j)} = 1$ *if* $i = j$, *and* 0 *otherwise.*

b) For every $A \in M(n, \mathbb{C})$, $tr(A)$ *is its trace,* A^T *is its transpose,* $A^* := \overline{A}^T$ *is its transpose conjugate,* $\det(A)$ *is its determinant and, provided that* $\det(A) \neq 0$, A^{-1} *is its inverse; furthermore* $\exp(A) := \sum_{i=0}^{+\infty} \frac{A^i}{i!}$ *denotes the exponential of* A.

If $C \in GL(n, \mathbb{R})$, *we denote by* Ad_C *the map from* $M(n, \mathbb{R})$ *onto itself, defined by* $Ad_C : X \mapsto Ad_C(X) := CXC^{-1}$. *Note that the maps* Ad_C *and* $\exp$ *commute.*

For every $\theta \in \mathbb{R}$, *we denote* $E_\theta := \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$ *and* $E := E_{\pi/2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$; *hence* $E_\theta = \cos(\theta) I_2 + \sin(\theta) E$.

It is easy to check that $\exp(\delta E) = E_\delta$, *for every* $\delta \in \mathbb{R}$, *and from this we get*

$$(\triangle) \qquad \exp(\alpha I_2 + \delta E) = e^\alpha E_\delta, \text{ for every } \alpha, \delta \in \mathbb{R}.$$

If $B_1, \ldots, B_m$ *are square matrices (of various orders), then* $B_1 \oplus \ldots \oplus B_m$ *denotes the block square matrix with* $B_1, \ldots, B_m$ *on its diagonal. If* B *is a square matrix then* $B^{\oplus m}$ *denotes* $B \oplus \cdots \oplus B$ *(m times). It is easy to check that* $\exp(B_1 \oplus \ldots \oplus B_m) = \exp(B_1) \oplus \ldots \oplus \exp(B_m)$, *for every* $B_1, \ldots, B_m$.

If $\mathcal{S}_1, \ldots, \mathcal{S}_m$ *are sets of square matrices, then* $\mathcal{S}_1 \oplus \cdots \oplus \mathcal{S}_m$ *denotes the set of all matrices* $B_1 \oplus \cdots \oplus B_m$ *with* $B_j \in \mathcal{S}_j$ *for every* j.

To give a full generality to the results of this paper (and to their proofs), it is necessary to establish agreements on the notations that we will use: if s *is a non-negative integer parameter, when we write a term as* $\sum_{i=1}^{s} (\cdots)$, $\bigoplus_{i=1}^{s} (\cdots)$ *or* $\prod_{i=1}^{s} (\cdots)$, *we mean that, if* $s = 0$, *this sum, this direct sum or this product does not appear in the related formula. Similar considerations hold for* $\sum_{i \in I} (\cdots)$ *and* $\bigoplus_{i \in I} (\cdots)$, *whenever the set* I *is empty. We also assign a meaning to the matrices of order zero* I_0, O_0 *and to the groups of order zero* $SO(0)$, $\mathcal{O}(0)$, $GL(0, \mathbb{R})$, $GL^+(0, \mathbb{R})$, *defining them all equal to a single (phantom) point* $\mathcal{Q}$ *which, conventionally, satisfies the following conditions:* $\lambda \mathcal{Q} = \mathcal{Q}$, *for any* $\lambda \in \mathbb{C}$; $\mathcal{Q} \oplus B = B \oplus \mathcal{Q} = B$, *for any complex square matrix* B; $\mathcal{Q} \oplus \mathcal{S} = \mathcal{S} \oplus \mathcal{Q} = \mathcal{S}$, *for any set* $\mathcal{S}$ *of complex square matrices. The eigenvalues of zero multiplicity of a given matrix* $X \in M(n, \mathbb{C})$ *are all the complex numbers which are not eigenvalues of* X. *Finally, a free abelian group* G *of rank zero means that* G *is reduced to the identity element.*

For all other notations and information on matrices, not explicitly mentioned here, we refer to [9] and to [8].

Remarks-Definitions 1.2 a) *The mapping* $\rho_1 : \mathbb{C} \to M(2, \mathbb{R})$ *defined by* $\rho_1(z) = Re(z) I_2 + Im(z) E$, *is a monomorphism of* $\mathbb{R}$*-algebras such that* $\rho_1(\overline{z}) = \rho_1(z)^T$ *and* $\rho_1(z) \in GL(2, \mathbb{R})$ *as soon as* $z \neq 0$. *More generally, for any* $h \geq 1$, *we define the* decomplexification mapping $\rho_h : M(h, \mathbb{C}) \to M(2h, \mathbb{R})$, *which maps the* $h \times h$ *complex matrix* $Z = (z_{ij})$ *to the* $(2h) \times (2h)$ *block real matrix* $(\rho_1(z_{ij}))$, *having* h^2 *square blocks of order* 2. *We have* $tr(\rho_h(Z)) = 2Re(tr(Z)))$, $\det(\rho_h(Z)) = |\det(Z)|^2$ *and, moreover,* ρ_h *is a monomorphism of* $\mathbb{R}$*-algebras, whose restriction to* $GL(h, \mathbb{C})$ *is a monomorphism of Lie groups from* $GL(h, \mathbb{C})$ *into* $GL(2h, \mathbb{R})$. *Since* $\rho_h(Z^*) = \rho_h(Z)^T$, *the restriction of the monomorphism* ρ_h *to* $U(h)$ *maps* $U(h)$ *into* $SO(2h)$ *and* $\rho_h(U(h)) = \rho_h(GL(h, \mathbb{C})) \cap SO(2h) = \rho_h(GL(h, \mathbb{C})) \cap \mathcal{O}(2h)$. *To simplify the notations and in absence of ambiguity, from now on, we omit to write the symbol* ρ_h, *so, for instance, we can consider* $M(h, \mathbb{C})$ *as an* $\mathbb{R}$*-subalgebra of* $M(2h, \mathbb{R})$, $GL(h, \mathbb{C})$ *and* $U(h)$ *as closed subgroups of* $GL(2h, \mathbb{R})$ *and* $SO(2h)$, *respectively.*

b) *For every matrix* $B \in M(n, \mathbb{R})$ *we denote* $\mathcal{C}_B := \{X \in GL(n, \mathbb{R}) : BX = XB\}$. *It is easy to prove that* $\mathcal{C}_B$ *is a closed subgroup of* $GL(n, \mathbb{R})$; *hence* $\mathcal{C}_B$ *is an embedded Lie* *subgroup of $GL(n, \mathbb{R})$ (see, for instance, [17, Thm. 3.21]). We also denote by $\sigma(B)$ the set of distinct complex eigenvalues of B.*

Lemma 1.3 *Let $D = \bigoplus_{j=1}^{t} D_j \in M(n, \mathbb{R})$, where each D_j is a semi-simple real square matrix of order n_j (with $\sum_{j=1}^{t} n_j = n$) and assume $\sigma(D_j) \cap \sigma(D_h) = \emptyset$, whenever $j \neq h$. Then we have $\mathcal{C}_D = \bigoplus_{j=1}^{t} \mathcal{C}_{D_j}$.*

Proof Any matrix $A \in \mathcal{C}_D$ can be written in blocks as $A = (A_{ij})$, where A_{ij} is an $n_i \times n_j$ real matrix, for $i, j = 1, \ldots, t$. The condition $AD = DA$ implies $A_{ij} D_j = D_i A_{ij}$, for every $1 \leq i, j \leq t$. Fix $i, j \in \{1, \ldots, t\}$ with $i \neq j$. Since D_j is semi-simple, there is a basis of $\mathbb{C}^{n_j}$, called $\{v_1, \cdots, v_{n_j}\}$, consisting of eigenvectors of D_j; hence, if v belongs to this basis, there exists $\lambda \in \sigma(D_j)$ such that $D_j(v) = \lambda v$, and therefore we get $D_i A_{ij}(v) = A_{ij} D_j(v) = \lambda A_{ij}(v)$. From the assumptions, $\lambda \notin \sigma(D_i)$, and so we conclude that $A_{ij}(v) = 0$, for every $v \in \{v_1, \ldots, v_{n_j}\}$; hence A_{ij} is the null matrix. Therefore we have $A = \bigoplus_{j=1}^{t} A_{jj}$ where $A_{jj} \in GL(n_j, \mathbb{R})$ and $A_{jj} D_j = D_j A_{jj}$, for every $j = 1, \cdots, t$, so we obtain $\mathcal{C}_D \subseteq \bigoplus_{j=1}^{t} \mathcal{C}_{D_j}$. Since the reverse inclusion is trivial, the statement has been proved. □

Using similar elementary arguments, it is easy to prove the following:

Lemma 1.4 *If $0 < \theta < \pi$, consider the matrix $E_\theta^{\oplus m} \in GL(2m, \mathbb{R})$ $(m \geq 1)$. We have $\mathcal{C}_{E_\theta^{\oplus m}} = GL(m, \mathbb{C}) \subset GL(2m, \mathbb{R})$ (remember Remarks–Definitions 1.2 (a)).*

Remarks-Definitions 1.5 a) *It is known that every closed subgroup H of a Lie group G is also a Lie group and that the set of left cosets G/H has a unique structure of differentiable manifold (of dimension equal to $\dim_{\mathbb{R}} G - \dim_{\mathbb{R}} H$) such that the projection $G \to G/H$ is differentiable and the natural action of G on G/H is transitive; moreover, G is a principal fiber bundle over G/H with group H (see, for instance, [17, pp. 120 -124]).*

b) *In the next Sections we will have to deal with the following homogeneous spaces:*

$$\widehat{\Theta}_{(\nu_1,\ldots,\nu_s)} = \frac{GL(\nu, \mathbb{C})}{\left(\bigoplus_{j=1}^{s} GL(\nu_j, \mathbb{C})\right)}, \qquad \Theta_{(\nu_1,\ldots,\nu_s)} = \frac{U(\nu)}{\left(\bigoplus_{j=1}^{s} U(\nu_j)\right)},$$

$$\widehat{\Gamma}_{(\zeta;\nu_1,\ldots,\nu_s)} = \begin{cases} \dfrac{GL^+(\zeta + 2\nu, \mathbb{R})}{GL^+(\zeta, \mathbb{R}) \oplus \left(\bigoplus_{j=1}^{s} GL(\nu_j, \mathbb{C})\right)} & if \quad \zeta \geq 1 \\ \dfrac{GL^+(2\nu, \mathbb{R})}{\left(\bigoplus_{j=1}^{s} GL(\nu_j, \mathbb{C})\right)} & if \quad \zeta = 0 \end{cases},$$

$$\Gamma_{(\zeta;\nu_1,\ldots,\nu_s)} = \begin{cases} \dfrac{SO(\zeta + 2\nu)}{SO(\zeta) \oplus \left(\bigoplus_{j=1}^{s} U(\nu_j)\right)} & if \quad \zeta \geq 1 \\ \dfrac{SO(2\nu)}{\left(\bigoplus_{j=1}^{s} U(\nu_j)\right)} & if \quad \zeta = 0 \end{cases},$$

where $\zeta, \nu_1, \ldots, \nu_s$ are integers such that $\zeta \geq 0$, $\nu_1, \ldots, \nu_s \geq 1$ $(s \geq 1)$ and $\nu = \sum_{j=1}^{s} \nu_j$. It is useful to define the spaces $\widehat{\Gamma}_{(\zeta;\nu_1,\ldots,\nu_s)}$ and $\Gamma_{(\zeta;\nu_1,\ldots,\nu_s)}$ even when $s = 0$ (i.e. when the multi-index $(\zeta; \nu_1, \ldots, \nu_s)$ reduces to (ζ)) and $\zeta \geq 0$, setting them, in all these cases, equal to a single point. Consequently, note that $\Gamma_{(\zeta;\nu_1,\ldots,\nu_s)}$ reduces to a single point if and only if either $s = 0$, $\zeta \geq 0$ or $s = 1$, $\nu_1 = \nu = 1$, $\zeta = 0$, while the space $\widehat{\Gamma}_{(\zeta;\nu_1,\ldots,\nu_s)}$ is a point if and only if $s = 0$, $\zeta \geq 0$. Also note that both spaces $\Theta_{(\nu_1,\ldots,\nu_s)}$ and $\widehat{\Theta}_{(\nu_1,\ldots,\nu_s)}$ are single points if and only if $s = 1$ (for every $\nu = \nu_1 \geq 1$).

