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Extension and tangential CRF conditions in quaternionic analysis

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Abstract

We prove some extension theorems for quaternionic holomorphic functions in the sense of Fueter. Starting from the existence theorem for the nonhomogeneous Cauchy–Riemann–Fueter problem, we prove that an $\mathbb{H}$-valued function f on a smooth hypersurface in $\mathbb{H}^2$, satisfying suitable tangential conditions, is locally a jump of two $\mathbb{H}$-holomorphic functions. From this, we obtain, in particular, the existence of the solution for the Dirichlet problem with smooth data. We extend these results to the continuous case. In the final part, we discuss the octonion case.

Keywords Quaternionic analysis · Cauchy–Riemann–Fueter operator · $\mathbb{H}$-holomorphic functions · Nonhomogeneous Cauchy–Riemann–Fueter system

Mathematics Subject Classification 30G35

1 Introduction

This paper aims to set forth the methods of complex analysis in the quaternionic analysis in several variables. The main objects of such a theory are the $\mathbb{H}$-*holomorphic* functions, i.e., those functions $f = f(q_1, \dots, q_n)$, $q_1, \dots, q_n \in \mathbb{H}$, which are (left) regular in the sense of Fueter with respect to each variable. For the basic results in the quaternionic analysis in one and several variables, we refer to the articles by Sudbery [22] and Pertici [19], respectively. As for a more geometric aspect of the theory, we refer to the book [11] and the rich bibliography quoted there.

Coming to the content of the paper, we are dealing with the boundary values and extension problems for $\mathbb{H}$-holomorphic functions. As it is well known, this is one of the

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central themes in complex analysis, which motivated the study of overdetermined systems of linear partial differential equations, the CR geometry, and the theory of extension of "holomorphic objects".

For the sake of simplicity, we restrict ourselves to the case $n = 2$, even if most of the main results proved in the paper hold in any dimension.

The paper is organized into three sections.

In Sect. 2, after fixing the main notations, we define the differential forms $\mathrm{d}q_\alpha$, $\mathrm{D}q_\alpha$ that play a fundamental role, and the Cauchy–Riemann–Fueter operator $\overline{\mathfrak{D}}$. As an application of the Cauchy–Fueter formula in one variable [10, 22], we prove a result of "Carleman type" (Proposition 2.1). We also recall the Bochner–Martinelli formula proved in [19], and we show that the Bochner–Martinelli kernel $\mathbf{K}^{\mathrm{BM}}(q, q_0)$ writes as a sum $\mathbf{K}_1^{\mathrm{BM}}(q, q_0) + \mathbf{K}_2^{\mathrm{BM}}(q, q_0)\mathrm{j}$, where $\mathbf{K}_1^{\mathrm{BM}}(q, q_0)$ and $\mathbf{K}_2^{\mathrm{BM}}(q, q_0)$ are complex differential forms and the latter is exact on $\{q \neq q_0\}$, see (2.12).

The section ends with a brief overview of the main results on $\mathbb{H}$-holomorphy, $\mathbb{H}$-convexity [21], and the $\overline{\mathfrak{D}}$-problem [1, 2, 6, 7].

Sections 3 and 4 are the bulk of the paper. In the first part of Sect. 3, using the differential forms $\mathrm{d}q_\alpha$, $\mathrm{D}q_\alpha$, we formulate the CRF condition on a smooth hypersurface S in terms of the tangential operators $\mathrm{D}q_1|_S \wedge \mathrm{d}_{(q_1)}f$, $\mathrm{D}q_2|_S \wedge \mathrm{d}_{(q_2)}f$ (Theorem 3.5). This allows us to give the notion of *admissible function* $f : S \to \mathbb{H}$, which is satisfied by the traces or, more generally, the "jumps" of $\mathbb{H}$-holomorphic functions, as done by the second author in [21]. Admissibility is a second-order condition, so, unlike the complex case, the traces or, more generally, the jumps of $\mathbb{H}$-holomorphic functions satisfy first- and second-order equations. This is not surprising since these problems are related to local solvability of the Cauchy–Riemann–Fueter problem $\overline{\mathfrak{D}}u = g$ and this requires a second-order differential condition for g. The main results of Sect. 3 are Theorems 3.10, and 3.12 reported below.

Let $\Omega \subset \mathbb{H}^2$ be a domain. A *domain splitting* (S, U^+, U^-) of Ω is given by a smooth (nonempty) hypersurface S closed in Ω and two open disjoint nonempty sets U^+, U^-, such that $\Omega \backslash S = U^+ \cup U^-$, where both U^+ and U^- have boundary S in Ω.

We say that a continuous (smooth) function $f : S \to \mathbb{H}$ is a *continuous (smooth) jump* relative to a domain splitting (S, U^+, U^-) of Ω, if there exist two $\mathbb{H}$-holomorphic functions F^+, F^-, on U^+, U^-, respectively, such that F^+, F^- are continuous (smooth) up to S and $f = F^+|_S - F^-|_S$.

Theorem *Let $\Omega \subset \mathbb{H}^2$ be a convex domain and (S, U^+, U^-) a domain splitting of Ω. Let $f : S \to \mathbb{H}$ a smooth admissible function. Then, f is a smooth jump.*

Theorem *Let Σ be an open half-space and $S \subset \mathbb{H}^2$ a connected closed smooth hypersurface of Σ. Assume that $\Sigma \backslash S$ splits into two connected components D and W, with D bounded. Let $f : S \to \mathbb{H}$ be a smooth admissible function. Then, f extends to D by an $\mathbb{H}$-holomorphic function, which is smooth up to S.*

In Sect. 4, we extend the previous results when the function f is admissible in a weak sense.

Finally, in "Appendix", we provide the characteristic conditions for the local solvability of the Cauchy–Riemann–Fueter problem $\overline{\mathfrak{D}}u = g$ in the case of $n = 2$ octonion variables. This allows us to generalize some of our constructions and results to the octonion case.

2 Generalities

In this section, we summarize some of the main notions and results contained in the seminal papers [19–21].

2.1 Fueter operators and $\mathbb{H}$-holomorphic functions

We fix some notations. Let $\mathbb{H}$ be the quaternion algebra over $\mathbb{R}$. For a generic $q \in \mathbb{H}$ we write

$$q = \sum_{\alpha=0}^{3} x_\alpha \mathrm{i}_\alpha, \quad \overline{q} = x_0 - \sum_{\alpha=1}^{3} x_\alpha \mathrm{i}_\alpha$$

$x_\alpha \in \mathbb{R}$, where $\mathrm{i}_0 = 1, \mathrm{i}_1 = \mathrm{i}, \mathrm{i}_2 = \mathrm{j}, \mathrm{i}_3 = \mathrm{k}$.

We also define the following $\mathbb{H}$-valued differential forms

$$\mathrm{d}q = \sum_{\alpha=0}^{3} \mathrm{i}_\alpha \mathrm{d}x_\alpha, \quad \overline{\mathrm{d}q} = \sum_{\alpha=0}^{3} \overline{\mathrm{i}}_\alpha \mathrm{d}x_\alpha \tag{2.1}$$

and

$$\mathrm{D}q = \sum_{\alpha=0}^{3} (-1)^\alpha \mathrm{i}_\alpha \mathrm{d}X_{\widehat{\alpha}}, \quad \overline{\mathrm{D}q} = \sum_{\alpha=0}^{3} (-1)^\alpha \overline{\mathrm{i}}_\alpha \mathrm{d}X_{\widehat{\alpha}}, \tag{2.2}$$

where $\mathrm{d}X_{\widehat{\alpha}} = \mathrm{d}x_0 \wedge \cdots \wedge \widehat{\mathrm{d}x_\alpha} \wedge \cdots \wedge \mathrm{d}x_3$.

Let F be a C^1 $\mathbb{H}$-valued function. Following Fueter, we define the operators

$$\frac{\partial F}{\partial q} = \sum_{\alpha=0}^{3} \overline{\mathrm{i}}_\alpha \frac{\partial F}{\partial x_\alpha}, \quad \frac{\partial F}{\partial \overline{q}} = \sum_{\alpha=0}^{3} \mathrm{i}_\alpha \frac{\partial F}{\partial x_\alpha}. \tag{2.3}$$

We have

$$\Delta F = \frac{\partial}{\partial q}\frac{\partial}{\partial \overline{q}} F = \frac{\partial}{\partial \overline{q}}\frac{\partial}{\partial q} F, \tag{2.4}$$

$$\mathrm{d}(\mathrm{D}q \cdot F) = \frac{\partial F}{\partial \overline{q}} \mathrm{d}x, \tag{2.5}$$

where $\mathrm{d}x = \mathrm{d}x_0 \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3$.

The function F is said to be (*left*) $\mathbb{H}$-*holomorphic* if

$$\frac{\partial F}{\partial \overline{q}} = 0.$$

The function F is said to be (*left*) $\mathbb{H}$-*antiholomorphic* if

$$\frac{\partial F}{\partial q} = 0.$$

Right $\mathbb{H}$-holomorphic and $\mathbb{H}$-antiholomorphic functions are defined interchanging in (2.3) $\partial F/\partial x_\alpha$ with i_α and $\bar{\mathsf{i}}_\alpha$, respectively. For the corresponding derivative, we adopt the notation

$$\frac{F\partial}{\partial\overline{q}}, \quad \frac{F\partial}{\partial q}.$$

For every $q_0 \in \mathbb{H}$, the function

$$G(q-q_0) = \frac{\overline{q}-\overline{q}_0}{|q-q_0|^4}$$

is left and right $\mathbb{H}$-holomorphic.

The function $G(q-q_0)$ is the Cauchy–Fueter kernel and is the main ingredient to prove the basic *Cauchy–Fueter formula*

$$F(q_0) = \frac{1}{2\pi^2}\int_{q\in \mathrm{b}\Omega} G(q-q_o)DqF(q),$$

where Ω is a bounded domain in $\mathbb{H}$ with $\mathrm{b}\,\Omega$ sufficiently smooth, $q_0 \in \Omega$, and $F : \overline{\Omega} \to \mathbb{H}$ a C^1 function which is $\mathbb{H}$-holomorphic in Ω and continuous on $\overline{\Omega}$.

From this formula and Eq. 2.4, one checks immediately that left, right $\mathbb{H}$-holomorphic and $\mathbb{H}$-antiholomorphic functions are harmonic.

For other general results in one quaternionic variable, we refer to [22]. Here we just want to mention the following "Carleman type" result.

Proposition 2.1 *Let Ω be a domain in the ball $B(r) = \{q \in \mathbb{H} : |q| < r\}$ such that $0 \notin \overline{\Omega}$ and $\mathrm{b}\,\Omega = \Gamma \cup \Sigma$, with $\Gamma \subset B(r)$ and $\Sigma \subset \mathrm{b}B(r)$. Let F be an $\mathbb{H}$-holomorphic function on a neighborhood of $\overline{\Omega}$. Then, $F_{|\Omega}$ depends only on $F_{|\Gamma}$.*

Proof Let $q \in \Omega$. By Cauchy–Fueter formula,

$$\begin{aligned} F(q) &= \frac{1}{2\pi^2}\int_{p\in \mathrm{b}\,\Omega} G(p-q)DpF(p) \\ &= \frac{1}{2\pi^2}\int_{p\in\Gamma} G(p-q)DpF(p) + \frac{1}{2\pi^2}\int_{p\in\Sigma} G(p-q)DpF(p). \end{aligned}$$

If $p \in \Sigma$, then $|q| < |p|$ and

$$G(p-q) = \sum_{m=0}^{+\infty}\sum_{\nu\in\sigma_m} P_\nu(q)G_\nu(p),$$

where $\sigma_m = \{(m_1, m_2, m_3) \in \mathbb{N}^3 : m_1 + m_2 + m_3 = m\}$, the P_ν are $\mathbb{H}$-holomorphic polynomials, the functions $G_\nu(p)$ are $\mathbb{H}$-holomorphic in $\mathbb{H}\backslash\{0\}$, and the series is totally convergent with respect to $p \in \Sigma$ (see [22, Proposition 10]).

