A CARTESIAN GRAPH IS “A THING OF MOVEMENT”

Bernardo Nannini[1] and Giulia Lisarelli [2]

[1]University of Florence, Italy, [2]University of Pisa, Italy

Adopting a multimodal semiotic perspective, we present the case study of Bea, a 10th grade low-achieving student with a story of difficulties with functions. We focus on a task designed within a Dynamic Geometry Environment (DGE) artefact involving graphs of functions. The analysis of her speech, inscriptions, gestures, and actions on the artefact highlights her productive struggle in coping with the dynamism of the proposed representation, which resulted in an evolution of her semiotic production culminating in a drawing of the Cartesian graph. Results highlight the potentials of DGE activities, fostering the use of multimodal resources for low-achieving students.

INTRODUCTION

Research shows that the study of signs can highlight crucial processes in the teaching-learning of mathematics. In the last decades different semiotic perspectives opened to a *multimodal* approach, focusing on a wide spectrum of signs, as gestures, sketches, manipulations of artefacts, etc. (Presmeg et al., 2016). This approach has enriched research in many areas of mathematics education, including the studies on teaching-learning of calculus with digital artefacts (Arzarello et al., 2009; Ng, 2016). The study reported in this paper aims to contribute to this stream of research, by adopting a multimodal perspective to analyse students’ processes during DGE-designed activities involving the construction of the Cartesian graphs of functions. The study is part of a wider research project investigating the impact of digital-integrated artefacts on the learning of high school students with a story of persistent difficulties in mathematics.

Expert mathematicians are able to interpret the Cartesian graph of a function as incorporating the functional relation between the two variables. They are also able to construct them starting from local and global properties of the function. Differently, for many students a Cartesian graph is a mute mark of ink on the paper, being them not able to recognise the two covarying variables and the functional relationship between them. Wide literature highlights the students’ need of opportunities to engage in activities on functions that emphasise the covariational aspects and give meaning to the graphical representation (e.g., Antonini et al., 2020; Thompson & Carlson, 2017).

THEORETICAL FRAMEWORK

This study is framed in a multimodal semiotic perspective. We use the notion of ‘semiotic bundle’ theorised by Arzarello (2006) to analyse a wide spectrum of signs simultaneously involved in thinking and learning processes. Arzarello (2006) firstly defines a *semiotic set* as triplet composed by (a) a set of signs which can be produced by intentional actions (speaking, drawing, gesturing, handling an artefact, etc.), (b) a set of modes for producing and transforming signs, and (c) a set of relationships

2023. In M. Ayalon, B. Koichu, R. Leikin, L. Rubel & M. Tabach (Eds.). *Proceedings of the 46th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 387-394). PME 46.

between these signs and their meanings. A *semiotic bundle* is defined as a couple made of a collection of semiotic sets and of relationships between them. A semiotic bundle is a dynamic structure since, during the semiotic activity of a subject, new signs can arise, and new relationships between semiotic sets can be created. Arzarello observes that semiotic sets may transform each other, when new signs in a semiotic set are formed as *genetic conversions* of signs belonging to other sets. A semiotic bundle can be analysed *synchronically*, focusing on the simultaneous relation between the semiotic sets, and *diachronically*, focusing on the development of the bundle over time.

In this study we will consider the semiotic bundle composed by four semiotic sets: speech, written inscriptions, gestures, and what we will call *DGE signs*, i.e. those signs produced by intentional actions on the artefact, as the construction or modification of a geometrical object, the dragging of a point, the zooming actions, etc. Our choice to introduce a specific semiotic set for DGE signs requires some remarks. Firstly, we observe that this set of signs fits in the definition of semiotic set, encompassing the (a-c) features presented above. For example, we can consider the dragging of a base point that causes the indirect motion of other objects, thus, modifying the system of signs already present on the screen and producing the dynamic sign corresponding to the observed movements. The activation of the *trace tool* allows to highlight these movements, however, as gestures are signs even without leaving a mark in the air, the dragging produces signs even if the trace is not activated. The choice of including the set of DGE signs allows us to consider dragging actions within the semiotic bundle, in addition to speaking, gesturing, and drawing. Research shows how dragging plays key roles in teaching-learning functions as covariation between two variables (e.g., Falcade et al., 2007). Ng, in a discursive perspective, defines the term *dragsturing* to refer to an “action subsuming both dragging and gesturing characteristics” (Ng, 2016, p. 130), showing how it can effectively foster an evolution of students’ discourse on functions. In our perspective, we can interpret dragsturing as an action allowing the subject to produce a gesture (the finger moving on the screen) and DGE signs (the direct/indirect movements caused by dragging). Moreover, the introduction of DGE signs is prompted by one of the objectives of the research project, in which this study is embedded, that is to provide insights into the impact of digital-integrated activities on students’ learning. By including this set we can contribute to the aim from a semiotic perspective, focusing on the roles of DGE signs for the evolution of students’ semiotic bundle.