c) *All the spaces we have defined in (b) are connected differentiable manifolds; moreover $\Gamma_{(\zeta;\nu_1,\ldots,\nu_s)}$ and $\Theta_{(\nu_1,\ldots,\nu_s)}$ are also compact. Their dimensions are the following:* $\dim_{\mathbb{R}} \widehat{\Gamma}_{(\zeta;\nu_1,\ldots,\nu_s)} = 4\nu(\nu + \zeta) - 2\sum_{j=1}^{s} \nu_j^2$, $\quad \dim_{\mathbb{R}} \widehat{\Theta}_{(\nu_1,\ldots,\nu_s)} = 2\nu^2 - 2\sum_{j=1}^{s} \nu_j^2$,

$\dim_{\mathbb{R}} \Gamma_{(\zeta;\nu_1,\ldots,\nu_s)} = \nu(2\nu + 2\zeta - 1) - \sum_{j=1}^{s} \nu_j^2$, $\quad \dim_{\mathbb{R}} \Theta_{(\nu_1,\ldots,\nu_s)} = \nu^2 - \sum_{j=1}^{s} \nu_j^2$. *We also observe that* $\Gamma_{(0,\nu)}, \widehat{\Gamma}_{(0,\nu)}, \Theta_{(\nu_1,\nu_2)}, \widehat{\Theta}_{(\nu_1,\nu_2)}$ *are symmetric spaces, for every* $\nu, \nu_1, \nu_2 \geq 1$. *Moreover, it can be easily seen that* $\Gamma_{(0,2)}$ *is diffeomorphic to the 2-dimensional sphere* S^2.

d) *Note that, if* $\zeta \geq 1$, *the homogeneous spaces* $\dfrac{GL(\zeta + 2\nu, \mathbb{R})}{GL(\zeta, \mathbb{R}) \oplus \left(\bigoplus_{j=1}^{s} GL(\nu_j, \mathbb{C})\right)}$ *and* $\dfrac{\mathcal{O}(\zeta + 2\nu)}{\mathcal{O}(\zeta) \oplus \left(\bigoplus_{j=1}^{s} U(\nu_j)\right)}$ *are diffeomorphic to* $\widehat{\Gamma}_{(\zeta;\nu_1,\ldots,\nu_s)}$ *and* $\Gamma_{(\zeta;\nu_1,\ldots,\nu_s)}$, *respectively (and so they are connected), while, for* $\zeta = 0$, *the spaces* $\dfrac{GL(2\nu, \mathbb{R})}{\left(\bigoplus_{j=1}^{s} GL(\nu_j, \mathbb{C})\right)}$ *and* $\dfrac{\mathcal{O}(2\nu)}{\left(\bigoplus_{j=1}^{s} U(\nu_j)\right)}$ *have two connected components both diffeomorphic to* $\widehat{\Gamma}_{(0;\nu_1,\ldots,\nu_s)}$ *and* $\Gamma_{(0;\nu_1,\ldots,\nu_s)}$, *respectively; hence we can say that* $\dfrac{GL(\zeta + 2\nu, \mathbb{R})}{GL(\zeta, \mathbb{R}) \oplus \left(\bigoplus_{j=1}^{s} GL(\nu_j, \mathbb{C})\right)}$ *and* $\dfrac{\mathcal{O}(\zeta + 2\nu)}{\mathcal{O}(\zeta) \oplus \left(\bigoplus_{j=1}^{s} U(\nu_j)\right)}$ *have* $2^{\delta_{(\zeta,0)}}$ *connected components (if* $\zeta + s \geq 1$*).*

Definition 1.6 Let $M \in M(n, \mathbb{R})$. We say that $X \in M(n, \mathbb{R})$ is a *real logarithm of* M, if $\exp(X) = M$; we denote by $\mathcal{L}og(M) := \{X \in M(n, \mathbb{R}) : \exp(X) = M\}$ the set of real logarithms of M. Moreover, if M is special orthogonal, we denote by $\mathcal{L}og_{\mathfrak{so}(n)}(M) := \mathcal{L}og(M) \cap \mathfrak{so}(n)$, the set of *real skew-symmetric logarithms of* M.

Remark 1.7 It is known that $\exp(X) \in GL^+(n, \mathbb{R})$, for every $X \in M(n, \mathbb{R})$; hence no matrix with non-positive determinant has real logarithms. Moreover, also the following fact is known (see, for instance, [2, Thm. 1] or [8, Thm. 1.23]):
$M \in GL^+(n, \mathbb{R})$ has at least one real logarithm if and only if it has an even number of Jordan blocks of each size, for every negative eigenvalue.
In particular, if $M \in GL^+(n, \mathbb{R})$ is semi-simple, then it has at least one real logarithm if and only if each of its (possible) negative eigenvalues has even multiplicity.

Remark 1.8 Let G be a Lie group acting smoothly on a differentiable manifold X. The orbit of each $x \in X$ is an immersed submanifold of X, diffeomorphic to the homogeneous space $\dfrac{G}{G_x}$, where G_x is the isotropy subgroup of G at x.
Furthermore, if G is compact, all orbits are embedded submanifolds of X(see [14]).

2 Real logarithms of semi-simple non-singular matrices

Let M be any semi-simple real square matrix of order n; by Remark 1.7, if we want $\mathcal{L}og(M)$ to be non-empty, we must assume that the matrix M is non-singular, with all (possible) negative eigenvalues of even multiplicity.

Aim of this section is to study $\mathcal{L}og(M)$, for any M satisfying these assumptions.

Lemma 2.1 *Let* $A \in M(n, \mathbb{R})$. *Then* A *is semi-simple if and only if* $\exp(A)$ *is semi-simple.*

Proof One implication is trivial. For the other, assume that $\exp(A)$ is semi-simple. By the additive Jordan-Chevalley decomposition, there exist a semi-simple matrix S and a nilpotent matrix N of index $k \geq 1$, such that $A = S + N$ with $SN = NS$ (see, for instance, [10, Ch. VI] and also [3]). Since N and S commute, we have $\exp(A) = \exp(S)\exp(N)$, with $\exp(S)$ semi-simple and $\exp(N)$ unipotent; so, by the uniqueness of the multiplicative Jordan-Chevalley decomposition, we get $\exp(A) = \exp(S)$ and $\exp(N) = \sum_{i=0}^{k-1} \frac{N^i}{i!} = I_n$. Since the minimal polynomial of N has degree k, this last equation implies $k = 1$, i.e. $N = O_n$, so A is semi-simple. □

Remarks-Definitions 2.2 a) *Let M be a semi-simple non-singular real matrix, having every (possible) negative eigenvalue of even multiplicity, and denote its eigenvalues as follows:*
◇ *the distinct positive eigenvalues are: $\lambda_1 < \lambda_2 < \ldots < \lambda_p$, with (positive) multiplicities $h_1, h_2, \ldots, h_p$ respectively $(p \geq 0)$;*
◇ *the distinct non-real eigenvalues are: $\rho_{(1,1)}\exp(\pm\mathbf{i}\theta_1), \ldots, \rho_{(1,a_1)}\exp(\pm\mathbf{i}\theta_1)$, $\rho_{(2,1)}\exp(\pm\mathbf{i}\theta_2), \ldots, \rho_{(2,a_2)}\exp(\pm\mathbf{i}\theta_2)$ up to $\rho_{(r,1)}\exp(\pm\mathbf{i}\theta_r), \ldots, \rho_{(r,a_r)}\exp(\pm\mathbf{i}\theta_r)$, where $\rho_{(l,t)}\exp(\pm\mathbf{i}\theta_l)$ both have (positive) multiplicity $m_{(l,t)}$, for every l, t, and where $0 < \theta_1 < \theta_2 < \ldots < \theta_r < \pi$, $a_l \geq 1$, $0 < \rho_{(l,1)} < \rho_{(l,2)} < \ldots < \rho_{(l,a_l)}$, for every $l = 1, \ldots, r$ $(r \geq 0)$;*
◇ *the distinct negative eigenvalues are: $-w_1 > -w_2 > \ldots > -w_q$, with (even positive) multiplicities $2k_1, 2k_2, \ldots, 2k_q$ respectively $(q \geq 0)$. Note that $\sum_{i=1}^p h_i + 2\sum_{l=1}^r \sum_{t=1}^{a_l} m_{(l,t)} + 2\sum_{j=1}^q k_j = n$. We also denote by $2A = 2\sum_{l=1}^r a_l$ the number of distinct non-real eigenvalues of M. We point out that one or two of the indices p, r, q can be zero. For instance, the index p vanishes when the matrix M has no positive eigenvalues. In this case, the numbers λ_i, h_i are not defined and it is understood that the term $\sum_{i=1}^p h_i$ (or any term of the same type), does not appear in the previous (or in a similar) equality, according to Notations 1.1 (b). Analogous remarks hold when r or q are zero.*

b) *Let M be as in (a) and let $Y \in M(n, \mathbb{R})$ denote a real logarithm of M. By Lemma 2.1, Y is semi-simple and its eigenvalues are (complex) logarithms of the eigenvalues of M. Hence there exist two finite sets, $\{\eta_{(i,x)}, \tau_{(l,t,z)}, \sigma_{(j,y)}\} \subset \mathbb{Z}$ and $\{u_{(i,x)}, b_i, \mu_{(l,t,z)}, d_{(l,t)}, v_{(j,y)}, c_j\} \subset \mathbb{N}$, such that the distinct eigenvalues of Y are precisely the following:*
◇ $\ln(\lambda_i) \pm 2\pi\eta_{(i,x)}\mathbf{i}$, *for $x = 0, 1, \ldots, b_i$, with $0 = \eta_{(i,0)} < \eta_{(i,1)} < \ldots < \eta_{(i,b_i)}$, where, if $b_i \geq 1$ and $x = 1, \ldots, b_i$, the eigenvalues $\ln(\lambda_i) \pm 2\pi\eta_{(i,x)}\mathbf{i}$ both have multiplicity $u_{(i,x)} \geq 1$, while, for $x = 0$, the multiplicity of $\ln(\lambda_i)$ is $g_i := h_i$, if $b_i = 0$, and $g_i := h_i - 2\sum_{x=1}^{b_i} u_{(i,x)} \geq 0$, if $b_i \geq 1$, for every $i = 1, \ldots, p$;*
◇ $\ln(\rho_{(l,t)}) \pm (\theta_l + 2\pi\tau_{(l,t,z)})\mathbf{i}$, *both with multiplicity $\mu_{(l,t,z)} \geq 1$, for $z = 1, \ldots, d_{(l,t)}$, where $\tau_{(l,t,1)} < \ldots < \tau_{(l,t,d_{(l,t)})}$ and $\sum_{z=1}^{d_{(l,t)}} \mu_{(l,t,z)} = m_{(l,t)}$ for every $t = 1, \ldots, a_l$ and $l = 1, \ldots, r$;*
◇ $\ln(w_j) \pm (\pi + 2\pi\sigma_{(j,y)})\mathbf{i}$, *both with multiplicity $v_{(j,y)} \geq 1$, for $y = 1, \ldots, c_j$, where $v_{(j,1)} < \ldots < v_{(j,c_j)}$ and $\sum_{y=1}^{c_j} v_{(j,y)} = k_j$, for every $j = 1, \ldots, q$.*

c) *Let M, Y (and their eigenvalues) be as in (a) and (b), respectively.*
In order to simplify notations and statements, we define the following sets:
$I := \{i : 1 \leq i \leq p,\ b_i \geq 1\}$, $\widehat{I} := \{i : 1 \leq i \leq p,\ b_i = 0\}$,
$J := \{i \in I : g_i = 0\} = \{i : 1 \leq i \leq p,\ g_i = 0\}$, $\widehat{J} := \{i \in I : g_i = 2\}$,
$K := \{i \in I : g_i = 0,\ b_i = u_{(i,1)} = 1\}$, $L := \{j : 1 \leq j \leq q,\ c_j = v_{(j,1)} = 1\}$,
and the following multi-indices:
$\eta := (0,\ \eta_{(1,1)}, \ldots, \eta_{(1,b_1)};\ \ldots;\ 0,\ \eta_{(p,1)}, \ldots, \eta_{(p,b_p)})$;
$u := (g_1, u_{(1,1)}, \ldots, u_{(1,b_1)};\ \ldots;\ g_p, u_{(p,1)}, \ldots, u_{(p,b_p)})$;
$\tau := (\tau_{(1,1,1)}, \ldots, \tau_{(1,1,d_{(1,1)})};\ \ldots;\ \tau_{(r,a_r,1)}, \ldots, \tau_{(r,a_r,d_{(r,a_r)})})$;