Since $0 \notin \overline{\Omega}$, by the Cauchy–Fueter theorem (see [10, 1. Hauptsatz]) we have

$$\int_{p\in \mathrm{b}\,\Omega} G_\nu(p)DpF(p) = \int_{p\in\Gamma} G_\nu(p)DpF(p) + \int_{p\in\Sigma} G_\nu(p)DpF(p) = 0,$$

for all ν. It follows that

$$\int_{p\in\Sigma} G(p-q)DpF(p) = \sum_{m=0}^{+\infty}\sum_{\nu\in\sigma_m} P_\nu(q)\int_{p\in\Sigma} G_\nu(p)DpF(p)$$
$$= -\sum_{m=0}^{+\infty}\sum_{\nu\in\sigma_m} P_\nu(q)\int_{p\in\Gamma} G_\nu(p)DpF(p),$$

whence the Carleman formula

$$F(q) = \frac{1}{2\pi^2}\int_{p\in\Gamma} G(p-q)DpF(p) - \frac{1}{2\pi^2}\sum_{m=0}^{+\infty}\sum_{\nu\in\sigma_m} P_\nu(q)\int_{p\in\Gamma} G_\nu(p)DpF(p)$$

proving the statement. □

2.2 Several variables

Fueter operators clearly extend to ($\mathbb{H}$-valued) functions of several quaternionic variables $q_1, q_2, \dots, q_n$.

For the sake of simplicity, from now on we assume $n = 2$, even if the most part of the results proved in the sequel hold for any n.

We denote $q = (q_1, q_2)$ the generic element of $\mathbb{H}^2$ and we set

$$q_1 = \sum_{\alpha=0}^{3} x_\alpha \mathrm{i}_\alpha, \quad q_2 = \sum_{\alpha=0}^{3} y_\alpha \mathrm{i}_\alpha.$$

The Cauchy–Riemann–Fueter operators $\overline{\mathfrak{D}}$ and $\mathfrak{D}$ are then defined, respectively, by

$$F \longmapsto (\partial F/\partial \overline{q}_1, \partial F/\partial \overline{q}_2), \quad F \longmapsto (\partial F/\partial q_1, \partial F/\partial q_2) \tag{2.6}$$

and F is said to be (*left*) $\mathbb{H}$-*holomorphic* if it is C^1 and $\overline{\mathfrak{D}}F = 0$.

We have the identity

$$\begin{aligned}&\frac{1}{2}\left(\overline{\mathrm{d}q_1}\wedge \mathrm{d}q_1 \wedge \mathrm{d}y \wedge \mathrm{d}F + \mathrm{d}x \wedge \overline{\mathrm{d}q_2}\wedge \mathrm{d}q_2 \wedge \mathrm{d}F\right)\\ &= -\left(\overline{\mathrm{D}q_1}\frac{\partial F}{\partial \overline{q}_1}\wedge \mathrm{d}y + \mathrm{d}x \wedge \overline{\mathrm{D}q_2}\frac{\partial F}{\partial \overline{q}_2}\right) + \star \mathrm{d}F,\end{aligned} \tag{2.7}$$

where $\mathrm{d}x = \mathrm{d}x_0 \wedge \cdots \wedge \mathrm{d}x_3$, $\mathrm{d}y = \mathrm{d}y_0 \wedge \cdots \wedge \mathrm{d}y_3$, and $\star$ is the Hodge operator.

In particular, by 2.7, we get that if F is $\mathbb{H}$-holomorphic,

$$\frac{1}{2}\left(\overline{\mathrm{d}q_1}\wedge \mathrm{d}q_1 \wedge \mathrm{d}y \wedge \mathrm{d}F + \mathrm{d}x \wedge \overline{\mathrm{d}q_2}\wedge \mathrm{d}q_2 \wedge \mathrm{d}F\right) = \star \mathrm{d}F. \tag{2.8}$$

Remark 2.2 Formula (2.8) holds, more generally, at those points where $\overline{\mathfrak{D}}F = 0$.

Let Δ_1 (Δ_2) denote the Laplacian in the coordinates x_α (y_α), $\alpha = 0, 1, 2, 3$. Then, if F is $\mathbb{H}$-holomorphic, $\Delta_1 F = \Delta_2 F = 0$. In particular, F is harmonic.

A useful way to construct $\mathbb{H}$-holomorphic functions in one quaternionic variable is to start by (complex) holomorphic functions $F = F(z) = u + \mathrm{i}v$ and define [10, 5. Satz]

$$F^{\#} = F^{\#}(q) := u(\operatorname{Re} q, |\operatorname{Im} q|) + \frac{\operatorname{Im} q}{|\operatorname{Im} q|} v(\operatorname{Re} q, |\operatorname{Im} q|). \tag{2.9}$$

In general, $F^{\#}$ is not $\mathbb{H}$-holomorphic, not even harmonic, but its Laplacian $\Delta F^{\#}$ is.

Example Let $F(z) = z^n$. Then,

$$F^{\#}(q) = (z^n)^{\#} = q^n.$$

In particular, for the cases $n = 3$ and $n = -1$, we get

$$\Delta q^3 = -4(2q + \bar{q}),$$
$$\Delta\left(\left(\frac{1}{z}\right)^{\#}\right) = -4\frac{\bar{q}}{|q|^4} = -4G(q).$$

2.3 Bochner–Martinelli Kernel

The *Bochner–Martinelli Kernel* $\mathbf{K}^{\mathrm{BM}}(q, q_0)$ was introduced in [19], where a representation formula for $\mathbb{H}$-holomorphic functions was proved:

$$F(q_0) = \int_{q \in \mathrm{b}\,\Omega} \mathbf{K}^{\mathrm{BM}}(q, q_0) F(q). \tag{2.10}$$

Here q_0 belongs to a bounded domain Ω in $\mathbb{H}^n$ with smooth boundary $\mathrm{b}\,\Omega$ and F is $\mathbb{H}$-holomorphic in Ω and continuous up to $\mathrm{b}\,\Omega$. We will use the notation $\mathbf{K}^{\mathrm{BM}}(q, q_0)$ instead of the original one.

Set $q_1 = z_1 + w_1\mathrm{j}$, $q_2 = z_2 + w_2\mathrm{j}$, $z_\alpha, w_\alpha \in \mathbb{C}$, $z = (z_1, z_2)$, $w = (w_1, w_2)$.

We use the notation

$$\mathbf{K}^{\mathrm{BM}}(q, q_0) := \mathbf{K}^{\mathrm{BM}}(z, w, z^0, w^0),$$

where $z^0 = (z_1^0, z_2^0)$, $w^0 = (w_1^0, w_2^0)$.

The $\mathbb{H}$-valued differential form $\mathbf{K}^{\mathrm{BM}}(q, q_0)$ is a real analytic form of degree 7 and

$$\mathbf{K}^{\mathrm{BM}}(q, q_0) = \mathbf{K}_1^{\mathrm{BM}}(q, q_0) + \mathbf{K}_2^{\mathrm{BM}}(q, q_0)\mathrm{j}, \tag{2.11}$$

where $\mathbf{K}_1^{\mathrm{BM}}, \mathbf{K}_2^{\mathrm{BM}}$ are real analytic complex-valued differential forms.

Observe that $\mathbf{K}_1^{\mathrm{BM}}(z, w, z_0, w_0)$ is the Bochner–Martinelli kernel for functions which are holomorphic with respect to z_1, z_2 and antiholomorphic with respect to w_1, w_2 and $\mathbf{K}_2^{\mathrm{BM}}(z, w, z_0, w_0)$ is exact on $\mathbb{H}^2 \backslash \{(z^0, w^0)\}$:

$$\mathbf{K}_2^{\mathrm{BM}}(z, w, z_0, w_0) = \mathrm{d}\omega_2 \tag{2.12}$$

where

$$\omega_2 = (8\pi^4)^{-1} \left|(z, w) - (z^0, w^0)\right|^{-6} \cdot$$
$$\left(\mathrm{d}\bar{z}_1 \wedge \mathrm{d}w_1 \wedge \mathrm{d}\bar{z}_2 \wedge \mathrm{d}z_2 \wedge \mathrm{d}\overline{w}_2 \wedge \mathrm{d}w_2 + \right.$$
$$\left. \mathrm{d}\bar{z}_1 \wedge \mathrm{d}z_1 \wedge \mathrm{d}\overline{w}_1 \wedge \mathrm{d}w_1 \wedge \mathrm{d}\bar{z}_2 \wedge \mathrm{d}w_2\right).$$

2.4 $\mathbb{H}$-holomorphy and $\mathbb{H}$-convexity

$\mathbb{H}$-*holomorphy* and $\mathbb{H}$-*convexity* are defined like in the complex case [21]. Kontinuitätssatz holds true [21, Theorem 2], as well as the following implications [21, Proposition 6, Theorem 3]

1. for a domain in $\mathbb{C}^4 \simeq \mathbb{H}^2$, holomorphy implies $\mathbb{H}$-holomorphy. The converse is not true in general (e.g., $\mathbb{H}\backslash\{(0,0)\}$ is a domain of $\mathbb{H}$-holomorphy, but it is not a domain of holomorphy in $\mathbb{C}^2 \simeq \mathbb{H}$);
2. $\mathbb{H}$-holomorphy implies $\mathbb{H}$-convexity;

For domains $\Omega \subset \mathbb{H}^n$, $n > 1$, with smooth boundary $\mathrm{b}\,\Omega$, a necessary condition for the $\mathbb{H}$-holomorphy can be given by the 2^{nd} fundamental form h of $\mathrm{b}\,\Omega$ with respect to the orientation of $\mathrm{b}\,\Omega$ determined by the inward unit normal vector. Precisely [21, Theorem 4],

3. given a point $q_0 \in \mathrm{b}\,\Omega$, there is no right $\mathbb{H}$-line ℓ tangent to $\mathrm{b}\,\Omega$ at q_0 such that $h(q_0)|_\ell < 0$.

In this case, we say that Ω (or its boundary) is *Levi* $\mathbb{H}$-*convex*. For $n = 2$, we say that Ω is *strongly Levi* $\mathbb{H}$-*convex*, if for all $q_0 \in \mathrm{b}\,\Omega$, we have $h(q_0)|_\ell > 0$, where ℓ is the only right $\mathbb{H}$ -line tangent to $\mathrm{b}\,\Omega$ at q_0.

In general, we say that a smooth hypersurface $S \subset \mathbb{H}^n$ is *nondegenerate*, if there exists a right $\mathbb{H}$-line ℓ such that the form $h(q_0)|_\ell$ has constant sign.

Two open problems:

1. Is a domain $\mathbb{H}$-convex a domain of $\mathbb{H}$-holomorphy?
2. Levi problem in $\mathbb{H}^n$.

2.5 $\overline{\mathfrak{D}}$-problem and Hartogs theorem

Let $q = (q_1, q_2) \in \mathbb{H}^2$ with

$$q_1 = \sum_{\alpha=0}^{3} x_\alpha i_\alpha, \quad q_2 = \sum_{\alpha=0}^{3} y_\alpha i_\alpha$$

and consider the Laplacians

$$\Delta_1 = \frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}, \quad \Delta_2 = \frac{\partial^2}{\partial y_0^2} + \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} + \frac{\partial^2}{\partial y_3^2}.$$

Then, since $\partial/\partial\overline{q}_s$ and Δ_h commute we have

$$\frac{\partial}{\partial\overline{q}_s}\frac{\partial}{\partial q_h}\frac{\partial}{\partial\overline{q}_h} = \frac{\partial}{\partial\overline{q}_s}\Delta_h = \Delta_h\frac{\partial}{\partial\overline{q}_s}. \tag{2.13}$$

It follows that, if u is a smooth (local) solution of the CRF *system*

$$\overline{\mathfrak{D}}u = g, \quad g = (g_1, g_2), \tag{2.14}$$

then

$$\Delta_h g_s = \frac{\partial}{\partial \overline{q}_s} \frac{\partial g_h}{\partial q_h}, \tag{2.15}$$

which is a nontrivial condition for $h \neq s$.

