A Unified Approach to Semantic and Soritical Paradoxes

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Abstract. The semantic paradoxes and the paradoxes of vagueness ('soritical paradoxes') display remarkable family resemblances. In particular, the same non-classical logics have been (independently) applied to both kinds of paradoxes. These facts have been taken by some authors to suggest that truth and vagueness require a unified logical framework (see e.g. [3,5]). Some authors go further, and argue that truth is itself a vague or indeterminate concept (see e.g. [4,7]). Importantly, however, there currently is no identification of what the common features of semantic and soritical paradoxes exactly consist in. This is what we aim to do in this work: we analyze semantic and soritical paradoxes, and develop our analysis into a *theory of paradoxicality*. The unification of the paradoxes of truth and vagueness we propose here has a wide scope, but for the sake of concreteness we focus on four three-valued logics.

Keywords: Semantic paradoxes · Paradoxes of vagueness · Three-valued logics

1 Paradoxes and Three-Valued Logics

Definition 1. $\mathcal{L}_{\mathrm{t,v}}$ *is a first-order language (including a propositional constant* $\bot$ *for 'absurdity') that satisfies the following requirements:*

(i) $\mathcal{L}_{\mathrm{t,v}}$ *includes a designated unary predicate* Tr*.*
(ii) $\mathcal{L}_{\mathrm{t,v}}$ *includes countably many designated unary predicates* P_1*,* P_2*, ...,* P_n*, ...*

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(iii) For every predicate P_i*,* $\mathcal{L}_{\mathrm{t,v}}$ *includes one designated binary relation constant* $\sim_{\mathsf{P_i}}$ *and countably many designated individual constants* $c_1^{\mathsf{P}_i}$*,* $c_2^{\mathsf{P}_i}$*, …,* $c_n^{\mathsf{P}_i}$*, …. (For simplicity, we will omit the superscript* $^{\mathsf{P}_i}$*.)*

(iv) It is possible to define in $\mathcal{L}_{\mathrm{t,v}}$ *an injective function* $\ulcorner\,\urcorner$ *s.t. for every* $\mathcal{L}_{\mathrm{t,v}}$*-formula* φ*,* $\ulcorner\varphi\urcorner$ *is a closed term.*

(v) There is at least one $\mathcal{L}_{\mathrm{t,v}}$*-structure* $\mathcal{M}$ *with support* M *s.t. (a)* M *is countable, (b)* $\mathcal{M}$ *is* acceptable[1]*, (c) for every* $a \in M$ *there is an* $\mathcal{L}_{\mathrm{t,v}}$*-constant* c_a *whose intended denotation is* a*.*

(vi) For every open $\mathcal{L}_{\mathrm{t,v}}$*-formula* $\varphi(x)$*, there is an* $\mathcal{L}_{\mathrm{t,v}}$*-term* t_φ *s.t.* $t_\varphi = \ulcorner\varphi(t_\varphi/x)\urcorner$ *in the selected acceptable model, where* $\varphi(t_\varphi/x)$ *is the result of uniformly replacing every occurrence of* x *with* t_φ *in* φ*.*

Requirements (iv)–(vi) make sure that our language can be used to formalize truth-predications (including the sentences used in semantic paradoxes). Consider the open formula $\neg\mathsf{Tr}(x)$, i.e. 'x is not true'. By requirement (vi), there is a term, call it t_λ, that denotes $\ulcorner\neg\mathsf{Tr}(t_\lambda)\urcorner$ (in the selected acceptable model). Let's use λ to abbreviate the sentence $\neg\mathsf{Tr}(t_\lambda)$. λ is a *Liar sentence* and can be informally interpreted as saying that t_λ is not true. But what is t_λ? It is a name of $\neg\mathsf{Tr}(t_\lambda)$, i.e. a name of λ itself. Therefore, there is a sense in which λ says of itself that it is not true. Since we wish to formulate sentences like λ, we impose requirement (vi) in order to employ the relevant sentence-formation process—called 'strong diagonalization'—and employ it in inferences, in any theory that we are going to consider. More explicitly, we are going to avail ourselves of a meta-rule of inference of the following kind (we exemplify it here with λ):

$$\textsc{MDiag}_\lambda\ \frac{\Gamma \vdash \lambda}{\Gamma \vdash \neg\mathsf{Tr}(t_\lambda)}$$

where $\vdash$ is whichever consequence relation we will be employing[2].

Terms, closed terms, formulae, and closed formulae (i.e. sentences) of $\mathcal{L}_{\mathrm{t,v}}$ are defined as usual. We use the (possibly indexed) letters s and t to range over $\mathcal{L}_{\mathrm{t,v}}$-terms, φ, ψ, and χ to range over $\mathcal{L}_{\mathrm{t,v}}$-formulae, and Γ and Δ to range over sets of $\mathcal{L}_{\mathrm{t,v}}$-formulae. We take $\neg$, $\wedge$, and $\forall$ as primitive logical operators. $\vee$, $\rightarrow$, $\leftrightarrow$, and $\exists$ are defined in the usual way. Open terms and formulae will be explicitly indicated (as in $t(x)$ and $\varphi(x)$). '$\varphi \in \mathcal{L}_{\mathrm{t,v}}$' and '$\Gamma \subseteq \mathcal{L}_{\mathrm{t,v}}$' are abbreviations for 'φ is an $\mathcal{L}_{\mathrm{t,v}}$-sentence' and '$\Gamma$ is a set of $\mathcal{L}_{\mathrm{t,v}}$-sentences' respectively. 'S.t.' abbreviates 'such that'.