$\mu := (\mu_{(1,1,1)}, \ldots, \mu_{(1,1,d_{(1,1)})}; \ \ldots; \ \mu_{(r,a_r,1)}, \ldots, \mu_{(r,a_r,d_{(r,a_r)})});$
$\sigma := (\sigma_{(1,1)}, \ldots, \sigma_{(1,c_1)}; \ \ldots; \ \sigma_{(q,1)}, \ldots, \sigma_{(q,c_q)});$
$v := (v_{(1,1)}, \ldots, v_{(1,c_1)}; \ \ldots; \ v_{(q,1)}, \ldots, v_{(q,c_q)}).$

If the set of multi-indices $(\eta, u, \tau, \mu, \sigma, v)$ satisfies the conditions stated in (b), we say that it is admissible with respect to the matrix$M$ *or simply* M-admissible. *Note that some multi-indices between $\eta, u, \tau, \mu, \sigma, v$ are necessarily empty when p, r or q vanish. For instance, if $p = 0$, then $\eta = u = \emptyset$, and something similar holds when the integer r or the integer q is zero. There is always a countable infinity of M-admissible sets of multi-indices, unless the eigenvalues of M are real, positive and simple. In this case, there exists a unique M-admissible set of multi-indices corresponding to the values $\tau = \mu = \sigma = v = \emptyset, \eta = (0, 0, \ldots, 0), u = (1, 1, \ldots, 1)$. We denote by $\mathcal{L}(M)^{(\eta,\tau,\sigma)}_{(u,\mu,v)}$ the subset of $\mathcal{L}og(M)$ of all real logarithms of M whose eigenvalues agree with the eigenvalues of the matrix Y (with the same multiplicities). We say that the eigenvalues of Y (each with its multiplicity) are the* eigenvalues (with related multiplicities) of $\mathcal{L}(M)^{(\eta,\tau,\sigma)}_{(u,\mu,v)}$. *Note also that, unless the eigenvalues of M are all real, positive and simple, we have*

$$\mathcal{L}og(M) = \bigsqcup \mathcal{L}(M)^{(\eta,\tau,\sigma)}_{(u,\mu,v)}$$

where the countable disjoint union is taken over all M-admissible sets of multi-indices $(\eta, u, \tau, \mu, \sigma, v)$, while, if the eigenvalues of M are all real, positive and simple, then $\mathcal{L}og(M)$ agrees with the set $\mathcal{L}(M)^{(\eta,\tau,\sigma)}_{(u,\mu,v)}$, where $\tau = \mu = \sigma = v = \emptyset$, and $\eta =$ $(0, 0, \ldots, 0)$, $u = (1, 1, \ldots, 1)$. *It is not difficult to prove that every $\mathcal{L}(M)^{(\eta,\tau,\sigma)}_{(u,\mu,v)}$ is an open and closed topological subspace of $\mathcal{L}og(M)$, and therefore, every connected component of $\mathcal{L}(M)^{(\eta,\tau,\sigma)}_{(u,\mu,v)}$ is also a connected component of $\mathcal{L}og(M)$.*

d) *If the semi-simple matrices M and Y (and their eigenvalues) are as in (a) and (b), the real Jordan forms, $\mathcal{J}_M$ of M and $\widetilde{\mathcal{J}}$ of Y, can be written, respectively, as follows:*

$$(\star) \qquad \mathcal{J}_M := \left[\bigoplus_{i\in I} \lambda_i I_{h_i}\right] \oplus \left[\bigoplus_{i\in \widehat{I}} \lambda_i I_{h_i}\right] \oplus \left[\bigoplus_{l=1}^{r} \ \bigoplus_{t=1}^{a_l} \rho_{(l,t)} E_{\theta_l}^{\oplus m_{(l,t)}}\right] \oplus \left[\bigoplus_{j=1}^{q} (-w_j) I_{2k_j}\right];$$

$$(\star\star)\widetilde{\mathcal{J}} := \left[\bigoplus_{i\in I} \left((\ln(\lambda_i) I_{g_i}) \oplus \left(\bigoplus_{x=1}^{b_i} (\ln(\lambda_i) I_{2u_{(i,x)}} + (2\pi\eta_{(i,x)}) E^{\oplus u_{(i,x)}})\right)\right)\right] \oplus \left[\bigoplus_{i\in\widehat{I}} \ln(\lambda_i) I_{h_i}\right] \oplus \left[\bigoplus_{l=1}^{r} \bigoplus_{t=1}^{a_l} \bigoplus_{z=1}^{d_{(l,t)}} \left(\ln(\rho_{(l,t)}) I_{2\mu_{(l,t,z)}} + (\theta_l + 2\pi\tau_{(l,t,z)}) E^{\oplus \mu_{(l,t,z)}}\right)\right] \oplus \left[\bigoplus_{j=1}^{q} \bigoplus_{y=1}^{c_j} \left(\ln(w_j) I_{2v_{(j,y)}} + (\pi + 2\pi\sigma_{(j,y)}) E^{\oplus v_{(j,y)}}\right)\right].$$

By [9, Cor. 3.4.1.10, p. 203], we know that there exist two matrices $C, T \in GL(n, \mathbb{R})$ such that $M = C\mathcal{J}_M C^{-1}$ and $Y = T\widetilde{\mathcal{J}}T^{-1}$. Since $\widetilde{\mathcal{J}}$ is a real Jordan form common to all matrices of $\mathcal{L}(M)^{(\eta,\tau,\sigma)}_{(u,\mu,v)}$, we say that $\widetilde{\mathcal{J}}$ is a real Jordan form of $\mathcal{L}(M)^{(\eta,\tau,\sigma)}_{(u,\mu,v)}$. *Taking into account Notations 1.1* (△)*, it is easy to check that we have* $\exp(\widetilde{\mathcal{J}}) = \mathcal{J}_M$. *Also note that this equality implies $\mathcal{C}_{\widetilde{\mathcal{J}}} \subseteq \mathcal{C}_{\mathcal{J}_M}$.*

Proposition 2.3 *Let M be a semi-simple non-singular real matrix, whose eigenvalues are as in Remarks-Definitions 2.2 (a) and fix $C \in GL(n, \mathbb{R})$ such that $M = C\mathcal{J}_M C^{-1}$, where $\mathcal{J}_M$ is the matrix defined by* ($\star$). *Choose any M-admissible set of multi-indices $(\eta, u, \tau, \mu, \sigma, v)$ as in Remarks–Definitions 2.2 (b),(c), and denote by $\widetilde{\mathcal{J}}$ the real Jordan form of $\mathcal{L}(M)^{(\eta,\tau,\sigma)}_{(u,\mu,v)}$ defined by* ($\star\star$).

Then we have $\mathcal{L}(M)^{(\eta,\tau,\sigma)}_{(u,\mu,v)} = \{CX\widetilde{\mathcal{J}}X^{-1}C^{-1} : X \in \mathcal{C}_{\mathcal{J}_M}\} = Ad_C\big(\mathcal{L}(J_M)^{(\eta,\tau,\sigma)}_{(u,\mu,v)}\big).$

Moreover, $\mathcal{L}(M)^{(\eta,\tau,\sigma)}_{(u,\mu,v)}$ *is a closed embedded submanifold of* $M(n,\mathbb{R})$*, diffeomorphic to the homogeneous space* $\dfrac{\mathcal{C}_{\mathcal{J}_M}}{\mathcal{C}_{\widetilde{\mathcal{J}}}}$.

Proof If $Y \in \mathcal{L}(M)^{(\eta,\tau,\sigma)}_{(u,\mu,v)}$, then $Y = T\widetilde{\mathcal{J}}T^{-1}$, for some $T \in GL(n,\mathbb{R})$, and $\exp(Y) = T\exp(\widetilde{\mathcal{J}})T^{-1} = T\mathcal{J}_M T^{-1} = M = C\mathcal{J}_M C^{-1}$. Hence $C^{-1}T \in \mathcal{C}_{\mathcal{J}_M}$, and so $T = CX$, for some $X \in \mathcal{C}_{\mathcal{J}_M}$. Conversely, if $Y = CX\widetilde{\mathcal{J}}X^{-1}C^{-1}$, with $X \in \mathcal{C}_{\mathcal{J}_M}$, we have $\exp(Y) = CX\mathcal{J}_M X^{-1}C^{-1} = M$, and so $Y \in \mathcal{L}(M)^{(\eta,\tau,\sigma)}_{(u,\mu,v)}$. Therefore $\mathcal{L}(M)^{(\eta,\tau,\sigma)}_{(u,\mu,v)} = \{CX\widetilde{\mathcal{J}}X^{-1}C^{-1} : X \in \mathcal{C}_{\mathcal{J}_M}\}$. The second equality of the statement follows directly from the definition of the mapping Ad_C.

Since Ad_C is a diffeomorphism of $M(n,\mathbb{R})$, it suffices to prove that the set $\mathcal{L}(\mathcal{J}_M)^{(\eta,\tau,\sigma)}_{(u,\mu,v)} = \{X\widetilde{\mathcal{J}}X^{-1} : X \in \mathcal{C}_{\mathcal{J}_M}\}$ has the properties stated for $\mathcal{L}(M)^{(\eta,\tau,\sigma)}_{(u,\mu,v)}$. The map: $(B,Z) \mapsto Ad_B(Z)$ defines an action of the Lie group $\mathcal{C}_{\mathcal{J}_M}$ on $M(n,\mathbb{R})$. The orbit of $\widetilde{\mathcal{J}}$ is the set $\mathcal{L}(\mathcal{J}_M)^{(\eta,\tau,\sigma)}_{(u,\mu,v)}$. By Remark 1.8, this set is an immersed submanifold of $M(n,\mathbb{R})$, diffeomorphic to the homogeneous space $\dfrac{\mathcal{C}_{\mathcal{J}_M}}{\mathcal{C}_{\widetilde{\mathcal{J}}}}$, since $\mathcal{C}_{\widetilde{\mathcal{J}}}$ is the isotropy subgroup of the action.