In this paper our goal is to investigate whether, and how, the semiotic bundle evolves in the students of the project when coping with a particular digital-integrated activity involving the construction of the Cartesian graph of a function.

METHODS AND TASK DESIGN

Data for this study were collected during an educational path focusing on functions and involving 12 students of 10th grade from three Italian high schools. Five sessions, lasting 120’ each, took place in an out-of-school learning centre and they were conducted by a researcher. Students worked with touch-screen tablets. Data consist of audio-video recordings of the sessions and screen recordings of the tablets.

In this paper we focus on an activity with a DGE artefact, called *dynagraph*, involving a dynamic representation of a function (for more detail, see Lisarelli, 2023). In this dynagraph two tick marks, labelled A and B, are bound to move on the Cartesian axes. A, representing the independent variable, can be directly dragged, whereas B, representing the dependent variable, moves only indirectly under the dragging of A in accordance with the involved function. The Cartesian graph of the function can be obtained by constructing the point (A,B), activating its *trace*, and dragging A (Fig. 1b). However, this feature of the dynagraph was hidden to students who could only see the tick marks A, B (Fig. 1a). The given task was: "While dragging the tick mark A, can you imagine the trajectory of the point (A,B) on the Cartesian plane? Try to draw it".

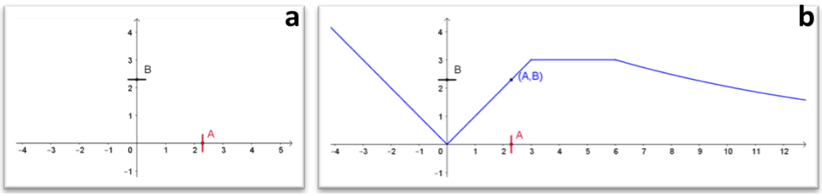

Figure 1: a) The given dynagraph; b) The trajectory of the point (A,B) obtained dragging A – not visible on the screen for students.

The task design involved also an a priori analysis, that allowed us to identify at least two aspects in this task that could generate complexities from cognitive and didactic points of view. The first one concerns the term 'trajectory' that has a twofold meaning (Falcade et al., 2007): either as a set of positions reached by the point (A,B), or as a static object that is the trace of the entire path taken by (A,B). The second one concerns the presence of different levels of dependency involving the elements of the dynagraph, and their movements. Indeed, since B is uniquely determined by A, the point (A,B) can be seen as uniquely dependent on A. However, the movement of (A,B) on the plane depends on both the direct movement of A, and the indirect movement of B.

CASE ANALYSIS

We present the case of Bea. In a preliminary interview, in front of the Cartesian graph of $y = -2x + 1$, she stated to have difficulties with functions ("To me they are a bit like the meanest topic"), without succeeding in working on the given graph. In this section we analyse four episodes from the fourth session. In the transcripts, 'I' stands for 'interviewer', and words in *(italics)* describe actions in the moment they are made.

Episode I.

In the first ten minutes, Bea explored the file (Fig. 1a) by dragging A, initially, on positive numbers and then focusing on the position of B when A is between 0 and 6, which she described by writing "0<A<6→0<B<3". When asked by the interviewer, Bea has correctly constructed a blue point corresponding to (A,B) for A=6 and B=3. Then Bea is asked to describe "where this point is when A varies between 0 and 6".

1 Bea: The blue point follows the direction of A. *(With the right hand she drags A from 6 to 3, and, at the same time, she moves her left index finger horizontally on the screen from the blue point toward left)*. However, it always remains at the same height. Except, indeed, until it arrives to the point three. *(She stops dragging and remains still)*. Where not only… I mean the point does not only follow the trajectory of A, so it is not only parallel to A, but also to B, so it goes down, it goes down. *(She drags A from 3 to 0, consequently B moves from 3 to 0. At the same time, she moves her left index finger on the screen diagonally toward the origin, Fig.2).*

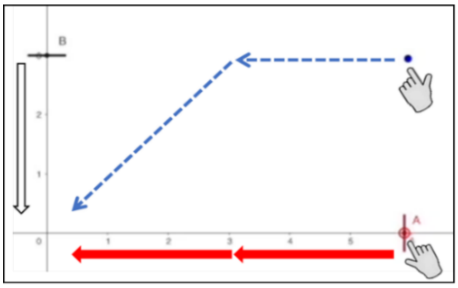

Figure 2: DGE signs and gesture of turn 1. A is dragged from 6 to 3 and then from 3 to 0 (bold arrows). B moves from 3 to 0 (outlined arrow). The blue point remains still. The dotted arrows show the path of Bea's left index finger on the screen.