For every pair $g = (g_1, g_2)$, we set

$$\begin{aligned} \overline{P}_1(g) &= \frac{\partial}{\partial \overline{q}_1} \frac{\partial g_2}{\partial q_2} - \Delta_2 g_1, \\ \overline{P}_2(g) &= \frac{\partial}{\partial \overline{q}_2} \frac{\partial g_1}{\partial q_1} - \Delta_1 g_2 \end{aligned} \tag{2.16}$$

and denote $\overline{P}$ the operator $g = (g_1, g_2) \mapsto (\overline{P}_1(g), \overline{P}_2(g))$. Then, if $g = \overline{\mathfrak{D}}u$ with u smooth, we have

$$\overline{P}(g) = 0, \tag{2.17}$$

i.e.,

$$\overline{P}_1(g) = 0, \quad \overline{P}_2(g) = 0. \tag{2.18}$$

Conditions (2.15) for $h, s = 1, \dots, n$ are still necessary in order to solve $\overline{\mathfrak{D}}u = g$ for $g = (g_1, \dots, g_n)$. If $g \in C_0^k$, $n, k \geq 2$, they are also sufficient and in such situation $\overline{\mathfrak{D}}u = g$ has a C_0^k solution u (see [20, Theorem 1]). In particular, this implies Hartogs theorem. We point out that Hartogs theorem was already proved by the second author [19, Teorema 6], by solving the equation $\overline{\mathfrak{D}}u = g$ with integral conditions on g, instead of (2.15). As for the system $\overline{\mathfrak{D}}u = g$, when $g \in C^\infty(\Omega, \mathbb{H})$, $\Omega \subset \mathbb{H}^n$, we have the following: if $n = 2$ and Ω is convex, the system has a smooth solution if and only if $\overline{P}(g) = 0$ (see [2]). If $n > 2$, conditions 2.15 are no longer sufficient in general. For $g \in C^\infty(\Omega, \mathbb{H})$, Ω convex, using the results of [1, 7], necessary and sufficient conditions were proved in [6].

Remark 2.3 The same is true if g is replaced by a distribution. This is a consequence of the "division of distributions" [4, 9, 15, 17, 18]. We will use this generalization in Sect. 4.

As far as we know, nothing is known about the equation $\overline{\mathfrak{D}}u = g$ in more general domains.

3 Riemann–Hilbert and Dirichlet problems for $\mathbb{H}$-holomorphic functions.

3.1 The operator $\overline{\mathfrak{D}}_b$ and the CRF condition

Let $\Omega \subset \mathbb{H}^2$ be a domain. A *domain splitting* (S, U^+, U^-) of Ω is given by a smooth (non-empty) hypersurface S closed in Ω and two open disjoint nonempty sets U^+, U^-, such that $\Omega \backslash S = U^+ \cup U^-$, where both U^+ and U^- have boundary S in Ω.

We say that a continuous (resp. smooth[1]) function $f : S \to \mathbb{H}$ is a *continuous (resp. smooth) jump* relative to a domain splitting (S, U^+, U^-) of Ω, if there exist two $\mathbb{H}$-holomorphic functions F^+, F^-, on U^+, U^-, respectively, such that F^+, F^- are continuous (resp. smooth) up to S and $f = F^+|_S - F^-|_S$.

A function $f : S \to \mathbb{H}$ (continuous or smooth) is *locally a jump* if, for every $q_0 \in S$, there exists a neighborhood U of q_0 such that $f|_{U \cap S}$ is a jump in U.

Observe that the functions F^+, F^- are determined up an $\mathbb{H}$-holomorphic function in U. In particular, if S is the boundary of a bounded domain in $\mathbb{H}^2$, Dirichlet problem reduces to Riemann–Hilbert problem via the Hartogs theorem.

Both these problems require conditions on the given function $f : S \to \mathbb{H}$ that we call CRF conditions.

Let S be defined by $\rho = 0$. We say that a smooth function $f : S \to \mathbb{H}$ is a (*left*) *CRF function*, if there is a smooth extension F of f on a neighborhood of S, such that we have

$$\overline{\mathfrak{D}}F = \rho \cdot A + \overline{\mathfrak{D}}\rho \cdot B, \tag{3.1}$$

with A and B smooth. The CRF condition is independent of the extension F, as well as of the equation of S.

The CRF condition can be given in a more intrinsic way, as shown in Theorem 3.5.

Remark 3.1 Observe that, f is a CRF function if and only if there exists a smooth extension F_1 of f with $\overline{\mathfrak{D}}F_1 = 0$ on S. (It is enough to take $F_1 = F - \rho \cdot B$, where F satisfies Eq. 3.1.)

Clearly, if F is an $\mathbb{H}$-holomorphic function on one-sided neighborhood of S, then $F|_S$ is a CRF function, in particular, every local jump f on S is a CRF function.

We will see below that, unlike the complex case, trace conditions on f involve both first-order and second-order differential equations (Remark 3.3).

This is not surprising, due to the fact that Riemann–Hilbert problem is related to local solvability of $\overline{\mathfrak{D}}u = g$ and this requires a second-order differential condition for g.

If $F = U + V\mathrm{j}$ is an extension of f, $q_1 = z_1 + w_1\mathrm{j}$, $q_2 = z_2 + w_2\mathrm{j}$, where U, V, z_1, w_1, z_2, w_2 are complex, then the CRF condition writes

$$\operatorname{rank}\begin{pmatrix} U_{\bar z_1} - \overline{V}_{\bar w_1} & \rho_{\bar z_1} & -\rho_{\bar w_1} \\ \overline{V}_{z_1} + U_{w_1} & \rho_{w_1} & \rho_{z_1} \\ U_{\bar z_2} - \overline{V}_{\bar w_2} & \rho_{\bar z_2} & -\rho_{\bar w_2} \\ \overline{V}_{z_2} + U_{w_2} & \rho_{w_2} & \rho_{z_2} \end{pmatrix} < 3. \tag{3.2}$$

3.1.1 CRF condition and extendability

Suppose S oriented. Denote ω the volume form of S and $\nu = (\nu_1, \nu_2)$, $\nu_1, \nu_2 \in \mathbb{H}$, the unit normal vector which gives the orientation of S.

Let $\langle , \rangle : \mathbb{H}^2 \times \mathbb{H}^2 \to \mathbb{H}$ be the scalar product

[1] For convenience of exposition, since our work reposes in an essential way to the theory of Ehrenpreis and its applications [7, 9], we restrict ourselves to the class of C^∞ functions, even if some definitions and constructions can be given in a more general setting.

$$\langle (q_1,q_2),(p_1,p_2)\rangle = \overline{q}_1 p_1 + \overline{q}_2 p_2.$$

By direct computation, one verifies that

$$\left(\overline{Dq_1}\wedge \mathrm{d}y\right)|_S = -\overline{\nu}_1\omega, \quad \left(\mathrm{d}x\wedge \overline{Dq_2}\right)|_S = -\overline{\nu}_2\,\omega. \tag{3.3}$$

Let $f : S \to \mathbb{H}$ be smooth and F a smooth extension of f on a neighborhood of S. Then, by restriction to S, from (2.7) we get

$$-\frac{1}{2}\left(\overline{\mathrm{d}q_1}\wedge \mathrm{d}q_1\wedge \mathrm{d}y + \mathrm{d}x\wedge \overline{\mathrm{d}q_2}\wedge \mathrm{d}q_2\right)|_S\wedge \mathrm{d}f = \left(-\langle \nu, \overline{\mathfrak{D}}F|_S\rangle + \frac{\partial F}{\partial \nu}\right)\omega, \tag{3.4}$$

where $\overline{\mathfrak{D}}F = \left(\frac{\partial F}{\partial \overline{q}_1}, \frac{\partial F}{\partial \overline{q}_2}\right)$.

Let $f^\perp : S \to \mathbb{H}$ be the smooth function defined by

$$-\frac{1}{2}\left(\overline{\mathrm{d}q_1}\wedge \mathrm{d}q_1\wedge \mathrm{d}y + \mathrm{d}x\wedge \overline{\mathrm{d}q_2}\wedge \mathrm{d}q_2\right)|_S\wedge \mathrm{d}f = f^\perp\cdot\omega \tag{3.5}$$

and set

$$\frac{\partial}{\partial x_\alpha}|_S = \tau_{x_\alpha} + \left(\frac{\partial}{\partial x_\alpha},\nu\right)\nu, \quad \frac{\partial}{\partial y_\alpha}|_S = \tau_{y_\alpha} + \left(\frac{\partial}{\partial y_\alpha},\nu\right)\nu \tag{3.6}$$

$\alpha = 0,1,2,3$, where $(\cdot,\cdot)$ denotes the Euclidean scalar product of $\mathbb{R}^8$ and τ_{x_α}, τ_{y_α} are the tangential components of $\frac{\partial}{\partial x_\alpha}|_S$, $\frac{\partial}{\partial y_\alpha}|_S$, respectively.

We set

$$\begin{aligned} f_{(x_\alpha)} &= \tau_{x_\alpha}(f) + \left(\frac{\partial}{\partial x_\alpha},\nu\right)f^\perp,\\ f_{(y_\alpha)} &= \tau_{y_\alpha}(f) + \left(\frac{\partial}{\partial y_\alpha},\nu\right)f^\perp,\\ f_{(\overline{q}_1)} &= f_{(x_0)} + \mathrm{i}f_{(x_1)} + \mathrm{j}f_{(x_2)} + \mathrm{k}f_{(x_3)},\\ f_{(\overline{q}_2)} &= f_{(y_0)} + \mathrm{i}f_{(y_1)} + \mathrm{j}f_{(y_2)} + \mathrm{k}f_{(y_3)}; \end{aligned} \tag{3.7}$$

they are smooth functions on S.

Proposition 3.2 *Let $f : S \to \mathbb{H}$ be a smooth* CRF *function and F a smooth local extension of f such that $\overline{\mathfrak{D}}F = 0$ on S. Then,*

$$\frac{\partial F}{\partial \nu} = f^\perp, \quad \frac{\partial F}{\partial x_\alpha}|_S = f_{(x_\alpha)}, \quad \frac{\partial F}{\partial y_\alpha}|_S = f_{(y_\alpha)}, \tag{3.8}$$

for $\alpha = 0,1,2,3$.

Proof Since $\overline{\mathfrak{D}}F = 0$ on S and $F|_S = f$, by Eq. 3.4.

$$-\frac{1}{2}\left(\overline{\mathrm{d}q_1}\wedge \mathrm{d}q_1\wedge \mathrm{d}y\wedge \mathrm{d}f + \mathrm{d}x\wedge \overline{\mathrm{d}q_2}\wedge \mathrm{d}q_2\wedge \mathrm{d}f\right)|_S = \frac{\partial F}{\partial \nu}\omega$$

and comparing with (3.5) we then have $\frac{\partial F}{\partial \nu} = f^\perp$. Formulas (3.6) now imply $\frac{\partial F}{\partial x_\alpha}|_S = f_{(x_\alpha)}$, $\frac{\partial F}{\partial y_\alpha}|_S = f_{(y_\alpha)}$, $\alpha = 0,1,2,3$. □

Remark 3.3 If f is the boundary value of an $\mathbb{H}$-holomorphic function F, then, by Proposition 3.2, we get

$$\left.\begin{aligned}\frac{\partial F}{\partial x_\alpha}|_S &= f_{(x_\alpha)}\\ \frac{\partial F}{\partial y_\alpha}|_S &= f_{(y_\alpha)}\end{aligned}\right. \quad \text{for } \alpha = 0, 1, 2, 3.$$

Since the operators $\overline{\mathfrak{D}}$, $\partial/\partial x_\alpha$, $\partial/\partial y_\alpha$ commute, $f_{(x_\alpha)}$ and $f_{(y_\alpha)}$ are restrictions of the $\mathbb{H}$-holomorphic functions $\frac{\partial F}{\partial x_\alpha}$ and $\frac{\partial F}{\partial y_\alpha}$, respectively, hence $f_{(x_\alpha)}$, $f_{(y_\alpha)}$ are CRF functions too.

A smooth CRF function $f : S \to \mathbb{H}$ is said to be *admissible* if $f_{(x_\alpha)}$,$f_{(y_\alpha)}$, $\alpha = 0, 1, 2, 3$, are CRF functions too. Unlike the complex case, a CRF function is not admissible in general. Here is a counterexample:

Example 3.4 Let $S = \{y_3 = 0\}$, $f = -x_1 y_0 \mathrm{j} + x_0 y_0 \mathrm{k}$. Since $\partial f/\partial \bar{q}_1 = 0$, f is CRF. Moreover, $f^\perp = f_{(y_3)} = -x_0 + x_1 \mathrm{i}$. In particular, if $f_{(y_3)}$ were CRF we should have $\partial f_{(y_3)}/\partial \bar{q}_1 = 0$, whereas $\partial f_{(y_3)}/\partial \bar{q}_1 = -2$.

3.1.2 The tangential operator $\overline{\mathfrak{D}}_{\mathrm{b}}$

The CRF condition determines a differential operator on S that will be denoted by $\overline{\mathfrak{D}}_{\mathrm{b}}$. We want to write explicitly the operator $\overline{\mathfrak{D}}_{\mathrm{b}}$.