The set $\mathcal{L}(\mathcal{J}_M)^{(\eta,\tau,\sigma)}_{(u,\mu,v)}$ is closed in $M(n,\mathbb{R})$. Indeed, if $\{Y_i\}_{i\in\mathbb{N}}$ is a sequence in $\mathcal{L}(\mathcal{J}_M)^{(\eta,\tau,\sigma)}_{(u,\mu,v)}$ converging to $Y \in M(n,\mathbb{R})$, then $\exp(Y) = \mathcal{J}_M$, and the characteristic polynomial of Y is the same characteristic polynomial of all matrices Y_i (constant with respect to $i \in \mathbb{N}$). Hence $Y \in \mathcal{L}(\mathcal{J}_M)^{(\eta,\tau,\sigma)}_{(u,\mu,v)}$; therefore this last set is closed and it is an embedded submanifold of $M(n,\mathbb{R})$ (see, for instance, [12, Thm.p. 65]). □

Lemma 2.4 *Let* $\mathcal{J}_M$ *and* $\widetilde{\mathcal{J}}$ *the matrices of Remarks–Definitions 2.2 (d) defined by* ($\star$) *and* ($\star\star$)*, respectively. Then the Lie groups consisting of the non-singular matrices commuting with* $\mathcal{J}_M$ *and* $\widetilde{\mathcal{J}}$ *are the following:*

$$\mathcal{C}_{\mathcal{J}_M} = \left[\bigoplus_{i\in I} GL(h_i,\mathbb{R})\right] \oplus \left[\bigoplus_{i\in\widehat{I}} GL(h_i,\mathbb{R})\right] \oplus \left[\bigoplus_{l=1}^{r} \bigoplus_{t=1}^{a_l} GL(m_{(l,t)},\mathbb{C})\right] \oplus \left[\bigoplus_{j=1}^{q} GL(2k_j,\mathbb{R})\right];$$

$$\mathcal{C}_{\widetilde{\mathcal{J}}} = \left[\bigoplus_{i\in I}\left(GL(g_i,\mathbb{R}) \oplus \left(\bigoplus_{x=1}^{b_i} GL(u_{(i,x)},\mathbb{C})\right)\right)\right] \oplus \left[\bigoplus_{i\in\widehat{I}} GL(h_i,\mathbb{R})\right] \oplus \left[\bigoplus_{l=1}^{r} \bigoplus_{t=1}^{a_l} \bigoplus_{z=1}^{d_{(l,t)}} GL(\mu_{(l,t,z)},\mathbb{C})\right] \oplus \left[\bigoplus_{j=1}^{q} \bigoplus_{y=1}^{c_j} GL(v_{(j,y)},\mathbb{C})\right].$$

Proof The statement follows directly from Lemmas 1.3 and 1.4. □

Theorem 2.5 *Let* M *be a semi-simple non-singular real matrix, whose eigenvalues are as in Remarks–Definitions 2.2 (a). Choose any* M*-admissible set of multi-indices* $(\eta, u, \tau, \mu, \sigma, v)$ *and let* $J = \{i : 1 \le i \le p, \quad g_i = 0\}$*; then* $\mathcal{L}(M)^{(\eta,\tau,\sigma)}_{(u,\mu,v)}$ *is a manifold with* $2^{(|J|+q)}$ *connected components, each of which is diffeomorphic to* $\left[\prod_{i\in I} \widehat{\Gamma}_{(g_i;u_{(i,1)},\ldots,u_{(i,b_i)})}\right] \times \left[\prod_{l=1}^{r}\prod_{t=1}^{a_l} \widehat{\Theta}_{(\mu_{(l,t,1)},\ldots,\mu_{(l,t,d_{(l,t)})})}\right] \times \left[\prod_{j=1}^{q} \widehat{\Gamma}_{(0;v_{(j,1)},\ldots,v_{(j,c_j)})}\right].$

Proof From Proposition 2.3 and Lemma 2.4, it follows that $\mathcal{L}(M)^{(\eta,\tau,\sigma)}_{(u,\mu,v)}$ is a manifold diffeomorphic to the following product of homogeneous spaces:

$$\left[\prod_{i\in I}\frac{GL(h_i,\mathbb{R})}{GL(g_i,\mathbb{R})\oplus\left(\bigoplus_{x=1}^{b_i}GL(u_{(i,x)},\mathbb{C})\right)}\right]\times\left[\prod_{l=1}^{r}\prod_{t=1}^{a_l}\frac{GL(m_{(l,t)},\mathbb{C})}{\left(\bigoplus_{z=1}^{d_{(l,t)}}GL(\mu_{(l,t,z)},\mathbb{C})\right)}\right]\times$$
$\left[\prod_{j=1}^{q}\frac{GL(2k_j,\mathbb{R})}{\left(\bigoplus_{y=1}^{c_j}GL(v_{(j,y)},\mathbb{C})\right)}\right]$; hence, recalling Remarks-Definitions 1.5 (d), we obtain the statement of the Theorem. □

Remark 2.6 Let M be a semi-simple non-singular real matrix, whose negative eigenvalues have even multiplicity. By Remarks–Definitions 2.2 (c), Theorem 2.5 and Remarks–Definitions 1.5 (b), $\mathcal{L}og(M)$ is a finite set if and only if the eigenvalues of M are all real, positive and simple, and in this case, it consists of a single point (see [2, Thm. 2]). Otherwise, the set $\mathcal{L}og(M)$ is countably infinite if and only if every manifold $\mathcal{L}(M)^{(\eta,\tau,\sigma)}_{(u,\mu,v)}$ has zero dimension, and so, taking into account Theorem 2.5 and Remarks-Definitions 1.5 (b),(c), we get that the set $\mathcal{L}og(M)$ is countably infinite if and only if all eigenvalues of M are simple and no eigenvalue of M is negative, as in [2, Cor. p. 1151].

Definition 2.7 Let $M \in M(n,\mathbb{R})$. We say that a matrix $X \in M(n,\mathbb{R})$ is a *generalized principal real logarithm* of M, if $\exp(X) = M$ and every eigenvalue of X has imaginary part in $[-\pi,\pi]$. Note that this definition is more general than the usual definition of *principal logarithm* (see, for instance, [8, Thm. 1.31 p. 20]).

We denote by $\mathcal{PL}og(M)$ the set of all generalized principal real logarithms of M. Of course this set can be empty, but this does not happen if the matrix M is non-singular, semi-simple and all its negative eigenvalues have even multiplicity.

Theorem 2.8 *Let M be a semi-simple non-singular real matrix, having exactly q distinct negative eigenvalues ($q \geq 0$), of multiplicity $2k_1,\ldots,2k_q$, respectively.*
If $q \geq 1$, the set $\mathcal{PL}og(M)$ is a manifold with 2^q connected components, each of which is diffeomorphic to the symmetric space $\prod_{j=1}^{q}\widehat{\Gamma}_{(0;k_j)}$.
If M has no negative eigenvalues, then the set $\mathcal{PL}og(M)$ is a single point.

Proof Using the same notations as in Remarks–Definition 2.2, and denoting by O any multi-index whose entries are all zero, we have $\mathcal{PL}og(M) = \mathcal{L}(M)^{(\eta,\tau,\sigma)}_{(u,\mu,v)}$, where $\eta = O$, $\tau = O$, $\sigma = O$, $u = (h_1;\ \ldots; h_p)$ $\mu = (m_{(1,1)};\ldots;\ m_{(r,a_r)})$, $v = (k_1;\ldots;\ k_q)$. Therefore, by Theorem 2.5, the manifold $\mathcal{PL}og(M)$ has 2^q connected components, which are, if $q \geq 1$, all diffeomorphic to $\prod_{j=1}^{q}\widehat{\Gamma}_{(0;k_j)}$. If $q = 0$ then $\mathcal{PL}og(M)$ consists of a single point (as in [8, Thm. 1.31]), since all sets $\widehat{\Gamma}_{(h_i)}$ and $\widehat{\Theta}_{(m_{(l,t)})}$ reduce to a point. □

3 Real skew-symmetric logarithms of special orthogonal matrices

Notations 3.1 *In this Section we assume $n \geq 2$. Let $Q \in SO(n)$. The eigenvalues of Q have unitary modulus, so the real Jordan form of Q can be written as follows:*
$(*)\quad \widehat{\mathcal{J}}_Q = I_h \oplus E_{\theta_1}^{\oplus m_1} \oplus \ldots \oplus E_{\theta_r}^{\oplus m_r} \oplus (-I_{2k})$
where $h,r,k \geq 0$, $\quad h + 2m_1 + \ldots + 2m_r + 2k = n$, $\quad 0 < \theta_1 < \theta_2 < \ldots < \theta_r < \pi$; so the eigenvalues of Q are: 1 with multiplicity $h \geq 0$, $\exp(\pm\mathbf{i}\theta_1)$ both with multiplicity m_1, …, up to $\exp(\pm\mathbf{i}\theta_r)$ both with multiplicity m_r ($m_j \geq 1$, for every j, if $r > 0$), and -1 with multiplicity $2k \geq 0$. Note that, also in this case, the integers h,r,k can vanish; so we assume, in this Section, the same agreements stated in Notations 1.1 (b). Also note that, if n is odd, h is also odd, so 1 is necessarily an eigenvalue of Q. It is known that there exists $K \in \mathcal{O}(n)$ such that $Q = K\widehat{\mathcal{J}}_Q K^T$ (see, for instance, [9, Cor. 2.5.11 p. 137]).

Remarks-Definitions 3.2 a) *Let Q be a matrix of $\mathcal{SO}(n)$ as in Notations 3.1, and choose any real skew-symmetric logarithm W of Q. The eigenvalues of W are complex logarithms of the eigenvalues of Q, so, as in Remarks–Definitions 2.2, there exist two finite sets, $\{\eta_i,\ \tau_{(l,t)},\ \sigma_j\} \subset \mathbb{Z}$ and $\{u_i,\ b,\ \mu_{(l,t)},\ d_l,\ v_j,\ c\} \subset \mathbb{N}$, such that the eigenvalues of W can be written as follows:*

$\diamond$ $\pm 2\pi\eta_i\mathbf{i}$, for $i = 0, 1, \ldots, b$ with $0 = \eta_0 < \eta_1 < \ldots < \eta_b$, where, if $b \geq 1$ and $i = 1, \ldots, b$ the eigenvalues $\pm 2\pi\eta_i\mathbf{i}$ both have multiplicity $u_i \geq 1$, while, for $i = 0$, the eigenvalue 0 has multiplicity $g := h - 2\sum_{i=1}^{b} u_i \geq 0$ if $b \geq 1$, and $g := h$, if $b = 0$;

$\diamond$ $\pm(\theta_l + 2\pi\tau_{(l,t)})\mathbf{i}$ both with multiplicity $\mu_{(l,t)} \geq 1$, for every $t = 1, \ldots, d_l$, where $\tau_{(l,1)} < \tau_{(l,2)} < \ldots < \tau_{(l,d_l)}$ and $\sum_{t=1}^{d_l} \mu_{(l,t)} = m_l$ for every $l = 1, \ldots, r$;

$\diamond$ $\pm(\pi + 2\pi\sigma_j)\mathbf{i}$ both with multiplicity $v_j \geq 1$, for every $j = 1, \ldots, c$ where $\sigma_1 < \ldots < \sigma_c$ and $\sum_{j=1}^{c} v_j = k$.

As in Sect. 2, to simplify the notations, we define the following multi-indices:

$\eta := (0, \eta_1, \ldots, \eta_b)$; $\quad u := (g, u_1, \ldots, u_b)$;

$\tau := (\tau_{(1,1)}, \ldots, \tau_{(1,d_1)}; \ldots; \tau_{(r,1)}, \ldots, \tau_{(r,d_r)})$;

$\mu := (\mu_{(1,1)}, \ldots, \mu_{(1,d_1)}; \ldots; \mu_{(r,1)}, \ldots, \mu_{(r,d_r)})$;

$\sigma := (\sigma_1, \ldots, \sigma_c)$; $\quad v := (v_1, \ldots, v_c)$.