In this short excerpt, Bea describes the movement of the point (A,B) for the first time, having until now only referred to the movements of A and B. Bea's description employs a series of multimodal semiotic resources. With her right hand she drags A, while B moves indirectly. With her left index finger she indicates the trajectory of the blue point corresponding to (A,B). Note that this point does not move while A is dragged, however Bea manages to represent this movement in a gesture embedded in the screen and co-timed with the movements of A and B. This dragsturing action allows Bea to represent the trajectory dynamically, as a series of positions reached by (A,B). Finally, in her speech, Bea refers to the dependency of the movement of (A,B) on the ones of A ("[it] follows the direction of A") and of B ("it is not only parallel to A, but also to B"), the latter when B is not constant. However, in her gesture the dependency on B is always present. In other terms, dragsturing adds semantic elements to Bea's speech. The bundle of speech, dragging, and gesturing allows Bea to represent the trajectory of (A,B) and the dependencies between the movements of A, B, and (A,B).

Episode II.

Right after the previous episode, the interviewer asks Bea to "try to draw it".

2 Bea: Eh…. [with concerned tone]

3 I: You already did it!

4 Bea: Yes, but the problem is to put it here *(she points to the white page of the tablet)*, how do I do? Because it is a thing of movement.

Encouraged by the interviewer, Bea draws a Cartesian plane on which she marks the values 3 and 6 on the x-axis and the value 3 on the y-axis (Fig. 3a). Then she continues:

5 Bea: Like…from here to here *(she draws an arrow between 3 and 6 on the x-axis, Fig. 3b)*, the point… it remains unchanged and so it remains always

here. *(She indicates over the just drawn arrow)*. Wait I'll try to draw it parallel. Here it is. *(She draws a red point over the arrow at the same height of the line '3' of the y-axis, Fig.3).*

At this point, Bea drags A between 3 and 0 in the two verses, then she adds:

6 Bea: From three to zero, instead… that is, from zero until three *(she draws an arrow from 0 to 3 on the x-axis, Fig. 3d)*, the point… it moves. So, it can go or towards down or towards up, but always on three. *(She draws two red vertical arrows over the x-axis, and then a horizontal line, Fig. 3e).*

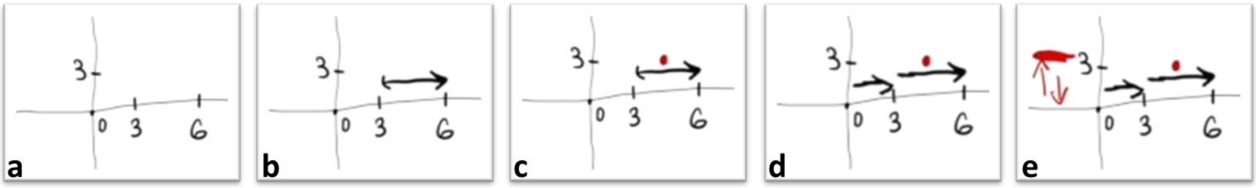

Figure 3: Inscriptions of turns 4-6.

This episode shows Bea's attempt to make a drawing of what she described in episode I. Initially, Bea makes explicit her difficulties in coping with this task (turns 2-4), which requires to represent with a written sign the "thing of movement" that she has previously described with other semiotic resources (the dragsturing of turn 1, Fig. 2). A genetic conversion to enrich the semiotic set of inscriptions is necessary and Bea obtains this by drawing a series of arrows. First, she focuses on 3<A<6, drawing an arrow representing the movement of A and a red point at height 3 (Fig. 3c). Then, focusing on 0<A<3, she draws one horizontal and two vertical arrows corresponding, respectively, to the movements of A and B (Fig. 3d-e). The bundle made by Bea's speech and inscriptions, with the temporal order of the drawings (the arrows for A's movement are followed by those for B's movement), allows her to represent the dynamism experienced and described in episode I. However, the arrows still refer to A and B, whereas the trajectory of (A,B) is not represented with a written inscription yet.

Episode III.

Bea explores the dynagraph for A<0 and observes that A and B "are opposite". When asked, she constructs (A,B) as a blue point (-6,6). Then the interviewer asks:

7 I: What trajectory did the blue point do to go from zero?

8 Bea: It enlarged, like it started from here and then it did like this. *(Gesture of Fig. 4).* I mean, it made a diagonal.