Consider on S the following $\mathbb{H}$-valued differential forms

$$\begin{aligned}\mathrm{d}_{(q_1)} f &= f_{(x_0)} \mathrm{d}x_0|_S + f_{(x_1)} \mathrm{d}x_1|_S + f_{(x_2)} \mathrm{d}x_2|_S + f_{(x_3)} \mathrm{d}x_3|_S\\ \mathrm{d}_{(q_2)} f &= f_{(y_0)} \mathrm{d}y_0|_S + f_{(y_1)} \mathrm{d}y_1|_S + f_{(y_2)} \mathrm{d}y_2|_S + f_{(y_3)} \mathrm{d}y_3|_S.\end{aligned} \tag{3.9}$$

The following equalities hold

$$\begin{aligned}\mathrm{D}q_1|_S \wedge \mathrm{d}_{(q_1)} f &= -f_{(\bar{q}_1)}\, \mathrm{d}x|_S\\ \mathrm{D}q_2|_S \wedge \mathrm{d}_{(q_2)} f &= -f_{(\bar{q}_2)}\, \mathrm{d}y|_S.\end{aligned} \tag{3.10}$$

We have the following

Theorem 3.5 *For a given smooth function f on S the following conditions are equivalent:*

(a) *f is a* CRF *function;*
(b) $f_{(\bar{q}_1)} \equiv f_{(\bar{q}_2)} \equiv 0$;
(c) $\mathrm{D}q_1|_S \wedge \mathrm{d}_{(q_1)} f \equiv \mathrm{D}q_2|_S \wedge \mathrm{d}_{(q_2)} f \equiv 0$.

Proof Let f be CRF. Then, there exists a smooth extension F of f with the property $\overline{\mathfrak{D}}F = 0$ on S (Remark 3.1). From (3.8), we get

$$\partial F/\partial \nu = f^\perp, \quad \frac{\partial F}{\partial x_\alpha}|_S = f_{(x_\alpha)}, \quad \frac{\partial F}{\partial y_\alpha}|_S = f_{(y_\alpha)},$$

$\alpha = 0, 1, 2, 3$. Consequently

$$\frac{\partial F}{\partial \overline{q}_1}|_S = f_{(\overline{q}_1)}, \quad \frac{\partial F}{\partial \overline{q}_2}|_S = f_{(\overline{q}_2)}. \tag{3.11}$$

By hypothesis, $\overline{\mathfrak{D}}F = 0$ on S hence $f_{(\overline{q}_1)} = f_{(\overline{q}_2)} = 0$ and therefore, by (3.10)

$$\mathrm{D}q_1|_S \wedge \mathrm{d}_{(q_1)}f = \mathrm{D}q_2|_S \wedge \mathrm{d}_{(q_2)}f = 0, \tag{3.12}$$

i.e., (b) and (c).

Assume that f satisfies (c). Then, by (3.10), we have $f_{(\overline{q}_1)}\,\mathrm{d}x|_S \equiv 0$, $f_{(\overline{q}_2)}\,\mathrm{d}y|_S \equiv 0$ and, if $\mathrm{d}x|_S(p) \neq 0$, $\mathrm{d}y|_S(p) \neq 0$, $p \in S$, then $f_{(\overline{q}_1)}(p) = f_{(\overline{q}_2)}(p) = 0$. Suppose, for instance, that $\mathrm{d}x|_S(p) = 0$. Then, the second of (3.3) implies $\nu_2(p) = 0$, i.e., $\nu(p) = \big(\nu_1(p), 0\big)$, where $\nu_1(p) \neq 0$. Thus, $\mathrm{d}y|_S(p) \neq 0$, otherwise [again by (3.3)], we should have $\nu_1(p) = 0$, consequently, $f_{(\overline{q}_2)}(p) = 0$.

Let us show that necessarily $f_{(\overline{q}_1)}(p) = 0$. By standard argument of differential topology, it is easy to construct a smooth extension F of f such that $\partial F/\partial \nu = f^{\perp}$. Identity (3.4) and definition of $f^{\perp}$ then imply that $\langle \nu(p), \overline{\mathfrak{D}}F(p)\rangle = 0$, i.e., $\overline{\nu}_1(p)\partial F/\partial \overline{q}_1(p) = 0$, whence $\partial F/\partial \overline{q}_1(p) = 0$. Arguing as in the first part of the proof, we get $\partial F/\partial \overline{q}_1|_S = f_{(\overline{q}_1)}$, $\partial F/\partial \overline{q}_2|_S = f_{(\overline{q}_2)}$, in particular, also $f_{(\overline{q}_1)}(p) = 0$ for every $p \in S$, and (c) imply (b). Furthermore, $\overline{\mathfrak{D}}F = 0$ on S, hence (c) implies (a) too. □

We denote $\overline{\mathfrak{D}}_{\mathrm{b}}$ the operator

$$\overline{\mathfrak{D}}_{\mathrm{b}} : f \longmapsto \big(\mathrm{D}q_1|_S \wedge \mathrm{d}_{(q_1)}f, \mathrm{D}q_2|_S \wedge \mathrm{d}_{(q_2)}f\big). \tag{3.13}$$

3.2 Solvability of the Riemann–Hilbert problem

We want to prove that for smooth admissible functions the local Riemann–Hilbert problem is always solvable.

We consider an orientable smooth hypersurface S, given as the zero set of a smooth function ρ such that $\nabla\rho \neq 0$ around S.

Proposition 3.6 *Let $f : S \to \mathbb{H}$ be a smooth function. The following properties are equivalent*

1. *f is admissible;*
2. *there exists a smooth extension F of f such that around S one has $\overline{\mathfrak{D}}F = \rho^2 u$, with u a smooth $\mathbb{H}^2$-valued map.*

Proof Let F as in (2). Clearly f is CRF. By Proposition 3.2, we have $\partial F/\partial x_\alpha|_S = f_{(x_\alpha)}$, $\partial F/\partial y_\alpha|_S = f_{(y_\alpha)}$, $\alpha = 0, 1, 2, 3$. Moreover,

$$\begin{aligned} \overline{\mathfrak{D}}\left(\frac{\partial F}{\partial x_\alpha}\right) &= \frac{\partial}{\partial x_\alpha}\left(\overline{\mathfrak{D}}F\right) = \rho\left(2\frac{\partial \rho}{\partial x_\alpha}u + \rho\frac{\partial u}{\partial x_\alpha}\right) \\ \overline{\mathfrak{D}}\left(\frac{\partial F}{\partial y_\alpha}\right) &= \frac{\partial}{\partial y_\alpha}\left(\overline{\mathfrak{D}}F\right) = \rho\left(2\frac{\partial \rho}{\partial y_\alpha}u + \rho\frac{\partial u}{\partial y_\alpha}\right), \end{aligned} \quad \text{for } \alpha = 0, 1, 2, 3. \tag{3.14}$$

Therefore, $\partial F/\partial x_\alpha$ $(\partial F/\partial y_\alpha)$, $\alpha = 0, 1, 2, 3$, is a smooth extension of $f_{(x_\alpha)}$ $(f_{(y_\alpha)})$, whose $\overline{\mathfrak{D}}$ is vanishing on S. It follows that f is admissible.

Assume now that f is admissible, in particular CRF. Therefore, there is a smooth extension G of f and a smooth $\mathbb{H}^2$-valued map ψ such that $\overline{\mathfrak{D}}G = \rho\psi$. Again, by Proposition 3.2, one has $\partial G/\partial x_\alpha|_S = f_{(x_\alpha)}$, $\partial G/\partial y_\alpha|_S = f_{(y_\alpha)}$, $\alpha = 0, 1, 2, 3$. Since also $f_{(x_\alpha)}$ is CRF, there is a smooth extension $F^{(x_\alpha)}$ of $f_{(x_\alpha)}$ such that $\overline{\mathfrak{D}}F^{(x_\alpha)} = \rho\eta^{(x_\alpha)}$ with $\eta^{(x_\alpha)}$ smooth, whence

$$\partial G/\partial x_\alpha - F^{(x_\alpha)} = \rho\psi^{(x_\alpha)} \tag{3.15}$$

with $\psi^{(x_\alpha)}$ smooth, $\alpha = 0, 1, 2, 3$.

Applying $\overline{\mathfrak{D}}$ to 3.15, and taking into account that $\overline{\mathfrak{D}}\circ(\partial/\partial x_\alpha) = (\partial/\partial x_\alpha)\circ\overline{\mathfrak{D}}$, we obtain

$$\frac{\partial(\overline{\mathfrak{D}}G)}{\partial x_\alpha} = \rho H^{(x_\alpha)} + \overline{\mathfrak{D}}\rho \cdot \psi^{(x_\alpha)} \tag{3.16}$$

with $H^{(x_\alpha)}$ smooth, $\alpha = 0, 1, 2, 3$.

In the same way,

$$\frac{\partial(\overline{\mathfrak{D}}G)}{\partial y_\alpha} = \rho H^{(y_\alpha)} + \overline{\mathfrak{D}}\rho \cdot \psi^{(y_\alpha)} \tag{3.17}$$

with $H^{(y_\alpha)}$ smooth, $\alpha = 0, 1, 2, 3$.

Let

$$\nu = \nabla\rho/|\nabla\rho| = |\nabla\rho|^{-1}\sum_{\alpha=0}^{3}\left(\frac{\partial\rho}{\partial x_\alpha}\frac{\partial}{\partial x_\alpha} + \frac{\partial\rho}{\partial y_\alpha}\frac{\partial}{\partial y_\alpha}\right).$$

By hypothesis, $\overline{\mathfrak{D}}G = \rho\psi$, so

$$\frac{\partial(\overline{\mathfrak{D}}G)}{\partial\nu} = \frac{\partial\rho}{\partial\nu}\psi + \rho\frac{\partial\psi}{\partial\nu} = |\nabla\rho|\psi + \rho\frac{\partial\psi}{\partial\nu} \tag{3.18}$$

On the other hand, from (3.16), (3.17), we derive

$$\frac{\partial(\overline{\mathfrak{D}}G)}{\partial\nu} = |\nabla\rho|^{-1}\left\{\rho\sum_{\alpha=0}^{3}A_\alpha + \overline{\mathfrak{D}}\rho \cdot \sum_{\alpha=0}^{3}B_\alpha\right\}. \tag{3.19}$$

Equalizing (3.18) and (3.19), we get

$$\psi = \rho\Phi + 2\overline{\mathfrak{D}}\rho \cdot \Theta \tag{3.20}$$

and consequently

$$\overline{\mathfrak{D}}G = \rho\psi = \rho^2\Phi + \overline{\mathfrak{D}}\rho^2 \cdot \Theta.$$

Then, $F := G - \rho^2\Theta$ is the desired extension of f. □

Lemma 3.7 *Let U be a domain in $\mathbb{H}^2$, $S = \{\rho = 0\}$ where $\rho : U \to \mathbb{H}$ is smooth and $\nabla\rho \neq 0$ on S. Let $\{h_k\}_{k\in\mathbb{N}}$ be a sequence of smooth functions $S \to \mathbb{H}$. Then, there exists a smooth function $E : U \to \mathbb{H}$ with the following properties*

1. $E|_S = h_0$;
2. $\frac{\partial^k E}{\partial\rho^k}|_S = h_k$, $\forall k \geq 1$.

This lemma is a straightforward generalization of [3, Proposition 2.2].

Proposition 3.8 *Let U be a domain in $\mathbb{H}^2$, $S=\{\rho=0\}$ where $\rho : U \to \mathbb{R}$ is smooth and $\nabla\rho \neq 0$ on S. Let $f : S \to \mathbb{H}$ be a smooth and admissible function. Then there are a smooth function $F : U \to \mathbb{H}$ and two sequences $\{\alpha_k\}_{k\geq 2}$, $\{\beta_k\}_{k\geq 2}$ of smooth functions $U \to \mathbb{H}$ and $U \to \mathbb{H}^2$, respectively, satisfying the following conditions:*

1. $F|_S = f$;
2. $(\partial^k \alpha_m/\partial\rho^k)|_S = 0, \forall k \geq 1, m \geq 2$;
3. $\overline{\mathfrak{D}}\big(F - \sum_{k=2}^m (\rho^k/k)\alpha_k\big) = \rho^m \beta_m, \forall m \geq 2.$

Proof Since f is admissible, by Proposition 3.6 there is a smooth extension $F : U \to \mathbb{H}$ of f such that $\overline{\mathfrak{D}}F = \rho^2\sigma$. We construct the sequences by recurrence assuming $\alpha_2 = 0$, $\beta_2 = \sigma$ in such a way that second and third conditions of the proposition are satisfied for $m = 2$.