The set of multi-indices $(\eta, u, \tau, \mu, \sigma, v)$ is Q-admissible (see Remarks–Definitions 2.2). Note that, since $n \geq 2$, there is a countable infinity of Q-admissible sets of multi-indices, for every $Q \in \mathcal{SO}(n)$. Also note that -1 is an eigenvalue of Q of multiplicity 2 if and only if $c = v_1 = 1$, for every Q-admissible set of multi-indices.

b) *Let W be as in (a). The assumptions on the eigenvalues of W are equivalent to saying that there is a real Jordan form $\widehat{\mathcal{J}}$ of W of the following type:*

$$(**)\ \ \widehat{\mathcal{J}} = \left[\mathrm{O}_g \oplus \left(\bigoplus_{i=1}^{b}(2\pi\eta_i E^{\oplus u_i})\right)\right] \oplus \left[\bigoplus_{l=1}^{r}\ \bigoplus_{t=1}^{a_l}(\theta_l + 2\pi\tau_{(l,t)})E^{\oplus\mu_{(l,t)}}\right] \oplus \left[\bigoplus_{j=1}^{c}(\pi + 2\pi\sigma_j)E^{\oplus v_j}\right].$$

Since the matrix W is skew-symmetric, there exists a matrix $Z \in \mathcal{O}(n)$ such that $W = Z\widehat{\mathcal{J}}Z^T$ (see again [9, Cor. 2.5.11 p. 136]). Note that $\widehat{\mathcal{J}}$ is also skew-symmetric.

c) *If the set of multi-indices $(\eta, u, \tau, \mu, \sigma, v)$ is Q-admissible, we denote by $\mathcal{L}_{\mathfrak{so}(n)}(Q)^{(\eta,\tau,\sigma)}_{(u,\mu,v)} = \mathfrak{so}(n) \cap \mathcal{L}(Q)^{(\eta,\tau,\sigma)}_{(u,\mu,v)}$ the set of real skew-symmetric logarithms of Q whose real Jordan form is the matrix $\widehat{\mathcal{J}}$ defined by $(**)$. As in Remarks-Definitions 2.2 (c),(d), we say that $\widehat{\mathcal{J}}$ is a* real Jordan form of $\mathcal{L}_{\mathfrak{so}(n)}(Q)^{(\eta,\tau,\sigma)}_{(u,\mu,v)}$ *and that the eigenvalues of $\widehat{\mathcal{J}}$ (with their multiplicities) are the* eigenvalues (with related multiplicities) of $\mathcal{L}_{\mathfrak{so}(n)}(Q)^{(\eta,\tau,\sigma)}_{(u,\mu,v)}$. *It is clear that*

$$\mathcal{L}og_{\mathfrak{so}(n)}(Q) = \bigsqcup \mathcal{L}_{\mathfrak{so}(n)}(Q)^{(\eta,\tau,\sigma)}_{(u,\mu,v)}$$

where the countable disjoint union is taken over all Q-admissible sets of multi-indices $(\eta, u, \tau, \mu, \sigma, v)$. As in Remarks–Definitions 2.2 (c), $\mathcal{L}_{\mathfrak{so}(n)}(Q)^{(\eta,\tau,\sigma)}_{(u,\mu,v)}$ is an open and closed topological subspace of $\mathcal{L}og_{\mathfrak{so}(n)}(Q)$; so we get that each connected component of $\mathcal{L}_{\mathfrak{so}(n)}(Q)^{(\eta,\tau,\sigma)}_{(u,\mu,v)}$ is also a connected component of $\mathcal{L}og_{\mathfrak{so}(n)}(Q)$.

Proposition 3.3 *Let $Q \in \mathcal{SO}(n)$; assume its real Jordan form $\widehat{\mathcal{J}}_Q$ is as in Notations 3.1 $(*)$ and fix $K \in \mathcal{O}(n)$ such that $Q = K\widehat{\mathcal{J}}_Q K^T$. Choose any Q-admissible set of multi-*

indices $(\eta, u, \tau, \mu, \sigma, v)$ *and let* $\widehat{\mathcal{J}}$ *be the real Jordan form of* $\mathcal{L}_{\mathfrak{so}(n)}(Q)^{(\eta,\tau,\sigma)}_{(u,\mu,v)}$ *defined in Remarks–Definitions 3.2* (∗∗)*. Then we have*
$\mathcal{L}_{\mathfrak{so}(n)}(Q)^{(\eta,\tau,\sigma)}_{(u,\mu,v)} = \{KX\widehat{\mathcal{J}}X^TK^T : X \in \mathcal{C}_{\widehat{\mathcal{J}}_Q} \cap \mathcal{O}(n)\} = Ad_K(\mathcal{L}_{\mathfrak{so}(n)}(\widehat{\mathcal{J}}_Q)^{(\eta,\tau,\sigma)}_{(u,\mu,v)})$.
Moreover $\mathcal{L}_{\mathfrak{so}(n)}(Q)^{(\eta,\tau,\sigma)}_{(u,\mu,v)}$ *is a compact submanifold of* $\mathfrak{so}(n)$*, diffeomorphic to the homogeneous space* $\dfrac{\mathcal{C}_{\widehat{\mathcal{J}}_Q} \cap \mathcal{O}(n)}{\mathcal{C}_{\widehat{\mathcal{J}}} \cap \mathcal{O}(n)}$.

Proof Let $W \in \mathcal{L}_{\mathfrak{so}(n)}(Q)^{(\eta,\tau,\sigma)}_{(u,\mu,v)}$. We know that there exists $Z \in \mathcal{O}(n)$ such that $W = Z\widehat{\mathcal{J}}Z^T$. By Notations 1.1 (△), we get $\exp(\widehat{\mathcal{J}}) = \widehat{\mathcal{J}}_Q$. Since $\exp(W) = Q$, we have $Z\widehat{\mathcal{J}}_QZ^T = K\widehat{\mathcal{J}}_QK^T$, so we get $Z = KX$, with $X \in \mathcal{C}_{\widehat{\mathcal{J}}_Q} \cap \mathcal{O}(n)$. Conversely, if $W = KX\widehat{\mathcal{J}}X^TK^T$, with $X \in \mathcal{C}_{\widehat{\mathcal{J}}_Q} \cap \mathcal{O}(n)$, then $W \in \mathfrak{so}(n)$ and $\exp(W) = KX\widehat{\mathcal{J}}_QX^TK^T = Q$. Hence $\mathcal{L}_{\mathfrak{so}(n)}(Q)^{(\eta,\tau,\sigma)}_{(u,\mu,v)} = \{KX\widehat{\mathcal{J}}X^TK^T : X \in \mathcal{C}_{\widehat{\mathcal{J}}_Q} \cap \mathcal{O}(n)\} = Ad_K(\{X\widehat{\mathcal{J}}X^T : X \in \mathcal{C}_{\widehat{\mathcal{J}}_Q} \cap \mathcal{O}(n)\}) = Ad_K(\mathcal{L}_{\mathfrak{so}(n)}(\widehat{\mathcal{J}}_Q)^{(\eta,\tau,\sigma)}_{(u,\mu,v)})$. Since Ad_K is a diffeomorphism of $\mathfrak{so}(n)$ which maps $\mathcal{L}_{\mathfrak{so}(n)}(\widehat{\mathcal{J}}_Q)^{(\eta,\tau,\sigma)}_{(u,\mu,v)}$ onto $\mathcal{L}_{\mathfrak{so}(n)}(Q)^{(\eta,\tau,\sigma)}_{(u,\mu,v)}$, it suffices to prove the final part of the Theorem for $\mathcal{L}_{\mathfrak{so}(n)}(\widehat{\mathcal{J}}_Q)^{(\eta,\tau,\sigma)}_{(u,\mu,v)}$. The compact Lie group $\mathcal{C}_{\widehat{\mathcal{J}}_Q} \cap \mathcal{O}(n)$ acts on $\mathfrak{so}(n)$ through the map: $(A, Y) \mapsto Ad_A(Y)$; $\mathcal{L}_{\mathfrak{so}(n)}(\widehat{\mathcal{J}}_Q)^{(\eta,\tau,\sigma)}_{(u,\mu,v)}$ is the orbit of $\widehat{\mathcal{J}}$, while the isotropy subgroup at $\widehat{\mathcal{J}}$ is $\mathcal{C}_{\widehat{\mathcal{J}}} \cap \mathcal{O}(n)$; hence, by Remark 1.8, $\mathcal{L}_{\mathfrak{so}(n)}(\widehat{\mathcal{J}}_Q)^{(\eta,\tau,\sigma)}_{(u,\mu,v)}$ (and therefore also $\mathcal{L}_{\mathfrak{so}(n)}(Q)^{(\eta,\tau,\sigma)}_{(u,\mu,v)}$) is a compact submanifold of $\mathfrak{so}(n)$ diffeomorphic to $\dfrac{\mathcal{C}_{\widehat{\mathcal{J}}_Q} \cap \mathcal{O}(n)}{\mathcal{C}_{\widehat{\mathcal{J}}} \cap \mathcal{O}(n)}$. This completes the proof. □

Theorem 3.4 *Let* $Q \in S\mathcal{O}(n)$*; assume its real Jordan form* $\widehat{\mathcal{J}}_Q$ *is as in Notations 3.1* (∗) *and choose any* Q*-admissible set of multi-indices* $(\eta, u, \tau, \mu, \sigma, v)$*. Then*

a) $\mathcal{L}_{\mathfrak{so}(n)}(Q)^{(\eta,\tau,\sigma)}_{(u,\mu,v)}$ *is a compact homogeneous submanifold of* $\mathfrak{so}(n)$*, whose connected components are all diffeomorphic to the product*
$\Gamma_{(g;u_1,\dots,u_b)} \times \left[\prod_{l=1}^{r} \Theta_{(\mu_{(l,1)},\dots,\mu_{(l,d_l)})}\right] \times \Gamma_{(0;v_1,\dots,v_c)}$;
b) *the manifold* $\mathcal{L}_{\mathfrak{so}(n)}(Q)^{(\eta,\tau,\sigma)}_{(u,\mu,v)}$ *is connected if and only if either* Q *has no real eigenvalues or* -1 *is not an eigenvalue of* Q *and* 0 *is an eigenvalue of* $\mathcal{L}_{\mathfrak{so}(n)}(Q)^{(\eta,\tau,\sigma)}_{(u,\mu,v)}$*; this manifold has two connected components if and only if either* 1 *is an eigenvalue of* Q*,* -1 *is not an eigenvalue of* Q *and* 0 *is not an eigenvalue of* $\mathcal{L}_{\mathfrak{so}(n)}(Q)^{(\eta,\tau,\sigma)}_{(u,\mu,v)}$*, or* -1 *is an eigenvalue of* Q *and* 0 *is an eigenvalue of* $\mathcal{L}_{\mathfrak{so}(n)}(Q)^{(\eta,\tau,\sigma)}_{(u,\mu,v)}$*, or* 1 *is not an eigenvalue of* Q *and* -1 *is an eigenvalue of* Q*;* $\mathcal{L}_{\mathfrak{so}(n)}(Q)^{(\eta,\tau,\sigma)}_{(u,\mu,v)}$ *has four connected components if and only if* 1 *and* -1 *are both eigenvalues of* Q *and* 0 *is not an eigenvalue of* $\mathcal{L}_{\mathfrak{so}(n)}(Q)^{(\eta,\tau,\sigma)}_{(u,\mu,v)}$.

Proof From Lemmas 1.3 and 1.4, we get
$\mathcal{C}_{\widehat{\mathcal{J}}_Q} \cap \mathcal{O}(n) = \{GL(h,\mathbb{R}) \oplus \left(\bigoplus_{l=1}^{r} GL(m_l,\mathbb{C})\right) \oplus GL(2k,\mathbb{R})\} \cap \mathcal{O}(n) = \mathcal{O}(h) \oplus \left(\bigoplus_{l=1}^{r} U(m_l)\right) \oplus \mathcal{O}(2k)$ and $\mathcal{C}_{\widehat{\mathcal{J}}} \cap \mathcal{O}(n) =$
$\left\{GL(g,\mathbb{R}) \oplus \left(\bigoplus_{i=1}^{b} GL(u_i,\mathbb{C})\right) \oplus \left(\bigoplus_{l=1}^{r} \bigoplus_{t=1}^{d_l} G(\mu_{(l,t)},\mathbb{C})\right) \oplus \left(\bigoplus_{j=1}^{c} GL(v_j,\mathbb{C})\right)\right\} \cap \mathcal{O}(n) = \mathcal{O}(g) \oplus \left(\bigoplus_{i=1}^{b} U(u_i)\right) \oplus \left(\bigoplus_{l=1}^{r} \bigoplus_{t=1}^{d_l} U(\mu_{(l,t)})\right) \oplus (\bigoplus_{j=1}^{c} U(v_j))$.