The interviewer invites Bea to "make a drawing of this". Bea draws a Cartesian plane, writing 'A', 'B', and 'A,B' all in correspondence of the origin. Then she continues:

9 Bea: Then, if I enlarge A B, I mean, if B goes here and A goes up *(she draws a horizontal arrow from B and a vertical arrow from A, Fig. 5a)* [omissis: Bea observes to have inverted A and B]. Basically, this moves in diagonal *(she draws an arrow in diagonal starting from the origin, Fig. 5b).*

10 Bea: If instead these lower *(she draws a vertical arrow and a horizontal one, opposite to the ones of turn 9)*, this does like this *(she draws a diagonal arrow toward the origin, Fig. 5c).*

Figure 4: Gesture of turn 8.

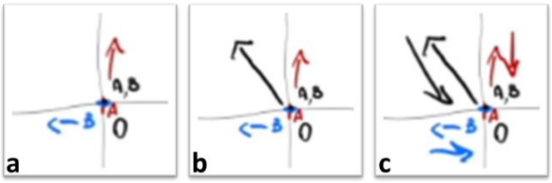

Figure 5: Inscriptions of turns 9-10.

This episode shows the presence of new signs which enrich, firstly, the semiotic set of gestures and then that of inscriptions. The first one is Bea's gesture of turn 8 representing the trajectory of (A,B). This is the second time that she performs a gesture referring to it – the first one occurred in episode I (Fig. 2) – however now it is not embedded in the screen with a dragging action as before, but it is performed in the air, detached from the tablet. It is an iconic gesture that represents, by itself, the movement of (A,B). Then (turns 9-10) we can observe the presence of a new genetic conversion in the semiotic set of inscriptions. Bea initially draws the same arrows of episode II to refer to A's and B's movement, but then she enriches the drawing with two diagonal arrows corresponding to the trajectory of (A,B) (Fig. 5b-c). This is a crucial point, since it is the first inscription made by Bea referring to the movement of (A,B). These arrows, despite being the same written mark than the others, have a very different genesis. They do not represent the movement of something visibly present on the screen, rather they are the conversion into written signs of the gestures previously used by Bea.

Episode IV.

The interviewer constructs the point (A,B) as the intersection of the lines perpendicular to the axes and passing by A and B, so that it moves in dependence of A and B. Then she asks Bea to drag A and to describe what she sees. Bea does not seem surprised by the movement of (A,B) and, referring to (A,B) for positive A-values, says:

11 Bea: It goes as a stair… I mean the shape is like this *(she rapidly makes a trace on the screen, without marking, close to the point (A,B))*. Wait I'll try to… *(Bea takes the other tablet and draws the inscription shown in Fig. 6a).*

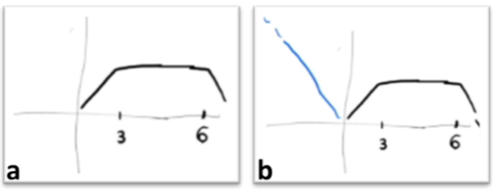

Figure 6: (a) Bea's inscription of turn 11 and (b) after turn 12.

Bea is then asked to consider negative A-values but before dragging A she says:

12 Bea: The point does…it follows the diagonal *(similar gesture of Fig.4).*

After dragging A on negative values, she adds a new line to her drawing (Fig. 6b).

This episode shows how, with the mediation of the point (A,B), Bea manages to produce an inscription that an expert would recognise as the Cartesian graph of the function defined in the dynagraph (Fig. 6b). In her drawing, "the shape" (turn 11) of the trajectory of (A,B) is represented by a series of lines without explicit reference to the movements of A and B and to the verse of movement of (A,B), as previously happened. Bea's speech also mirrors this aspect (turn 12), because the verb 'to follow' that in episode I was used to refer the dependency of (A,B) on A ("[it] follows the direction of A"), is instead used here to refer to the shape of the movement itself.

DISCUSSION

The synchronic analysis highlighted the fundamental role played by the bundle of different semiotic sets in Bea's activity becoming a crucial resource for her to cope with the task. In all the four episodes, by using different signs, mostly personal and not mathematically coded, she effectively represented the trajectory of (A,B).