Suppose that $\alpha_2, \dots, \alpha_m, \beta_2, \dots, \beta_m$ are already constructed in such a way that conditions (2) and (3) are satisfied for all integers $s \leq m$, $k \geq 1$, in order to define α_{m+1} and β_{m+1}.

Set

$$G = F - \sum_{k=2}^{m} (\rho^k/k)\alpha_k, \quad \beta_m = (\zeta_1, \zeta_2) \in \mathbb{H}^2.$$

By definition, $\partial G/\partial\overline{q}_h = \rho^m \zeta_h$, $h = 1, 2$, hence $\overline{\mathfrak{D}}G$ satisfies the condition (2.15), that is

$$\frac{\partial}{\partial\overline{q}_s}\frac{\partial}{\partial q_s}\frac{\partial G}{\partial\overline{q}_h} = \frac{\partial}{\partial\overline{q}_h}\frac{\partial}{\partial q_s}\frac{\partial G}{\partial\overline{q}_s},$$

which gives

$$\frac{\partial}{\partial\overline{q}_s}\left(m\rho^{m-1}\frac{\partial\rho}{\partial q_s}\zeta_h + \rho^m\frac{\partial\zeta_h}{\partial q_s}\right) = \frac{\partial}{\partial\overline{q}_h}\left(m\rho^{m-1}\frac{\partial\rho}{\partial q_s}\zeta_s + \rho^m\frac{\partial\zeta_s}{\partial q_s}\right).$$

Taking into account that ρ is real and $m \geq 2$ we get

$$\begin{aligned} &m(m-1)\rho^{m-2}\frac{\partial\rho}{\partial\overline{q}_s}\frac{\partial\rho}{\partial q_s}\zeta_h + m\rho^{m-1}\frac{\partial}{\partial\overline{q}_s}\left(\frac{\partial\rho}{\partial q_s}\zeta_h\right) + m\rho^{m-1}\frac{\partial\rho}{\partial\overline{q}_s}\frac{\partial\zeta_h}{\partial q_s} + \rho^m\frac{\partial}{\partial\overline{q}_s}\left(\frac{\partial\zeta_h}{\partial q_s}\right) = \\ &m(m-1)\rho^{m-2}\frac{\partial\rho}{\partial\overline{q}_h}\frac{\partial\rho}{\partial q_s}\zeta_s + m\rho^{m-1}\frac{\partial}{\partial\overline{q}_h}\left(\frac{\partial\rho}{\partial q_s}\zeta_s\right) + m\rho^{m-1}\frac{\partial\rho}{\partial\overline{q}_h}\frac{\partial\zeta_s}{\partial q_s} + \rho^m\frac{\partial}{\partial\overline{q}_h}\left(\frac{\partial\zeta_s}{\partial q_s}\right). \end{aligned} \tag{3.21}$$

Summing with respect to s the above equalities and dividing by $m(m-1)$, for fixed $h = 1, 2$ we get

$$\rho^{m-2}|\nabla\rho|^2\zeta_h = \rho^{m-2}\frac{\partial\rho}{\partial\overline{q}_h}\left(\sum_{s=1}^{2}\frac{\partial\rho}{\partial q_s}\zeta_s\right) + \rho^{m-1}l_h \tag{3.22}$$

with $l_h \in \mathrm{C}^\infty(U)$ $h = 1, 2$ whence

$$|\nabla\rho|^2\zeta_h = \frac{\partial\rho}{\partial\overline{q}_h}\left(\sum_{s=1}^{2}\frac{\partial\rho}{\partial q_s}\zeta_s\right) + \rho l_h \tag{3.23}$$

$h = 1, 2$. Since $\nabla\rho \neq 0$ on S, on an open neighborhood $V \subset U$ of S we have

$$\zeta_h = \frac{\partial\rho}{\partial\bar{q}_h} g + \rho\gamma_h$$

$h = 1, 2$, with $g, \gamma_h \in C^\infty(V)$.

Setting $\gamma = (\gamma_1, \gamma_2)$, and recalling that $\beta_m = (\zeta_1, \zeta_2)$, on V we have $\beta_m = (\overline{\mathfrak{D}}\rho)g + \rho\gamma$, so, by the beginning assumption, we derive

$$\begin{aligned}\overline{\mathfrak{D}}\left(F - \sum_{k=2}^{m}(\rho^k/k)\alpha_k\right) &= \rho^m\beta_m = \rho^m\overline{\mathfrak{D}}\rho \cdot g + \rho^{m+1}\gamma \\ &= \overline{\mathfrak{D}}\big(\rho^{m+1}g/(m+1)\big) - \frac{\rho^{m+1}}{m+1}\overline{\mathfrak{D}}g + \rho^{m+1}\gamma.\end{aligned} \tag{3.24}$$

With

$$\theta = \gamma - \overline{\mathfrak{D}}g/(m+1)$$

equation (3.24) rewrites

$$\overline{\mathfrak{D}}\left(F - \sum_{k=2}^{m}(\rho^k/k)\alpha_k - \rho^{m+1}g/(m+1)\right) = \rho^{m+1}\theta. \tag{3.25}$$

Observe that g and θ can be chosen in such a way that an equality like (3.25) holds on U. (It is enough to consider a closed neighborhood $V' \subset V$ of S, a smooth extension of $g|_{V'}$ to U and take θ according to (3.25)).

By Lemma 3.7, there exists a smooth function $\alpha_{m+1} : U \to \mathbb{H}$, such that $\alpha_{m+1}|_S = g|_S$, $\partial^k\alpha_{m+1}/\partial\rho^k|_S = 0$ for every $k \geq 1$. Then, $\alpha_{m+1} - g = \rho\varepsilon$ with $\varepsilon : U \to \mathbb{H}$ smooth and, consequently,

$$\begin{aligned}-\overline{\mathfrak{D}}\left(\frac{\rho^{m+1}g}{m+1}\right) &= \overline{\mathfrak{D}}\left(\frac{\rho^{m+2}\varepsilon}{m+1}\right) - \overline{\mathfrak{D}}\left(\frac{\rho^{m+1}}{m+1}\alpha_{m+1}\right) \\ &= -\overline{\mathfrak{D}}\left(\frac{\rho^{m+1}}{m+1}\alpha_{m+1}\right) + \frac{\rho^{m+1}}{m+1}\Big((m+2)\overline{\mathfrak{D}}\rho \cdot \varepsilon + \rho\overline{\mathfrak{D}}\varepsilon\Big).\end{aligned}$$

If we define

$$\zeta_{m+1} = \theta - \frac{1}{m+1}\Big((m+2)\overline{\mathfrak{D}}\rho \cdot \varepsilon + \rho\overline{\mathfrak{D}}\varepsilon\Big)$$

α_{m+1} and ζ_{m+1} satisfy conditions (2) and (3) of the proposition for $m + 1$. □

Let U, ρ, S be as in Proposition 3.8 and let $G : U \to \mathbb{H}^r$ be a smooth map. We say that G *vanishes of infinite order* on S or that G is *flat* on S if, for any integer k,

$$\lim_{\rho\to 0} G/\rho^k = 0$$

uniformly on the compact sets of U.

Proposition 3.9 *With U, ρ, S as above, let $f : S \to \mathbb{H}$ be a smooth admissible function. Then, there exists a smooth extension G of f to U such that $\overline{\mathfrak{D}}G$ is flat on S.*

Proof By Proposition 3.8, there exist smooth functions $F : U \to \mathbb{H}$, $\alpha_j : U \to \mathbb{H}$, $\beta_j : U \to \mathbb{H}^2$, $(j \geq 2)$ such that

- $F|_S = f$;
- $(\partial^k \alpha_j / \partial \rho^k)|_S = 0, \forall k \geq 1, j \geq 2$;
- $\overline{\mathfrak{D}}\big(F - \sum_{k=2}^m (\rho^k/k)\alpha_k\big) = \rho^m \beta_m, \forall m \geq 2.$

Moreover, by Lemma 3.7, there exists a smooth function $E : U \to \mathbb{H}$ such that $E|_S = 0$ and

$$\frac{\partial^k E}{\partial \rho^k}|_S = \frac{k!}{k+1}\alpha_{k+1}|_S$$

for all $k \geq 1$.

Let $T = \rho E$. Then, since $T|_S = 0$ and

$$\frac{\partial^k T}{\partial \rho^k} = k\frac{\partial^{k-1} E}{\partial \rho^{k-1}} + \rho\frac{\partial^k E}{\partial \rho^k}$$

for $k \geq 1$ in a neighborhood of S, we get

$$\alpha_k|_S = \frac{k}{(k-1)!}\frac{\partial^{k-1} E}{\partial \rho^{k-1}}|_S = \frac{1}{(k-1)!}\frac{\partial^k T}{\partial \rho^k}|_S$$

for all $k \geq 2$ and $\frac{\partial T}{\partial \rho}|_S = E|_S = 0$.

Now, fix a point p of S and let W_p be a neighborhood of p where ρ is one of the real coordinates, say the first, and denote $\xi_1, \dots, \xi_7$ the remaining. Let $\pi : W_p \to W_p \cap S$ denote the projection $(\rho, \xi_1, \dots, \xi_7) \to (0, \xi_1, \dots, \xi_7)$. By what is preceding, we deduce that in W_p, for all $m \geq 2$, the following holds true

$$T - \sum_{k=2}^m \frac{\rho^k}{k}(\alpha_k \circ \pi) = T - \sum_{k=0}^m \frac{\rho^k}{k!}\left(\frac{\partial^k T}{\partial \rho^k}\circ \pi\right) = \rho^{m+1}\zeta$$

with $\zeta : W_p \to \mathbb{H}$ smooth. Consequently,

$$\overline{\mathfrak{D}}\left(T - \sum_{k=2}^m \frac{\rho^k}{k}(\alpha_k \circ \pi)\right) = \rho^m v \tag{3.26}$$

with $v : W_p \to \mathbb{H}^2$ smooth. Moreover, since $(\partial^k \alpha_j / \partial \rho^k)|_S = 0, \forall k \geq 1, j \geq 2$, we get

$$\sum_{k=2}^m \frac{\rho^k}{k}\big(\alpha_k - \alpha_k \circ \pi\big) = \rho^{m+1}\theta,$$

$\theta : W_p \to \mathbb{H}$ smooth. It follows that

$$\overline{\mathfrak{D}}\left(\sum_{k=2}^m (\rho^k/k)\alpha_k - \sum_{k=2}^m (\rho^k/k)(\alpha_k \circ \pi)\right) = \rho^m u, \tag{3.27}$$

where $u : W_p \to \mathbb{H}^2$ is smooth.

Finally, we define $G = F - T$. Clearly $G|_S = f$ and by (3.26), (3.27) we get

$$\begin{aligned}\overline{\mathfrak{D}}G =&\overline{\mathfrak{D}}\left(F - \sum_{k=2}^{m}(\rho^k/k)\alpha_k\right) + \overline{\mathfrak{D}}\left(\sum_{k=2}^{m}(\rho^k/k)\alpha_k - \sum_{k=2}^{m}(\rho^k/k)(\alpha_k\circ\pi)\right)\\ &-\overline{\mathfrak{D}}\left(T - \sum_{k=2}^{m}(\rho^k/k)(\alpha_k\circ\pi)\right)\\ =&\rho^m\left(\beta_m + u - v\right) = \rho^m w_p.\end{aligned}$$

Here $w_p : W_p \to \mathbb{H}^2$ is smooth and uniquely determined by the condition $\overline{\mathfrak{D}}G = \rho^m w_p$. If $p \notin S$, we take W_p such that $W_p \cap S = \emptyset$ and $w_p = \overline{\mathfrak{D}}G/\rho^m$. Therefore, the family of the local maps w_p defines a smooth map $w_m : U \to \mathbb{H}^2$ such that $\overline{\mathfrak{D}}G = \rho^m w_m$ for every integer $m \geq 2$, i.e., G is a smooth extension of f to U such that $\overline{\mathfrak{D}}G$ is flat on S.