Therefore the homogeneous space $\dfrac{\mathcal{C}_{\widehat{\mathcal{J}}_Q} \cap \mathcal{O}(n)}{\mathcal{C}_{\widehat{\mathcal{J}}} \cap \mathcal{O}(n)}$ is diffeomorphic to the product

$$\left[\frac{\mathcal{O}(h)}{\mathcal{O}(g) \oplus \left(\bigoplus_{i=1}^{b} U(u_i)\right)} \right] \times \left[\prod_{l=1}^{r} \frac{U(m_l)}{\left(\bigoplus_{t=1}^{d_l} U(\mu_{(l,t)})\right)} \right] \times \left[\frac{\mathcal{O}(2k)}{\left(\bigoplus_{j=1}^{c} U(v_j)\right)} \right] .$$

(We have assumed, without losing generality, that $h, k \geq 1$.) By Proposition 3.3 and Remarks-Definitions 1.5 (d), we get the statement (a), while (b) still follows, by means of simple arguments, from Remarks-Definitions 1.5 (d). □

Remark 3.5 a) Now, keeping in mind Notations 3.1 and Remarks–Definitions 3.2 (a), if the set of multi-indices $(\eta, u, \tau, \mu, \sigma, v)$ is Q-admissible and the order n of Q is odd, then 0 is necessarily an eigenvalue of $\mathcal{L}_{\mathfrak{so}(n)}(Q)^{(\eta,\tau,\sigma)}_{(u,\mu,v)}$.

b) If $Q \in S\mathcal{O}(n)$ $(n \geq 2)$, the set of real skew-symmetric logarithms of Q is never finite and it is countably infinite if and only if, for every Q-admissible set of multi-indices $(\eta, u, \tau, \mu, \sigma, v)$, the manifold $\mathcal{L}_{\mathfrak{so}(n)}(Q)^{(\eta,\tau,\sigma)}_{(u,\mu,v)}$ has zero dimension. By Theorem 3.4 and Remarks–Definitions 1.5 (b), (c), it follows that $\mathcal{L}og_{\mathfrak{so}(n)}(Q)$ is countably infinite if and only if all non-real eigenvalues of Q are simple and the multiplicity of 1 and -1 as (possible) eigenvalues of Q is less than or equal to 2.

Definition 3.6 Let $Q \in S\mathcal{O}(n)$. As in Definition 2.7, we say that $X \in \mathfrak{so}(n)$ is a *generalized principal skew-symmetric logarithm* of Q, if $\exp(X) = Q$ and each eigenvalue of X has imaginary part in $[-\pi, \pi]$. We denote by $\mathcal{PL}og_{\mathfrak{so}(n)}(Q)$ the set of generalized principal skew-symmetric logarithms of Q.

Theorem 3.7 *Let $Q \in S\mathcal{O}(n)$. If -1 is not an eigenvalue of Q, then the set $\mathcal{PL}og_{\mathfrak{so}(n)}(Q)$ consists of a single point, while, if -1 is an eigenvalue of Q of multiplicity $2k \geq 2$, then $\mathcal{PL}og_{\mathfrak{so}(n)}(Q)$ is a compact submanifold of $\mathfrak{so}(n)$, diffeomorphic to the homogeneous space $\dfrac{\mathcal{O}(2k)}{U(k)}$. In this last case, $\mathcal{PL}og_{\mathfrak{so}(n)}(Q)$ has two connected components, each of which is diffeomorphic to the symmetric space $\Gamma_{(0;k)}$.*

Proof The set $\mathcal{PL}og_{\mathfrak{so}(n)}(Q)$ agrees with $\mathcal{L}_{\mathfrak{so}(n)}(Q)^{(\eta,\tau,\sigma)}_{(u,\mu,v)}$, where $\eta = O$, $\tau = O$, $\sigma = O$, $u = (h)$, $\mu = (m_1; \ldots; m_r)$, $v = (k)$, so the statement follows from the proof of Theorem 3.4, taking into account Remarks–Definitions 1.5 (b), (c), (d). □

4 Remarks on the homotopy groups of some homogeneous spaces

As we have seen, the homogeneous spaces defined in Remarks–Definitions 1.5 describe the structure of the set of real logarithms of an arbitrary matrix. For this reason, in this section we will study some of their topological properties. We start with some general properties concerning homogeneous spaces.

Remark 4.1 Let G be a connected Lie group with identity e and let H be any connected closed subgroup of G. Denoting by G/H the related homogeneous space and by $\{e\} = H$ the equivalence class of e in the quotient G/H, it is known that we have the following homotopy exact sequence, induced by the fibration on the quotient (see [16, p.90]):
$\ldots \xrightarrow{\delta} \pi_i(H) \xrightarrow{\psi} \pi_i(G) \xrightarrow{\xi} \pi_i(G/H) \xrightarrow{\delta} \ldots \xrightarrow{\xi} \pi_2(G/H) \xrightarrow{\delta} \pi_1(H) \xrightarrow{\psi} \pi_1(G) \xrightarrow{\xi} \pi_1(G/H) \to 0.$

In this sequence, the homotopy groups are based at the point e for G and H and at the point $\{e\}$ for G/H; the mappings ψ and ξ are, respectively, the homomorphisms induced by the natural inclusion: $H \to G$ and by the projection of G onto the quotient G/H, while the mappings δ are the connecting homomorphisms.

Lemma 4.2 *Let G', H, H' be connected closed subgroups of a connected Lie group G, such that $H' \subset G' \cap H$. Suppose G' is a deformation retract of G and H' is a deformation retract of H. Then $\pi_i(G/H) \cong \pi_i(G'/H')$, for every $i \geq 1$.*

Proof From the assumptions, it follows that the natural inclusion: $G' \to G$ is a bundle morphism, i.e. there exist a (natural) inclusion map: $G'/H' \to G/H$ such that the diagram
$$\begin{array}{ccc} G' & \longrightarrow & G \\ \downarrow & & \downarrow \\ G'/H' & \longrightarrow & G/H \end{array}$$
commutes. Then, for every $i \geq 2$, we get the following commutative diagram, where the rows are exact sequences (see [16, p.90]):

$$\begin{array}{ccccccccc} \cdots\pi_i(H') & \xrightarrow{\psi'} & \pi_i(G') & \xrightarrow{\xi'} & \pi_i(G'/H') & \xrightarrow{\delta'} & \pi_{i-1}(H') & \xrightarrow{\psi'} & \pi_{i-1}(G')\ldots \\ \downarrow f_i & & \downarrow j_i & & \downarrow l_i & & \downarrow f_{i-1} & & \downarrow j_{i-1} \\ \cdots\pi_i(H) & \xrightarrow{\psi} & \pi_i(G) & \xrightarrow{\xi} & \pi_i(G/H) & \xrightarrow{\delta} & \pi_{i-1}(H) & \xrightarrow{\psi} & \pi_{i-1}(G)\cdots \end{array}$$

Here the maps f_i, j_i and l_i are the homomorphisms induced by the natural inclusions. Since all groups are connected, if we define, as usual, $\pi_0(G) = \pi_0(G') = \pi_0(H) = \pi_0(H') = \{0\}$, the previous commutative diagram remains valid also for $i = 1$. Furthermore, since G' and H' are deformation retracts of G and H, respectively, all maps f_r, j_r are isomorphisms, therefore, by the classical Five-Lemma (see, for instance, [7, p. 129], where the proof also works for non-abelian groups), all maps l_i are also isomorphisms, for every $i \geq 1$. This concludes the proof. □

Proposition 4.3 *Let $\zeta, \nu_1, \ldots, \nu_s$ be integers such that $\zeta \geq 0$ and $\nu_j \geq 1$, for $j = 1, \ldots, s$ $(s \geq 1)$. Then, for every $i \geq 1$, we have*
$\pi_i(\widehat{\Gamma}_{(\zeta;\nu_1,\ldots,\nu_s)}) \cong \pi_i(\Gamma_{(\zeta;\nu_1,\ldots,\nu_s)})$ *and* $\pi_i(\widehat{\Theta}_{(\nu_1,\ldots,\nu_s)}) \cong \pi_i(\Theta_{(\nu_1,\ldots,\nu_s)})$.

Proof It is well-known that $SO(n)$ is a deformation retract of $GL^+(n,\mathbb{R})$. Indeed, if $X \in GL^+(n,\mathbb{R})$, by polar decomposition (see [9, Thm. 7.3.1 p.449]), we can write $X = (XX^T)^{1/2}\big((XX^T)^{-1/2}\cdot X\big)$, where $(XX^T)^{1/2}$ is a positive definite symmetric real matrix of order n and $\big((XX^T)^{-1/2}\cdot X\big) \in SO(n)$. Denoting by $\log((XX^T)^{1/2})$ the unique symmetric real logarithm of the positive definite matrix $(XX^T)^{1/2}$, by $j : SO(n) \to GL^+(n,\mathbb{R})$ the natural inclusion, and by $\widehat{r} : GL^+(n,\mathbb{R}) \to SO(n)$ the retraction: $X \mapsto \big((XX^T)^{-1/2}\cdot X\big)$, we define $H(X,t) = \exp\big(t\cdot\log((XX^T)^{1/2})\big)\big((XX^T)^{-1/2}\cdot X\big)$, for every $X \in GL^+(n,\mathbb{R})$ and $t \in [0,1]$. H is a C^∞ homotopy between $j \circ \widehat{r}$ and the identity map of $GL^+(n,\mathbb{R})$, so $SO(n)$ is a deformation retract of $GL^+(n,\mathbb{R})$. Similarly, it can be proved that $U(n)$ is a deformation retract of $GL(n,\mathbb{C})$, $\big(\bigoplus_{j=1}^s U(\nu_j)\big)$ is a deformation retract of $\big(\bigoplus_{j=1}^s GL(\nu_j,\mathbb{C})\big)$, and $SO(\zeta) \oplus \big(\bigoplus_{j=1}^s U(\nu_j)\big)$ is a deformation retract of
$GL^+(\zeta,\mathbb{R}) \oplus \big(\bigoplus_{j=1}^s GL(\nu_j,\mathbb{C})\big)$; so the Proposition follows from Lemma 4.2. □

Remark 4.4 The spaces $\Gamma_{(0;1)}$, $\Gamma_{(\zeta)}$ and $\Theta_{(\nu)}$ reduce to a single point (remember Remarks–Definitions 1.5 (b)), so their homotopy groups are trivial.

We also recall that the so-called *stable* homotopy groups of the symmetric spaces $\Gamma_{(0,\nu)} = \dfrac{SO(2\nu)}{U(\nu)}$ have been computed by R. Bott in his fundamental work [1], while results about *unstable* homotopy groups of $\Gamma_{(0,\nu)}$ have been obtained by various other authors (see, for instance, [6, 15] and [11]). Among the known results, we will use the following:

Proposition 4.5 *The manifold* $\Gamma_{(0,\nu)}$ *is simply connected and* $\pi_2(\Gamma_{(0,\nu)}) \cong \mathbb{Z}$ *for every* $\nu \geq 2$.

We will study some other cases in the next Propositions of this Section.