The diachronic analysis highlights an evolution of the semiotic bundle in response to the task of drawing the trajectory of (A,B), characterised by an enrichment of signs in the semiotic set of inscriptions. Initially, no inscription is made by Bea who represents the movement of (A,B) with bundles of speech, gestures, and DGE signs (episode I). Then, drawings of arrows, coloured tick marks, and dots appear (episode II). This is Bea's initial attempt to represent in the semiotic set of inscriptions the dynamism of the dynagraph. Then this semiotic set further evolves (episode III) when Bea adds to her drawing the arrows representing the movement of (A,B) which, until now, she has only referred to with gestures on the tablet (turn 1) or in the air (turn 8). This is a crucial genetic conversion which introduces the trajectory of (A,B) as a new character within the inscriptions. The evolution culminates when Bea's drawing corresponds, for an expert, to the Cartesian graph (episode IV). In this inscription, mediated by the new constructed point (A,B), some elements of the previous ones are lost (the arrows for the movement of A and B), and others are transformed (the arrows for (A,B) become a straight line). We interpret this as a form of *semiotic contraction*, "the mechanism for reducing attention to those aspects that appear to be relevant […] We need to forget to be able to focus" (Radford, 2008, p. 94). Bea, in drawing the final inscription, 'focuses' on "the shape" of the trajectory of (A,B), 'forgetting' the directions of movements of A, B, and (A,B) and the role of time characterising the dragging actions.

CONCLUDING REMARKS

The multimodal approach and the theoretical lens of the semiotic bundle allowed us to observe, analyse, and describe the story of Bea dealing with a dynagraph. This story is, firstly, a story of struggle. Her question "How do I do it? Because it's a thing of movement", synthetises the core of the problem. The task of drawing the trajectory of (A,B) requires Bea to face a semiotic complexity (to represent with a static inscription the dynamic and co-timed movements of A, B, and (A,B)) and a logical complexity (to take into account the dependencies between these movements). The analysis conducted in this study enriches the literature on functions in relation to students' difficulties with graphs (e.g., Thompson & Carlson, 2017), by allowing these complexities to come into focus and thus providing insights for further research. On the other side, this is also a story of productive struggle. At the end, after a rich intertwining of signs, genetic conversions and external mediations (as the interviewer's requests to focus on (A,B), or the construction of (A,B) as a dynamic point), Bea's semiotic production converged towards the educational aim of the activity: the Cartesian graph. This is highly relevant from a didactic point of view, considering Bea's initial difficulties. Her graph is not a mute trace of ink, as in the preliminary interview, but a sign echoing all her semiotic production and thus rich of personal significance. This result confirms the didactic potentials of dynagraphs, also for low-achieving students (Antonini et al., 2020).

Lastly, our choice of extending the analysis of the semiotic bundle to the set of DGE signs enabled us to observe that many signs produced by Bea developed as genetic conversions of DGE signs. Therefore, our analysis provides an initial contribution to the issue of investigating the impact of digital-integrated activities on students' learning by showing that such activities can have a genetic role, i.e. they foster students' generation of new signs that enrich not only the set of DGE signs, but also other semiotic sets. These signs could be distant from mathematical ones, as in the case of Bea, nevertheless they can be effective didactic resources allowing the teacher to engage a *semiotic game* (Arzarello et al., 2009) bridging them with mathematical signs.

ACKNOWLEDGEMENT

Study part of the funded project DynaMat (PRIN 2020BKWEXR); experimental work conducted at CARME (www.carme.center), UNISER Pistoia Srl, Italy.

REFERENCES

Antonini, S., Baccaglini-Frank, A., & Lisarelli, G. (2020). From experiences in a dynamic environment to written narratives on functions. *Digital Experiences in Mathematics Education, 6*(3), 1-29.

Arzarello, F. (2006). Semiosis as a multimodal process. *Relime,* Número Especial, 267-299.

Arzarello, F., Paola, D., Robutti, O., & Sabena, C. (2009). Gestures as semiotic resources in the mathematics classroom. *Educational Studies in Mathematics, 70,* 97-109.

Falcade, R., Laborde, C., & Mariotti, M.A. (2007). Approaching functions: Cabri tools as instruments of semiotic mediation. *Educational Studies in Mathematics*, *66*(3), 317-333.

Lisarelli, G. (2023). Transition task for building bridges between dynamic digital representations and Cartesian graphs of functions. *Digital Experiences in Mathematics Education.*

Ng, O. (2016). Comparing calculus communication across static and dynamic environments using a multimodal approach. *Digital Experiences in Mathematics Education, 2*(2), 115-141.

Presmeg, N., Radford, L., Roth, W.-M., & Kadunz, G. (2016). *Semiotics in Mathematics Education*. Springer International Publishing.

Radford, L. (2008). Iconicity and contraction: a semiotic investigation of forms of algebraic generalizations of patterns in different contexts. *ZDM, 40*(1), 83-96.

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421-456). National Council of Teachers of Mathematics.