This proves Proposition 3.9. □

We apply Proposition 3.9 in order to prove that the local Riemann–Hilbert problem is always solvable. This will follow from the following

Theorem 3.10 *Let $\Omega \subset \mathbb{H}^2$ be a convex domain and (S, U^+, U^-) a domain splitting of Ω. Let $f : S \to \mathbb{H}$* be *a smooth admissible function. Then, f is a smooth jump.*

Proof Observe that S is orientable, so S is defined by $\rho = 0$, where $\rho \in C^\infty(\Omega)$. Let $G : \Omega \to \mathbb{H}$ be a smooth extension of f, with $\overline{\mathfrak{D}}G$ flat on S (Proposition 3.9). Define $\eta : \Omega \to \mathbb{H}^2$ by

$$\eta = \begin{cases} -\overline{\mathfrak{D}}G & \text{on } U^+\\ 0 & \text{on } S\\ \overline{\mathfrak{D}}G & \text{on } U^-.\end{cases}$$

η is smooth in Ω, since $\overline{\mathfrak{D}}G$ is flat on S. Set $\eta = (\eta_1, \eta_2)$. Then, the conditions

$$\Delta_1\eta_2 = \frac{\partial}{\partial\bar{q}_2}\frac{\partial\eta_1}{\partial q_1}, \qquad \Delta_2\eta_1 = \frac{\partial}{\partial\bar{q}_1}\frac{\partial\eta_2}{\partial q_2}$$

are satisfied on $U^+ \cup U^-$ [see (2.17)] whence on Ω. Since Ω is convex, there exists $\psi : \Omega \to \mathbb{H}$ smooth such that $\overline{\mathfrak{D}}\psi = \eta$ [2]. Defining $F^+ = (\psi + G)/2$, $F^- = (\psi - G)/2$, we have the following: F^+ and F^- are smooth up to S, $\overline{\mathfrak{D}}F^+ = 0$ ($\overline{\mathfrak{D}}F^- = 0$) in U^+ (U^-) and $F^+|_S - F^-|_S = f$.

This ends the proof of Theorem 3.10. □

3.2.1 Two applications

Theorem 3.11 *Let Ω be a bounded domain with connected smooth boundary* $\mathrm{b}\,\Omega$. *Then, every smooth admissible function $f : \mathrm{b}\,\Omega \to \mathbb{H}$ extends to Ω by an $\mathbb{H}$-holomorphic function, smooth up to* $\mathrm{b}\,\Omega$.

Proof In our hypothesis, $\mathbb{H}^2\backslash\overline{\Omega}$ is connected with boundary $\mathrm{b}\,\Omega$. Since $(\mathrm{b}\,\Omega, \Omega, \mathbb{H}^2\backslash\overline{\Omega})$ is a domain splitting of $\mathbb{H}^2$, by Theorem 3.10, $f = F^+|_S - F^-|_S$, where F^+, F^- are $\mathbb{H}$-holomorphic. By Hartogs' theorem F^- extends to all of $\mathbb{H}^2$ by an $\mathbb{H}$-holomorphic function $\widehat{F}^-$. And this implies that f is the boundary value of $F^+ - \widehat{F}^-$. □

Theorem 3.12 *Let Σ be an open half-space and $S \subset \mathbb{H}^2$ a connected closed smooth hypersurface of Σ. Assume that $\Sigma\backslash S$ splits into two connected components D and W with D bounded. Let $f : S \to \mathbb{H}$ be a smooth admissible function. Then, f extends to D by an $\mathbb{H}$-holomorphic function F which is smooth up to S.*

Proof Without loss of generality, we can assume that Σ be the half-space $\{y_3 > 0\}$. Let B be an open ball centered at origin such that S divides $B \cap \Sigma$ into two connected components U^+ and $U^- = D$ and D is relatively compact in B. By Theorem 3.10, there are $\mathbb{H}$-holomorphic functions $F^+ : U^+ \to \mathbb{H}$, $F^- : D \to \mathbb{H}$, smooth up to S, such that $f = F^+|_S - F^-|_S$. It is enough to show that F^+ extends $\mathbb{H}$-holomorphically to $B \cap \{y_3 > 0\}$. We may assume that F^+ is defined on an neighborhood of $\mathrm{b}\,B \cap \Sigma$ in Σ.

Fix $\varepsilon > 0$ sufficiently small. For every $c > 0$, let S_c be the sphere centered at $(0, -c\mathrm{k})$ and passing through $\mathrm{b}\,\overline{B} \cap \{\underline{y_3} = \varepsilon\}$. Consider the set $\mathcal{C}$ of $c \in \mathbb{R}$ such that $F^+|_{S_c\cap\overline{B}}$ extends to a neighborhood of $\overline{S}_c \cap \overline{B}$ in $\overline{B}$. We have $\mathcal{C} \neq \emptyset$. Let $c_0 = \sup \mathcal{C}$, and assume by contradiction that c_0 is finite. Observe that F^+ is defined in a neighborhood of $\mathrm{b}B \cap \{y_3 = \varepsilon\}$ in $\overline{B}$. Consider B_{c_0}, the open ball having S_{c_0} as its boundary and let $U = B\backslash\overline{B}_{c_0}$. Then, the second fundamental form of $S_{c_0} \cap \Sigma$ (as part of the boundary of U) is negative definite. Hence, as in the proof of Theorem 4 of [21], we get that, for every $q_0 \in S_{c_0} \cap \overline{B}$, there exists a domain $\Delta_{q_0} \subset U$ such that every $\mathbb{H}$-holomorphic function in Δ_{q_0} extends to a bigger domain $\widehat{\Delta}_{q_0}$ containing q_0. It follows that F^+ extends $\mathbb{H}$-holomorphically to a neighborhood of $S_{c_0} \cap \overline{B}$ in $\overline{B}$: contradiction. This means that $c_0 = +\infty$, thus F^+ extends to $B \cap \{y_3 > \varepsilon\}$, for every ε near 0^+. By analytic continuation (see Theorem 3.14), this completes the proof. □

Remark 3.13 With the same notations of the above theorem, let F be the $\mathbb{H}$-holomorphic extension of f. If $|f|$ is bounded on S, then for every $q \in D$

$$|F(q)| \leq \sup_S |f|.$$

We mention that, an extension theorem of different type has been recently found by Baracco, Fassina and Pinton [5].

Let S be a connected smooth hypersurface in $\mathbb{H}^2$. We say that the *analytic continuation principle* holds for smooth admissible functions on S when the following is true: if $f : S \to \mathbb{H}$ is a smooth admissible function which vanishes on a nonempty open set of S, then $f \equiv 0$.

Theorem 3.14 *Let S be a connected smooth hypersurface in $\mathbb{H}^2$. Then, the analytic continuation principle for smooth admissible functions holds on S in the following two cases:*

1. *S is the boundary of a domain $\Omega \Subset \mathbb{H}^2$ satisfying the hypothesis of Theorem 3.11;*
2. *S is nondegenerate.*

Proof (1) Consider a smooth admissible function f on S, F its $\mathbb{H}$-holomorphic extension on Ω, and let $Z = \{f = 0\}$. Let $q_0 \in \mathring{Z}$ and U be a neighborhood of q_0 relatively compact in $\mathring{Z}$.

Then there exists a domain Ω_1 with smooth boundary, satisfying the hypothesis of Theorem 3.11, such that $\Omega \subset \Omega_1$, $\mathrm{b}\,\Omega_1 \backslash \mathrm{b}\,\Omega \subset \mathbb{H}^2 \backslash \overline{\Omega}$, and $\mathrm{b}\,\Omega \backslash U = \mathrm{b}\,\Omega_1 \cap \mathrm{b}\,\Omega$. The function f_1 on $\mathrm{b}\,\Omega_1$ that coincides with f on $\mathrm{b}\,\Omega_1 \cap \mathrm{b}\,\Omega$ and is zero elsewhere, is smooth admissible and, by Theorem 3.11, extends to an $\mathbb{H}$-holomorphic function F_1 on Ω_1, smooth up to the boundary. By the Bochner–Martinelli formula, it follows immediately that F_1 is an extension of F. By construction, F_1 vanishes on the boundary of $\Omega_1 \backslash \Omega$, and then, F_1 vanishes on $\Omega_1 \backslash \Omega$. By analytic continuation, $F_1 \equiv 0$ and therefore $F \equiv 0$ and $f \equiv 0$ too.

(2) Let f be a smooth admissible function on $S = \{\rho = 0\}$ and let $Z = \{f = 0\}$. Assume f is not identically zero. By Theorem 6 [21], there exists a neighborhood U of S in, say, $\{\rho \le 0\}$, such that the function f extends by an $\mathbb{H}$-holomorphic function F. Take a point $p \in S$, there exists a domain $\Omega \subset \{\rho < 0\}$, whose boundary contains p, such that the set $\mathrm{b}\,\Omega \cap Z$ has interior points in S, and the hypothesis of (1) holds for Ω. Using (1), $F|_{\mathrm{b}\,\Omega} = 0$, in particular, $f(p) = 0$. This concludes the proof, p being a generic point of S. □

Remark 3.15 The analytic continuation principle does not hold for an arbitrary smooth hypersurface S. For instance, all smooth functions $f = f(y_0, y_1, y_2)$ are admissible on $S = \{y_3 = 0\}$.

4 The CRF condition in weak form

In order to treat the Riemann–Hilbert problem (in particular the boundary problem) for $\mathbb{H}$-holomorphic functions with continuous boundary data we need to give the CRF conditions in a weak form. We need some preliminaries.

Let (S, U^+, U^-) be a domain splitting of a domain Ω in $\mathbb{R}^n$ and $\varrho \in \mathrm{C}^\infty(\Omega)$ such that

$$S = \{\rho = 0\}, \quad U^+ = \{\rho > 0\}, \quad U^- = \{\rho < 0\}, \quad \nabla\rho|_S \neq 0$$

and consider on S the orientation determined on the boundary of U^+ by the inward normal vector.

By the existence of tubular neighborhoods, we may assume that for $-\varepsilon_0 < \varepsilon < \varepsilon_0$, $\varepsilon_0 > 0$, the hypersurface $S_\varepsilon = \{\varrho = \varepsilon\}$ is diffeomorphic to S by a diffeomorphism $\pi_\varepsilon : S_\varepsilon \to S$.

Let T be a distribution on S. We say that T is the *trace* or the *boundary value* (in the sense of distributions) of a function $u \in \mathrm{L}^1_{\mathrm{loc}}(U^+)$ following $\{S_\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ if

$$\lim_{\varepsilon \to 0^+} \int_{S_\varepsilon} \pi^*_\varepsilon(\phi) u = (T, \phi)$$

for every real-valued test form ϕ of class C^∞_0 on S of degree $n-1$. In such a situation, we set $\gamma^+(u) = T$.

In the same manner, we give the notion of trace $\gamma^-(u)$ if $u \in \mathrm{L}^1_{\mathrm{loc}}(U^-)$.

The following result was proved in [12, Corollary I. 2. 6]. Let $P(D)$ be a linear elliptic operator on U^+ with smooth coefficients and $u \in \mathrm{C}^\infty(U^+)$ a solution of the equation $P(D)u = 0$. Then u has a boundary value $\gamma^+(u)$ if and only if u extends as distribution through S.

Now we are in position to state the CRF condition in a weak form.

Proposition 4.1 *Let $f : S \to \mathbb{H}$ be continuous function and $T_1 = T_{1f}$, $T_2 = T_{2f}$ the distributions on Ω (supported by S) defined by*

$$\begin{cases} \phi \mapsto \int_S \mathrm{D}q_1 \wedge f\phi \,\mathrm{d}y \\ \phi \mapsto \int_S \mathrm{D}q_2 \wedge f\phi \,\mathrm{d}x, \end{cases} \tag{4.1}$$

(where $\phi \in \mathrm{C}_0^\infty(\Omega)$ is a real-valued test function). If f is locally a jump of $\mathbb{H}$-holomorphic functions continuous up to S, then the system

$$\begin{cases} \dfrac{\partial v}{\partial \overline{q}_1} = T_1 \\ \dfrac{\partial v}{\partial \overline{q}_2} = T_2 \end{cases} \tag{4.2}$$

is locally solvable along S.