Proposition 4.6 *Let* $\zeta, \nu_1, \ldots, \nu_s$ *be integers such that* $\zeta \geq 0$, $\nu_1, \ldots, \nu_s \geq 1$ *(*$s \geq 1$*), and assume either* $\zeta \geq 1$ *or* $s \geq 2$*. Then*

a) $\Gamma_{(\zeta;\nu_1,\ldots,\nu_s)}$ *is simply connected;*
b) $\pi_2(\Gamma_{(\zeta;\nu_1,\ldots,\nu_s)})$ *is isomorphic to* $\mathbb{Z}^s$, *if* $\zeta \neq 2$, *while* $\pi_2(\Gamma_{(2;\nu_1,\ldots,\nu_s)})$ *is isomorphic to* $\mathbb{Z}^{s+1}$*;*
c) *if* $\nu_1 = \ldots = \nu_s = 1$ *and* $\zeta = 0, 1, 2,$ *then* $\pi_i(\Gamma_{(\zeta;1,\ldots,1)})$ *is isomorphic to* $\pi_i(SO(\zeta + 2s))$, *for every* $i \geq 3$.

Proof The assumptions about ζ and s imply that $\zeta + 2\nu \geq 3$, where $\nu := \sum_{j=1}^{s} \nu_j$; then $\pi_1(SO(\zeta + 2\nu))$ is a cyclic group of order two. Furthermore, since $\pi_2(SO(\zeta + 2\nu)) = \{0\}$, the final part of the homotopy exact sequence reduces to: $0 \to \pi_2(\Gamma_{(\zeta;\nu_1,\ldots,\nu_s)}) \xrightarrow{\delta} \pi_1\big(SO(\zeta) \oplus (\bigoplus_{h=1}^{s} U(\nu_h))\big) \xrightarrow{\psi} \pi_1(SO(\zeta + 2\nu)) \xrightarrow{\xi} \pi_1(\Gamma_{(\zeta;\nu_1,\ldots,\nu_s)}) \to 0$. In this sequence the homomorphism ψ is induced by the inclusion determined by the decomplexification mapping. Now we set $\phi_1 = 0$, $\phi_j = \sum_{r=1}^{j-1} \nu_r$, for $j = 2, \ldots, s$, and we define, for $j = 1, \ldots, s$, the following loops:
$\alpha_j : t \mapsto I_{\zeta} \oplus I_{\phi_j} \oplus (e^{2\pi t \mathbf{i}}) \oplus I_{(\nu-\phi_j-1)} \in SO(\zeta) \oplus \big(\bigoplus_{h=1}^{s} U(\nu_h)\big)$,

$\beta_j : t \mapsto I_{(2\phi_j+\zeta)} \oplus \begin{pmatrix} \cos(2\pi t) & -\sin(2\pi t) \\ \sin(2\pi t) & \cos(2\pi t) \end{pmatrix} \oplus I_{2(\nu-\phi_j-1)} \in SO(\zeta + 2\nu)$, for every $t \in [0, 1]$. Hence, denoting by $[\alpha_j]$ and $[\beta_j]$ the equivalence classes of the loops α_j and β_j in $\pi_1\big(SO(\zeta) \oplus (\bigoplus_{h=1}^{s} U(\nu_h))\big)$ and $\pi_1(SO(\zeta + 2\nu))$, respectively, we have $\psi([\alpha_j]) = [\beta_j]$, for every $j = 1, \ldots, s$. The mapping ψ is surjective, since $[\beta_1]$ is the generator (of order two) of $\pi_1(SO(\zeta + 2\nu))$; so, by the exactness of the previous sequence, $\pi_1(\Gamma_{(\zeta;\nu_1,\ldots,\nu_s)})$ is the trivial group and therefore (a) is proved.
Moreover, all loops β_j are homotopic to loop β_1. Indeed, if Q_j is a (special orthogonal) permutation matrix such that $Q_j \beta_1(t) Q_j^T = \beta_j(t)$ (for every $t \in [0, 1]$) and $\gamma : [0, 1] \to SO(\zeta + 2\nu)$ is a continuous path joining $I_{(\zeta+2\nu)}$ and Q_j, then the mapping H defined by $H(t, s) = \gamma(s)\beta_1(t)\gamma(s)^T$ (with $t, s \in [0, 1]$) is a homotopy between the loops β_1 and β_j. Therefore $\psi([\alpha_j]) = [\beta_j] = [\beta_1]$, for $j = 1, \ldots, s$.
If $\zeta = 0, 1$, the fundamental group $\pi_1\big(SO(\zeta) \oplus (\bigoplus_{h=1}^{s} U(\nu_h))\big)$ is a free abelian group of rank s and its generators are the homotopy classes of the loops α_j, for $j = 1, \ldots, s$, and we have $\psi(\sum_{j=1}^{s} n_j[\alpha_j]) = (\sum_{j=1}^{s} n_j)[\beta_1]$, for every $n_1, \ldots, n_s \in \mathbb{Z}$.
Furthermore, $\ker\psi = \{\sum_{j=1}^{s} n_j[\alpha_j] : \sum_{j=1}^{s} n_j$ is even$\}$ is a free abelian group, whose rank is less than or equal to $s = \text{rank}\big(\pi_1(SO(\zeta) \oplus (\bigoplus_{h=1}^{s} U(\nu_h)))\big)$. Since the elements $2[\alpha_1], \ldots, 2[\alpha_s] \in \ker\psi$ are linearly independent over $\mathbb{Z}$, it follows that rank$(\ker\psi) = s$, and so, $\pi_2(\Gamma_{(\zeta;\nu_1,\ldots,\nu_s)}) \cong \ker\psi \cong \mathbb{Z}^s$.
If $\zeta \geq 2$, we denote by $\omega : [0, 1] \to SO(\zeta) \oplus \big(\bigoplus_{h=1}^{s} U(\nu_h)\big)$ and $\tilde{\omega} : [0, 1] \to SO(\zeta + 2\nu)$ the loops defined, respectively, as follows:

$\omega(t) = \begin{pmatrix} \cos(2\pi t) & -\sin(2\pi t) \\ \sin(2\pi t) & \cos(2\pi t) \end{pmatrix} \oplus I_{(\zeta-2)} \oplus I_{\nu}$ and $\tilde{\omega}(t) = \begin{pmatrix} \cos(2\pi t) & -\sin(2\pi t) \\ \sin(2\pi t) & \cos(2\pi t) \end{pmatrix} \oplus I_{(2\nu+\zeta-2)}$, for every $t \in [0, 1]$. Since $\psi([\omega]) = [\tilde{\omega}]$, as in the previous case we get $\psi([\omega]) = [\beta_1]$. Furthermore, the elements $[\omega], [\alpha_1], \ldots, [\alpha_s]$ are independent generators of $\pi_1\big(SO(\zeta) \oplus (\bigoplus_{h=1}^{s} U(\nu_h))\big)$.

If $\zeta = 2$, all these elements have infinite order and $\pi_1\big(SO(\zeta) \oplus (\bigoplus_{h=1}^{s} U(\nu_h))\big)$ is a free abelian group of rank $s+1$. Therefore
$\ker\psi = \{n_0[\omega] + \sum_{j=1}^{s} n_j[\alpha_j] : \sum_{j=0}^{s} n_j \text{ is even}\}$ is a free abelian group of rank $\leq s+1$; since $2[\omega], 2[\alpha_1], \ldots, 2[\alpha_s]$ are $\mathbb{Z}$-linearly independent elements of $\ker\psi$, we conclude that $\text{rank}(\ker\psi) = s+1$, and so $\pi_2(\Gamma_{(2;\nu_1,\ldots,\nu_s)}) \cong \ker\psi \cong \mathbb{Z}^{s+1}$.
If $\zeta \geq 3$, we have $\pi_1\big(SO(\zeta) \oplus (\bigoplus_{h=1}^{s} U(\nu_h))\big) \cong \mathbb{Z}_2 \oplus \mathbb{Z}^s$ and hence $\text{rank}\,\big(\pi_1(SO(\zeta) \oplus (\bigoplus_{h=1}^{s} U(\nu_h)))\big) = s$. As before, $2[\alpha_1], \ldots, 2[\alpha_s]$ are $\mathbb{Z}$-linearly independent elements of $\ker\psi \subset \pi_1\big(SO(\zeta) \oplus (\bigoplus_{h=1}^{s} U(\nu_h))\big)$. Therefore
$s \leq \text{rank}(\ker\psi) \leq \text{rank}\big(\pi_1(SO(\zeta) \oplus (\bigoplus_{h=1}^{s} U(\nu_h)))\big) = s$, and so $\text{rank}(\ker\psi) = s$. Note that $\ker\psi$ is a torsion-free finitely generated abelian group. Indeed, $[\omega]$ is the unique non-trivial torsion element of the group $\pi_1\big(SO(\zeta) \oplus (\bigoplus_{h=1}^{s} U(\nu_h))\big)$ and $[\omega] \notin \ker\psi$. Hence $\ker\psi$ is a free abelian group of rank s, and therefore
$\pi_2(\Gamma_{(\zeta;\nu_1,\ldots,\nu_s)}) \cong \ker\psi \cong \mathbb{Z}^s$. So the proof of (b) is complete.
Finally, if $\nu_j = 1$, for every $j = 1, \ldots, s$, and $\zeta = 0, 1, 2$, since we have $\pi_r(U(1)) = \pi_r(SO(1)) = \pi_r(SO(2)) = \{0\}$, for every $r \geq 2$, the sequence:
$0 \to \pi_i(SO(\zeta + 2s)) \xrightarrow{\xi} \pi_i(\Gamma_{(\zeta;1,\ldots,1)}) \to 0$ is exact, for $i \geq 3$; so (c) holds. □

Proposition 4.7 *Let $\nu_1, \ldots, \nu_s$ be positive integers ($s \geq 1$). Then*

a) $\Theta_{(\nu_1,\ldots,\nu_s)}$ *is simply connected;*
b) $\pi_2(\Theta_{(\nu_1,\ldots,\nu_s)})$ *is a free abelian group of rank $s-1$*
c) *if $\nu_1 = \ldots = \nu_s = 1$, then $\pi_i(\Theta_{(1,\ldots,1)})$ is isomorphic to $\pi_i(U(s))$, for every $i \geq 3$. In particular, if $s \geq 2$, the group $\pi_3(\Theta_{(1,\ldots,1)})$ is isomorphic to $\mathbb{Z}$.*

Proof We set $\nu := \sum_{j=1}^{s} \nu_j$. Taking into account that $\pi_2(U(\nu)) = \{0\}$ and arguing as in Proposition 4.6, we obtain (a). Then we have the following short exact sequence:
$0 \to \pi_2(\Theta_{(\nu_1,\ldots,\nu_s)}) \xrightarrow{\delta} \pi_1(\bigoplus_{j=1}^{s} U(\nu_j)) \xrightarrow{\psi} \pi_1(U(\nu)) \to 0.$
The group $\pi_2(\Theta_{(\nu_1,\ldots,\nu_s)})$ is free abelian, since it is a subgroup of the free abelian group $\pi_1(\bigoplus_{j=1}^{s} U(\nu_j)) \cong \mathbb{Z}^s$. Furthermore, the previous short exact sequence splits, because $\pi_1(U(\nu)) \cong \mathbb{Z}$ is a free abelian group. We can therefore conclude that $\pi_2(\Theta_{(\nu_1,\ldots,\nu_s)}) \cong \mathbb{Z}^{s-1}$, and so (b) holds.
Since $\pi_r(U(1)) = \{0\}$, for every $r \geq 2$, the exactness of the sequence
$0 \to \pi_i(U(s)) \xrightarrow{\xi} \pi_i(\Theta_{(1,\ldots,1)}) \to 0$ implies $\pi_i(\Theta_{(1,\ldots,1)}) \cong \pi_i(U(s))$, for $i \geq 3$. The last statement of (c) follows from the fact that $\pi_3(U(s)) \cong \mathbb{Z}$, for all $s \geq 2$. □

Remark 4.8 Using the Kronecker delta and taking into account Proposition 4.3, it is possible to summarize Remark 4.4 and Propositions, 4.5, 4.6 (b), 4.7 (b), by saying that $\pi_2(\widehat{\Gamma}_{(\zeta;\nu_1,\ldots,\nu_s)})$ and $\pi_2(\Gamma_{(\zeta;\nu_1,\ldots,\nu_s)})$ are free abelian groups of rank
$s - \delta_{(\zeta,0)}\delta_{(s,1)}\delta_{(\nu_1,1)} + \delta_{(\zeta,2)}(1 - \delta_{(s,0)})$ (for $\zeta, s \geq 0$, $\zeta + s \geq 1$), while $\pi_2(\widehat{\Theta}_{(\nu_1,\ldots,\nu_s)})$ and $\pi_2(\Theta_{(\nu_1,\ldots,\nu_s)})$ are free abelian groups of rank $s-1$, for every $s \geq 1$. Furthermore, all these homogeneous spaces are simply connected.

5 Some topological properties of $\mathcal{L}og(M)$ and $\mathcal{L}og_{\mathfrak{so}(n)}(Q)$

Theorem 5.1 *Let M be a semi-simple non-singular real matrix, whose eigenvalues are as in Remarks–Definitions 2.2 (a), denote by $2A$ the number of distinct non-real eigenvalues of M, choose any M-admissible set of multi-indices $(\eta, u, \tau, \mu, \sigma, \upsilon)$, and define the sets $\widehat{J}$, K, L as in Remarks–Definitions 2.2 (c).*

If $\mathcal{C}$ is an arbitrary connected component of $\mathcal{L}(M)^{(\eta,\tau,\sigma)}_{(u,\mu,\upsilon)}$, then

a) *$\mathcal{C}$ is simply connected and $\pi_2(\mathcal{C})$ is a free abelian group whose rank is* $\sum_{i=1}^{p} b_i - |K| + |\widehat{J}| + \sum_{l=1}^{r} \sum_{t=1}^{a_l} d_{(l,t)} - A + \sum_{j=1}^{q} c_j - |L|$;

b) *assume that all non-real eigenvalues of $\mathcal{L}(M)_{(u,\mu,v)}^{(\eta,\tau,\sigma)}$ are simple; then the rank of $\pi_2(\mathcal{C})$ is* $\frac{1}{2}\left(n - \sum_{i=1}^{p} g_i\right) - |K| + |\widehat{J}| - A - |L|$; *if, in addition, the multiplicity of all real eigenvalues of $\mathcal{L}(M)_{(u,\mu,v)}^{(\eta,\tau,\sigma)}$ is less than or equal to 2, then, for every $\alpha \geq 3$, $\pi_\alpha(\mathcal{C})$ is isomorphic to the direct sum*
$$\left[\bigoplus_{i=1}^{p} \pi_\alpha\left(SO(h_i)\right)\right] \oplus \left[\bigoplus_{l=1}^{r} \bigoplus_{t=1}^{a_l} \pi_\alpha\left(U(m_{(l,t)})\right)\right] \oplus \left[\bigoplus_{j=1}^{q} \pi_\alpha\left(SO(2k_j)\right)\right].$$

c) *If $\mathcal{W}$ is any connected component of $\mathcal{PL}og(M)$, then $\mathcal{W}$ is simply connected and $\pi_2(\mathcal{W})$ is a free abelian group of rank B, where B is the number of distinct negative eigenvalues of M, the multiplicity of which is greater than or equal to 4.*

Proof By Theorem 2.5 and Remark 4.8, the component $\mathcal{C}$ is simply connected and the rank of the free abelian group $\pi_2(\mathcal{C})$ is
$\sum_{i\in I}\left(b_i - \delta_{(g_i,0)}\delta_{(b_i,1)}\delta_{(u_{(i,1)},1)} + \delta_{(g_i,2)}(1-\delta_{(b_i,0)})\right) + \sum_{l=1}^{r}\sum_{t=1}^{a_l} d_{(l,t)} - \sum_{l=1}^{r} a_l + \sum_{j=1}^{q}\left(c_j - \delta_{(c_j,1)}\delta_{(v_{(j,1)},1)}\right) = \sum_{i=1}^{p} b_i - |K| + |\widehat{J}| + \sum_{l=1}^{r}\sum_{t=1}^{a_l} d_{(l,t)} - A + \sum_{j=1}^{q} c_j - |L|$
where I is the set defined in Remarks–Definitions 2.2 (c). This proves (a).
For part (b), we note that the condition on the non-real eigenvalues of $\mathcal{L}(M)_{(u,\mu,v)}^{(\eta,\tau,\sigma)}$ is equivalent to $u_{(i,x)} = \mu_{(l,t,z)} = v_{(j,y)} = 1$, for any possible choice of indices i, x, l, t, z, j, y, so, under this condition, we have
(•) $h_i = g_i + 2b_i$, $m_{(l,t)} = d_{(l,t)}$, $k_j = c_j$, for all possible indices i, l, t, j.
Hence $\sum_{i=1}^{p} b_i + \sum_{l=1}^{r}\sum_{t=1}^{a_l} d_{(l,t)} + \sum_{j=1}^{q} c_j = \sum_{i=1}^{p} \frac{(h_i - g_i)}{2} + \sum_{l=1}^{r}\sum_{t=1}^{a_l} m_{(l,t)} + \sum_{j=1}^{q} k_j = \frac{1}{2}\left(n - \sum_{i=1}^{p} g_i\right)$. From this and (a), we get the asserted formula for the rank of $\pi_2(\mathcal{C})$. If, in addition, the condition on the real eigenvalues holds, we have $g_i \leq 2$, for $i = 1, \ldots, p$, so the statement about $\pi_\alpha(\mathcal{C})$ ($\alpha \geq 3$) follows from Theorem 2.5 and Propositions 4.3, 4.6 (c), 4.7 (c), taking into account the equalities (•) and the fact that, if $i \notin I$, then $\pi_\alpha(SO(h_i)) = \pi_\alpha(SO(g_i)) = \{0\}$, for every $\alpha \geq 3$.
Part (c) follows from (a), since, in this case, the set I is empty, $d_{(l,t)} = c_j = 1$, for all possible indices l, t, j, $L = \{j : 1 \leq j \leq q, \ k_j = 1\}$ and $B = q - |L|$. □

Theorem 5.2 *Let $Q \in SO(n)$; assume its real Jordan form $\widehat{\mathcal{J}}_Q$ is as in Notations 3.1 (∗) and choose any Q-admissible set of multi-indices $(\eta, u, \tau, \mu, \sigma, v)$. Denote by $\mathcal{B}$ an arbitrary connected component of $\mathcal{L}_{\mathfrak{so}(n)}(Q)_{(u,\mu,v)}^{(\eta,\tau,\sigma)}$. Then*

a) *$\mathcal{B}$ is simply connected and $\pi_2(\mathcal{B})$ is a free abelian group whose rank is*
$b - \delta_{(g,0)}\delta_{(b,1)}\delta_{(u_1,1)} + \delta_{(g,2)}(1-\delta_{(b,0)}) + \sum_{l=1}^{r} d_l - r + c - \delta_{(c,1)}\delta_{(v_1,1)}$;

b) *assume that all non-real eigenvalues of $\mathcal{L}_{\mathfrak{so}(n)}(Q)_{(u,\mu,v)}^{(\eta,\tau,\sigma)}$ are simple and that the multiplicity of 0 as eigenvalue of $\mathcal{L}_{\mathfrak{so}(n)}(Q)_{(u,\mu,v)}^{(\eta,\tau,\sigma)}$ is less than or equal to 2; then, for every $\alpha \geq 3$, $\pi_\alpha(\mathcal{B})$ is isomorphic to the direct sum* $\pi_\alpha\left(SO(h)\right) \oplus \left[\bigoplus_{l=1}^{r} \pi_\alpha\left(U(m_l)\right)\right] \oplus \pi_\alpha\left(SO(2k)\right)$.

c) *If $\mathcal{Z}$ is any connected component of $\mathcal{PL}og_{\mathfrak{so}(n)}(Q)$, then*
◇ *$\mathcal{Z}$ is a single point, if either -1 is not an eigenvalue of Q or it is an eigenvalue of Q of multiplicity 2;*
◇ *$\mathcal{Z}$ is diffeomorphic to a 2-dimensional sphere, if -1 is an eigenvalue of Q of*

multiplicity 4*;*
◇ $\mathcal{Z}$ *is simply connected and* $\pi_2(\mathcal{C})$ *is an infinite cyclic group, if* -1 *is an eigenvalue of* Q *of multiplicity greater than or equal to* 6*.*

Proof Part (a) easily follows from Theorem 3.4 (a) and Remark 4.8. Part (b) follows from Theorem 3.4 (a) and Propositions 4.6 (c), 4.7 (c), since, in this case, we have $g = h - 2b \leq 2$, $c = k$ and $m_l = d_l$, for every $l = 1, \ldots, r$. Part (c) follows from Theorem 3.7 and Proposition 4.5, also remembering that $\Gamma_{(0,2)} \cong S^2$. □

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Declarations

Conflicts of interest The author states that there is no conflict of interest.

References

1. Bott, R.:"The stable homotopy of the classical groups", Ann. Math.(2) 70 , 313–337 (1959)
2. Culver, W.J.: On the existence and uniqueness of the real logarithm of a matrix. Proc. Am. Math Soc. **17**, 1146–1151 (1966)
3. Dolcetti, A., Pertici, D.: "Some remarks on the Jordan-Chevalley decomposition", São Paulo J. Math. Sci, 11, No.2, 385–404 (2017)
4. Dolcetti, A., Pertici, D.:"Skew symmetric logarithms and geodesics on $O_n(\mathbb{R})$", Adv. Geom. 18, No.4, 495–507 (2018)
5. Dolcetti, A., Pertici D.: "Real square roots of matrices: differential properties in semi-simple, symmetric and orthogonal cases", Riv. Mat. Univ. Parma (N.S.), 11, No.2, 315–333 (2020)
6. Harris, B.: Some calculations of homotopy groups of symmetric spaces. Trans. Am. Math. Soc. **106**, 174–184 (1963)
7. Hatcher, A.: Algebraic Topology. Cambridge University Press, Cambridge (2001)
8. Higham, N.J.: Functions of Matrices : Theory and Computation. SIAM Society for Industrial and Applied Mathematics, Phildelphia (2008)
9. Horn, R. A., Johnson, C. R: Matrix Analysis, Second Edition, Cambridge University Press, Cambridge, (2013)
10. Humphreys, J. E.: Linear Algebraic Groups, GTM 21 (Corrected fourth printing), Springer-Verlag, New York-Berlin, (1995)
11. Mukai, J.: Remarks on homotopy groups of symmetric spaces. Math. J. Okayama Univ. **32**, 159–164 (1990)
12. Montgomery, D., Zippin, L.: Topological Transformation Groups. Interscience Tracts in Pure and Applied Mathematics, New York (1955)
13. Nunemacher, J.: "Which real matrices have real logarithms?", Math. Mag. 62, No.2, 132–135 (1989)
14. Orbit. Encyclopedia of Mathematics. http://encyclopediaofmath.org/index.php?title=Orbit_form&oldid=48062(accessed on 17 February) (2022)
15. Oshima, H.:"A homotopy group of the symmetric space SO(2n)/U(n)", Osaka J. Math. 21, No.3, 473–475 (1984)
16. Steenrod, N.E.: The Topology of Fibre Bundles. Princeton University Press, Princeton (1951)

17. Warner, F.W.: Foundations of Differentiable Manifolds and Lie Groups, GTM 94 (Corrected reprint of the 1971 edn.). Springer-Verlag, New York-Berlin (1983)