Proof Let $q_0 \in S$ and $F^\pm$ $\mathbb{H}$-holomorphic functions in $U^\pm$, smooth up to S such that $f|_{U\cap S} = F^+|_S - F^-|_S$. Let

$$F = \begin{cases} -F^+ & \text{in } U^+ \\ -F^- & \text{in } U^- \end{cases}$$

and denote by the same letter F the distribution

$$\phi \mapsto \int_\Omega \phi\, F \,\mathrm{d}x \wedge \mathrm{d}y \;:$$

here ϕ is a real-valued test function. Then

$$\begin{aligned} \frac{\partial F}{\partial \overline{q}_1}(\phi) &= -\int_\Omega \phi_{\overline{q}_1} F \,\mathrm{d}x \wedge \mathrm{d}y \\ &= \int_{U^+} \phi_{\overline{q}_1} F^+ \mathrm{d}x \wedge \mathrm{d}y + \int_{U^-} \phi_{\overline{q}_1} F^- \mathrm{d}x \wedge \mathrm{d}y. \end{aligned} \tag{4.3}$$

Denote d_x (d_y) the differential with respect to the x (y)-variables. Then, since $\mathrm{D}q_1$ is closed, we have

$$\begin{aligned} \int_{U^+} \phi_{\overline{q}_1} F^+ \mathrm{d}x \wedge \mathrm{d}y &= \int_{U^+} \mathrm{d}_x(\mathrm{D}q_1 \phi) F^+ \wedge \mathrm{d}y \\ &= \int_{U^+} \mathrm{d}_x(\mathrm{D}q_1 \cdot F^+ \cdot \phi) \wedge \mathrm{d}y - \int_{U^+} \mathrm{d}_x(\mathrm{D}q_1 \cdot F^+)\phi \wedge \mathrm{d}y \\ &= \int_{U^+} \mathrm{d}(\mathrm{D}q_1 \cdot F^+ \phi \wedge \mathrm{d}y) - \int_{U^+} \mathrm{d}_x(\mathrm{D}q_1 \cdot F^+)\phi \wedge \mathrm{d}y \\ &= \int_S \mathrm{D}q_1 \cdot F^+ \phi \wedge \mathrm{d}y - \int_{U^+} \frac{\partial F^+}{\partial \bar{q}_1} \phi \,\mathrm{d}x \wedge \mathrm{d}y \\ &= \int_S \mathrm{D}q_1 \cdot F^+ \phi \wedge \mathrm{d}y \end{aligned}$$

by (2.5) and the $\mathbb{H}$-holomorphy of F^+.

In the same manner,

$$\int_{U^-} \phi_{\overline{q}_1} F^- \mathrm{d}x \wedge \mathrm{d}y = -\int_S \mathrm{D}q_1 \cdot F^- \phi \wedge \mathrm{d}y$$

($\mathrm{b}\, U^- \cap S = -\mathrm{b}\, U^+ \cap S$), whence

$$\begin{aligned} \frac{\partial F}{\partial \overline{q}_1}(\phi) &= \int_S \mathrm{D}q_1 \cdot F^+ \phi \wedge \mathrm{d}y - \int_S \mathrm{D}q_1 \cdot F^- \phi \wedge \mathrm{d}y \\ &= \int_S \mathrm{D}q_1 \cdot f \phi \wedge \mathrm{d}y. \end{aligned} \tag{4.4}$$

Analogously, we get

$$\frac{\partial F}{\partial \overline{q}_2}(\phi) = \int_S \mathrm{D}q_2 \cdot f \phi \wedge \mathrm{d}x. \tag{4.5}$$

Equations (4.4) and (4.5) show that the distribution F is a solution of (4.2).

If f is only continuous, we approximate S by hypersurfaces S_ε, with $0 < |\varepsilon| < \varepsilon_0$, and we apply the previous argument to F^+ on $\{\rho \geq \varepsilon\}$ when $\varepsilon > 0$, and F^- on $\{\rho \leq \varepsilon\}$ when $\varepsilon < 0$. Hence, by taking the limits, identities (4.4) and (4.5) hold in the continuous case too.

□

From Proposition and (2.15), it follows

Corollary 4.2 *If f is a jump of $\mathbb{H}$-holomorphic functions continuous up to S, then in Ω we have*

$$\begin{aligned} &\frac{\partial}{\partial \overline{q}_1} \frac{\partial T_2}{\partial q_2} - \Delta_2 T_1 = 0 \\ &\frac{\partial}{\partial \overline{q}_2} \frac{\partial T_1}{\partial q_1} - \Delta_1 T_2 = 0 \end{aligned} \tag{4.6}$$

in the distribution sense, i.e.,

$$\begin{aligned} &\int_S \left[\left(\frac{\partial}{\partial \overline{q}_1} \frac{\partial}{\partial q_2} \phi \right) \mathrm{d}x \wedge \mathrm{D}q_2 - (\Delta_2 \phi) \mathrm{d}y \wedge \mathrm{D}q_1 \right] f = 0 \\ &\int_S \left[\left(\frac{\partial}{\partial \overline{q}_2} \frac{\partial}{\partial q_1} \phi \right) \mathrm{d}y \wedge \mathrm{D}q_1 - (\Delta_1 \phi) \mathrm{d}x \wedge \mathrm{D}q_2 \right] f = 0 \end{aligned} \tag{4.7}$$

for every $\phi \in \mathrm{C}_0^\infty(\Omega)$.

We say that a continuous function $f : S \to \mathbb{H}$ is a *weakly admissible function* if it satisfies (4.7).

We have the

Theorem 4.3 *A continuous function $f : S \to \mathbb{H}$ is locally a jump of $\mathbb{H}$-holomorphic functions, continuous up to S if and only if it is a weakly admissible function. In particular, assume that S is the connected boundary of a bounded domain Ω. Then, every continuous weakly admissible function $f : \mathrm{b}\,\Omega \to \mathbb{H}$ extends to Ω by an $\mathbb{H}$-holomorphic function which is continuous up to* $\mathrm{b}\,\Omega$.

Proof We only have to prove that if f is weakly admissible then it is locally a jump of $\mathbb{H}$-holomorphic functions continuous up to S.

Let $q_0 \in S$, and $(S \cap \Omega, U^+, U^-)$ be a splitting domain of a convex domain Ω containing q_0. Since f is weakly admissible and Ω is convex, by (4.6) and Remark 2.3 there exists a distribution F in Ω such that

$$\begin{cases} \dfrac{\partial F}{\partial \overline{q}_1} = T_1 \\ \dfrac{\partial F}{\partial \overline{q}_2} = T_2. \end{cases} \tag{4.8}$$

Since $\frac{\partial F}{\partial \overline{q}_1}|_{\Omega \setminus S} = \frac{\partial F}{\partial \overline{q}_2}|_{\Omega \setminus S} = 0$, F is $\mathbb{H}$-holomorphic on $\Omega \setminus S$.

Let $F^\pm = F|_{U^\pm}$. Since $F^\pm$ are pluriharmonic, the results of [12] apply. In particular, $F^\pm$ have traces $\gamma(F^\pm)$ on S in the sense of distributions and $\gamma(F^+) - \gamma(F^-) = f$ [12, Corollaire I.2.6 and Théorème II.1.3].

Let $V^\pm$ be domains with smooth boundary such that $V^\pm \Subset \Omega$, $V^\pm \subset U^\pm$ and $(\mathrm{b}V^+ \cap S) = (\mathrm{b}V^- \cap S)$ is a relative neighborhood S_0 of q_0 in S. Again, by [12, Corollaire I.2.6], $F^\pm$ have traces $\theta^\pm$ on $\mathrm{b}V^\pm$ in the sense of distributions, and, by the Bochner–Martinelli formula for $\mathbb{H}$-holomorphic functions, we have

$$F^\pm(q) = \langle \theta^\pm, \mathbf{K}^{\mathrm{BM}}(\cdot, q)\rangle$$

for $q \in V^\pm$.

Let $\psi \in \mathrm{C}_0^\infty(S_0)$ such that $\psi = 1$ on a neighborhood of q_0. Then,

$$\left\langle \psi\theta^+, \mathbf{K}^{\mathrm{BM}}(\cdot, q)\right\rangle + \left\langle (1-\psi)\theta^+, \mathbf{K}^{\mathrm{BM}}(\cdot, q)\right\rangle = \begin{cases} F^+(q) & q \in V^+ \\ 0 & q \notin \overline{V}^+ \end{cases}$$
$$\left\langle \psi\theta^-, \mathbf{K}^{\mathrm{BM}}(\cdot, q)\right\rangle + \left\langle (1-\psi)\theta^-, \mathbf{K}^{\mathrm{BM}}(\cdot, q)\right\rangle = \begin{cases} F^-(q) & q \in V^- \\ 0 & q \notin \overline{V}^- . \end{cases}$$

The functions $\langle (1-\psi)\theta^\pm, \mathbf{K}^{\mathrm{BM}}(\cdot, q)\rangle$, $\langle \psi\theta^\pm, \mathbf{K}^{\mathrm{BM}}(\cdot, q)\rangle$ are smooth near q_0 and, since $\theta^+ - \theta^- = f$, we have

$$\begin{aligned} F^+(q) &= \langle \psi f, \mathbf{K}^{\mathrm{BM}}(\cdot, q)\rangle + \langle \psi\theta^-, \mathbf{K}^{\mathrm{BM}}(\cdot, q)\rangle + \langle (1-\psi)\theta^+, \mathbf{K}^{\mathrm{BM}}(\cdot, q)\rangle \\ F^-(q) &= -\langle \psi f, \mathbf{K}^{\mathrm{BM}}(\cdot, q)\rangle + \langle \psi\theta^+, \mathbf{K}^{\mathrm{BM}}(\cdot, q)\rangle + \langle (1-\psi)\theta^-, \mathbf{K}^{\mathrm{BM}}(\cdot, q)\rangle \end{aligned}$$

and consequently

$$\begin{aligned} F^+(q) &= \int_{\mathrm{b}V^+} \psi f \mathbf{K}^{\mathrm{BM}}(\cdot, q) + u(q) \\ F^-(q) &= -\int_{\mathrm{b}V^-} \psi f \mathbf{K}^{\mathrm{BM}}(\cdot, q) + v(q) \end{aligned}$$

with $u = u(q)$, $v = v(q)$ smooth near q_0. Now, as a consequence of the classical potential theory [16], $F^\pm$ are continuous up to the boundary and this concludes the proof of the general case. In the particular case when S is the boundary of Ω, the proof runs as in the smooth case of Theorem 3.11. □

Remark 4.4 Theorem 3.12 also generalizes.

Remark 4.5 A smooth function $f : S \to \mathbb{H}$ is weakly admissible if and only if it is admissible.

Proof First assume that f is C^∞ and weakly admissible. By Remark 3.3, the functions $F^\pm$ of the previous Proposition are smooth up to the boundary S, hence f is admissible.

Next, assume f is C^∞ and admissible, then $f = F^+ - F^-$. Since $F^\pm$ are smooth up to S, then f is weakly admissible. □

Let S be a connected smooth hypersurface in $\mathbb{H}^2$. We say that the *analytic continuation principle* holds for weakly admissible functions on S when the following is true: if $f : S \to \mathbb{H}$ is a continuous weakly admissible function which vanishes on an nonempty open set of S, then $f \equiv 0$.

Theorem 4.6 *Let S be a connected smooth hypersurface in $\mathbb{H}^2$. Then, analytic continuation principle holds for weakly admissible functions on S in the following two cases:*

(i) *S is the boundary of a domain $\Omega \Subset \mathbb{H}^2$ satisfying the hypothesis of Theorem 3.11;*
(ii) *S is nondegenerate.*

Proof The proof is analogous to the one of Theorem 3.14 using Theorem 4.3 instead of Theorem 3.11. □

Proposition 4.7 *Let $S = \{\rho = 0\}$ be a smooth hypersurface in $\mathbb{H}^2$, $\nabla\rho \neq 0$ on S, and $\Omega^- = \{\rho < 0\}$. Assume that Ω^- is strongly Levi $\mathbb{H}$-convex along S. Then, every weakly admissible function $f : S \to \mathbb{H}$ extends to a neighborhood U of S in $S \cup \Omega^-$ by an $\mathbb{H}$-holomorphic function in U, continuous up to S.*

Proof By Theorem 4.3, using Kontinuitätssatz as in the proof of Theorem 4 of [21], for every point of $p \in S$ there exists a ball $B(p)$ such that $f|_{B(p)\cap\{\rho=0\}}$ extends $\mathbb{H}$-holomorphically on $B(p) \cap \Omega^-$ by a function F_p. This implies that there exists an open covering $B(p_j)$ of S and $\mathbb{H}$-holomorphic functions $F_j : B(p_j) \cap \Omega^- \to \mathbb{H}$, continuous up to S, such that F_j and f agree on $B(p_j) \cap S$. By construction, $F_j = F_k$ on the intersection $B(p_j) \cap B(p_k) \cap S$, hence, by the analytic continuation principle, $F_j = F_k$ on $B(p_j) \cap B(p_k) \cap \Omega^-$. Thus the functions $\{F_k\}$ define the required extension of f. □

Appendix: Some generalizations to octonions

We sketch some generalizations of our results to octonion regular functions. We denote by $\mathsf{i}_0 = 1$ the real unit and by $\mathsf{i}_1, \dots, \mathsf{i}_7$ the imaginary units of the division algebra of the octonions $\mathbb{O}$. Thus, every element p of $\mathbb{O}$ can be written in the form

$$p = \sum_{\alpha=0}^{7} x_\alpha \mathsf{i}_\alpha \quad \text{with } x_\alpha \in \mathbb{R}.$$

As usual, we set $\mathsf{Re}(p) = x_0$, $\mathsf{Im}(p) = \sum_{\alpha=1}^{7} x_\alpha \mathsf{i}_\alpha$ and $\bar{p} = \mathsf{Re}(p) - \mathsf{Im}(p)$. We recall that the product of octonions is noncommutative and nonassociative.

Let U be an open set in $\mathbb{O}$ and $u : U \to \mathbb{O}$ a smooth function. The Cauchy–Riemann–Fueter operator $\partial_{\bar{p}}$ acts on u in the following way:

$$\partial_{\bar{p}} u = \sum_{\alpha=0}^{7} \mathrm{i}_\alpha \frac{\partial u}{\partial x_\alpha} = \left(\sum_{\alpha=0}^{7} \mathrm{i}_\alpha \partial_{x_\alpha} \right) u.$$

We say that u is *(left) $\mathbb{O}$-holomorphic* in U if $\partial_{\bar{p}} u = 0$ on U. We also consider the conjugate operator

$$\partial_p u = \overline{\partial_{\bar{p}}} u = \sum_{\alpha=0}^{7} \bar{\mathrm{i}}_\alpha \frac{\partial u}{\partial x_\alpha} = \left(\sum_{\alpha=0}^{7} \bar{\mathrm{i}}_\alpha \partial_{x_\alpha} \right) u.$$

In the case of several octonion variables $p_1, \dots, p_n$, we set

$$p_h = \sum_{\alpha=0}^{7} x_{h,\alpha} \mathrm{i}_\alpha, \quad \text{with } x_{h,\alpha} \in \mathbb{R}, \tag{5.1}$$

and given an open subset U of $\mathbb{O}^n$, we consider the set $\mathscr{E}^r(U)$ of the smooth maps $U \to \mathbb{O}^r$.

Let us consider a function $u \in \mathscr{E}^1(U)$, $u = u(p_1, \dots, p_n)$. We define the operator

$$\overline{\mathfrak{D}} u = \left(\partial_{\bar{p}_1} u, \dots, \partial_{\bar{p}_n} u \right). \tag{5.2}$$

We have $\overline{\mathfrak{D}} u \in \mathscr{E}^n(U)$. The kernel of the operator $\overline{\mathfrak{D}}$ consists of the *(left) $\mathbb{O}$-holomorphic functions* in the sense of Fueter.

For some of the basic results in octonion analysis, we refer to [8, 13, 23].

Lef $f = (f_1, \dots, f_n) \in \mathscr{E}^n(U)$. The *nonhomogeneous Cauchy–Riemann–Fueter problem* asks for the existence of a solution of

$$\overline{\mathfrak{D}} u = f \tag{5.3}$$

that is

$$\partial_{\bar{p}_h} u = f_h \tag{5.4}$$

for $h = 1, \dots n$.

In this "Appendix", we aim to study conditions on $U \subseteq \mathbb{O}^2$ and $f = (f_1, f_2)$ which guarantee the existence of a solution $u \in \mathscr{E}^1(U)$ of (5.3). In other words, to characterize the image of the operator $\overline{\mathfrak{D}}$ for $n = 2$.

We start by looking at the necessary conditions on the datum f for arbitrary n. Let us recall that $\partial_{p_m} \partial_{\bar{p}_m} = \partial_{\bar{p}_m} \partial_{p_m} = \Delta_{p_m}$ is the Laplacian with respect to the real coordinates of the octonion variable p_m. If the system of Cauchy–Riemann–Fueter (5.3) has a solution, the datum f must satisfy the equations

$$\Delta_{p_m} f_l = \partial_{\bar{p}_l} (\partial_{p_m} f_m) \tag{5.5}$$

for $l, m = 1, \dots, n$. Indeed, if u is a solution of (5.3), i.e, $\partial_{\bar{p}_i} u = f_i$, $i = 1, \dots, n$, then

$$\Delta_{p_m} f_l = \Delta_{p_m} (\partial_{\bar{p}_l} u) = \partial_{\bar{p}_l} (\Delta_{p_m} u) = \partial_{\bar{p}_l} (\partial_{p_m} (\partial_{\bar{p}_m} u)) = \partial_{\bar{p}_l} (\partial_{p_m} f_m).$$

Wang and Ren proved in [23] that such conditions are actually sufficient when the data $f_1, \dots, f_n$ have a compact support. For the sufficience in the general case, we follow the method of Ehrenpreis [9].

Once written the system (5.3) in the form $\overline{D}u = f$, where $\overline{D}$ is the real matrix of differential operator $\overline{\mathfrak{D}}$, the problem reduces to find the generators of the module of relations of the rows of $\overline{D}$.

In real coordinates, given the function

$$u(p) = \sum_{\alpha=0}^{7} u_\alpha(p)\mathrm{i}_\alpha = (u_0(p), \dots, u_7(p)),$$

the Cauchy–Riemann–Fueter operator (in one variable) takes the form

$$\overline{\mathfrak{D}}u = \overline{D}\begin{bmatrix} u_0 \\ \vdots \\ u_7 \end{bmatrix} = \begin{bmatrix} \partial_{x_0} & -\partial_{x_1} & -\partial_{x_2} & -\partial_{x_3} & -\partial_{x_4} & -\partial_{x_5} & -\partial_{x_6} & -\partial_{x_7} \\ \partial_{x_1} & \partial_{x_0} & -\partial_{x_3} & \partial_{x_2} & -\partial_{x_5} & \partial_{x_4} & \partial_{x_7} & -\partial_{x_6} \\ \partial_{x_2} & \partial_{x_3} & \partial_{x_0} & -\partial_{x_1} & -\partial_{x_6} & -\partial_{x_7} & \partial_{x_4} & \partial_{x_5} \\ \partial_{x_3} & -\partial_{x_2} & \partial_{x_1} & \partial_{x_0} & -\partial_{x_7} & \partial_{x_6} & -\partial_{x_5} & \partial_{x_4} \\ \partial_{x_4} & \partial_{x_5} & \partial_{x_6} & \partial_{x_7} & \partial_{x_0} & -\partial_{x_1} & -\partial_{x_2} & -\partial_{x_3} \\ \partial_{x_5} & -\partial_{x_4} & \partial_{x_7} & -\partial_{x_6} & \partial_{x_1} & \partial_{x_0} & \partial_{x_3} & -\partial_{x_2} \\ \partial_{x_6} & -\partial_{x_7} & -\partial_{x_4} & \partial_{x_5} & \partial_{x_2} & -\partial_{x_3} & \partial_{x_0} & \partial_{x_1} \\ \partial_{x_7} & \partial_{x_6} & -\partial_{x_5} & -\partial_{x_4} & \partial_{x_3} & \partial_{x_2} & -\partial_{x_1} & \partial_{x_0} \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} \tag{5.6}$$

(see also [13]). Analogously, in the multivariate case, the Cauchy–Riemann–Fueter operator can be written in real components as

$$\overline{\mathfrak{D}}u = \overline{D}\begin{bmatrix} u_0 \\ \vdots \\ u_7 \end{bmatrix},$$

where

$$\overline{D} = \begin{bmatrix} \overline{D}_{p_1} \\ \vdots \\ \overline{D}_{p_n} \end{bmatrix} \tag{5.7}$$

is an $8n \times 8$ matrix with entries in the polynomial ring with $8n$ indeterminates

$$R_n = \mathbb{R}[\partial_{x_{1,0}}, \dots, \partial_{x_{1,7}}, \dots, \partial_{x_{n,0}}, \dots, \partial_{x_{n,7}}],$$

and $\overline{D}_{p_i}$ denotes the matrix $\overline{D}$ relative to the variable p_i.

We denote by Syz the module of syzygies of the rows of the matrix $\overline{D}$, which is a graded module with grading inherited by the polynomial ring R_n. By taking the real components, we get eight such real syzygies from each one of the octonion conditions (5.5).

Proposition 5.1 *The $n(n-1)$ conditions $\Delta_{p_m} f_l = \partial_{\bar{p}_l}(\partial_{p_m} f_m)$, of (5.5) for $l, m = 1, \dots n$, $l \neq m$, give $8n(n-1)$ real quadratic relations. These relations correspond to linearly independent elements over $\mathbb{R}$ in* Syz.

Proof We want to prove that the operators $z_{l,m}(f) = \Delta_{p_m} f_l - \partial_{\bar{p}_l}(\partial_{p_m} f_m)$ for $l, m = 1, \dots n$, $l \neq m$ are linear independent on $\mathbb{R}$. Given $a, b = 1, \dots, n$, $a \neq b$, we will prove that $z_{a,b}$ is not a linear combination of the other $z_{l,m}$. Indeed, consider the test data $g = (g_1, \dots, g_n)$ where

$$g_k(p_1,\dots,p_n) = \begin{cases} x_{b,0}^2 & \text{for } k = a, \\ 0 & \text{otherwise} \end{cases} \tag{5.8}$$

with notations as in (5.1). Then, $z_{l,m}(g)$ is nonzero if and only if $(l,m) = (a,b)$. □

Now we focus on the case of $n = 2$ octonion variables p_1, p_2. Conditions (5.5) become

$$\begin{aligned} \Delta_{p_2} f_1 &= \partial_{\bar{p}_1}(\partial_{p_2} f_2), \\ \Delta_{p_1} f_2 &= \partial_{\bar{p}_2}(\partial_{p_1} f_1). \end{aligned}$$

Using a computer program that calculates the generators and the Betti numbers of a graded module, one checks directly that the module Syz is generated in degree 2, and Syz_2, its component of degree 2, has real dimension 16.[2] From this we get

Proposition 5.2 *For $n = 2$ octonion variables, the conditions (5.5) correspond to 16 real relations that form a basis of the module of syzygies Syz as a real vector space.*

Proof It follows immediately from the above computer verification and Proposition 5.1. □

Proposition 5.2 and Ehrenpreis' theorem [9, Theorem 6.2, p. 176] now imply

Theorem 5.3 *Let $U \subset \mathbb{O}^2$ be a convex domain and $f \in \mathscr{E}^2(U)$. Then, the Cauchy–Riemann–Fueter problem $\overline{\mathfrak{D}}u = f$ has a solution $u \in \mathscr{E}^1(U)$ if and only if f satisfies conditions (5.5).*

Remark 5.4 We stress that conditions (5.5) do not generate the module of syzygies Syz for $n > 2$. For $n = 3$, this can be directly checked (employing a computer algebra system), hence conditions (5.5) are not sufficient to guarantee the existence of a solution to (5.3).

As in the quaternionic case, we can introduce the notion of admissible octonion function. Thus, in view of Theorem 5.3, we can run through the proofs of Theorems 3.10 and 3.11 and we get:

Theorem 5.5 *Let $\Omega \subset \mathbb{O}^2$ be a domain.*

(1) *If Ω is convex and (S, U^+, U^-) is a domain splitting of Ω, then every smooth admissible function $f : S \to \mathbb{O}$ is a smooth jump.*
(2) *If Ω is bounded with connected smooth boundary $\mathrm{b}\,\Omega$, then every smooth admissible function $f : \mathrm{b}\,\Omega \to \mathbb{O}$ extends to Ω by an $\mathbb{O}$-holomorphic function, smooth up to $\mathrm{b}\,\Omega$.*

[2] We performed the mentioned computation, as well as the one of Remark 5.4, using the commands `syz` and `betti` of the computer algebra system Macaulay 2 [14].

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