The truth predicate is often argued to satisfy a property of *naïveté* or *transparency*, to the effect that, for any sentence φ, φ and $\mathsf{Tr}(\ulcorner\varphi\urcorner)$ are in some sense equivalent. One way to spell out naïveté more precisely consists in requiring that all the instances of the following schema be validated:

(Tr-Schema) $\varphi \leftrightarrow \mathsf{Tr}(\ulcorner\varphi\urcorner)$,

[1] In the sense of [8, Chap. 5].

[2] This meta-inferential formulation of diagonalization is required, because the 'usual' form of diagonalization ('weak diagonalization') involving a biconditional is not available in some of the theories we consider.

where $\leftrightarrow$ is a biconditional connective. Alternatively, one can require the truth predicate to obey an *inter-substitutivity* requirement, to the effect that φ and $\mathsf{Tr}(\ulcorner\varphi\urcorner)$ are always intersubstitutable (in all non-opaque contexts). More precisely, it is required that from ψ one can always infer any formula ψ^{t} that results from ψ by replacing, possibly non-uniformly, a subformula φ of ψ with $\mathsf{Tr}(\ulcorner\varphi\urcorner)$ or *vice versa*. Let's call ψ^{t} a *truth-theoretic substitution* of ψ. Naïveté famously gives rise to semantic paradoxes.

Vague predicates (such as 'rich', 'tall', 'red', ...) are often argued to satisfy a property of *tolerance*. Let P be a vague predicate. Tolerance for P dictates that, if s is P and t is very similar to s as far as P is concerned (in symbols, $s \sim_{\mathsf{P}} t$), then t is P as well. Suppose for instance that Sarah is tall (say that she is 194cm tall). If Lois is only 1mm shorter than Sarah, and therefore is very similar to Sarah as far as height is concerned, then Lois is tall as well. Of course, the relevant similarity between Sarah and Lois only concerns how tall they are—it does not matter how different Sarah and Lois are in other respects. Tolerance can be formalized as the following schema:

(Tolerance) $\forall x \forall y(\mathsf{P}(x) \wedge x \sim_{\mathsf{P}} y \rightarrow \mathsf{P}(y))$

As with naïveté, tolerance can also be formulated as an inference rule or as a meta-inference, but for simplicity, we will use Tolerance by default, keeping in mind that its inferential or meta-inferential formulation might be required in some of the three-valued logics we consider later. Just like transparency, tolerance also gives rise to paradoxes.

In order to avoid semantic and soritical paradoxes, several authors have advocated the use of non-classical logics. Here, we focus on *many-valued* truth-functional logics, and in particular on *three-valued* logics (also called *trivalent*, and sometimes *partial*).

Definition 2. *A* three-valued model $\mathcal{M}$ *is a pair* $\langle M, f\rangle$*, where* M *is a non-empty set and* f *is a multi-function from closed* $\mathcal{L}_{\mathrm{t,v}}$*-terms to* M *and from atomic* $\mathcal{L}_{\mathrm{t,v}}$*-sentences to the set* $\{0, 1/2, 1\}$.

Definition 3. *For every three-valued model* $\mathcal{M} = \langle M, f\rangle$*, the* strong Kleene evaluation *induced by* $\mathcal{M}$ *is the function* $e_{\mathcal{M}}$ *from sentences to* $\{0, 1/2, 1\}$ *s.t.:*

$$
\begin{aligned}
e_{\mathcal{M}}(R(t_0, \ldots, t_n)) &:= f(R(t_0, \ldots, t_n))\\
e_{\mathcal{M}}(\neg\varphi) &:= 1 - e_{\mathcal{M}}(\varphi)\\
e_{\mathcal{M}}(\varphi \wedge \psi) &:= \min(e_{\mathcal{M}}(\varphi), e_{\mathcal{M}}(\psi))\\
e_{\mathcal{M}}(\forall x\varphi(x)) &:= \inf\{e_{\mathcal{M}}(\varphi(t)) \in \{0, 1/2, 1\} \,|\, t \textit{ is a closed } \mathcal{L}_{\mathrm{t,v}}\textit{-term}\}
\end{aligned}
$$

Definitions 2 and 3 provide a semantics for $\mathcal{L}_{\mathrm{t,v}}$ but not yet a logic. Using many-valued evaluations, several notions of logical consequence are definable. We now present four logics that can be defined using strong Kleene semantics (our presentation follows [2]).

Definition 4. *For every* $\Gamma \subseteq \mathcal{L}_{\mathrm{t,v}}$*, an evaluation* e *makes* Γ strictly true *(*S-true*) if for every* $\varphi \in \Gamma$*,* $e(\varphi) = 1$*, and* e *makes* Γ tolerantly true *(*T-true*) if for every* $\varphi \in \Gamma$*,* $e(\varphi) \geq 1/2$*.*

Definition 5. *For* $\mathsf{M}, \mathsf{N} \in \{\mathsf{S}, \mathsf{T}\}$*, for every* $\Gamma \subseteq \mathcal{L}_{\mathrm{t,v}}$ *and for every* $\varphi \in \mathcal{L}_{\mathrm{t,v}}$*, we say that* Γ MN*-entail* φ *(in symbols* $\Gamma \models_{\mathsf{MN}} \varphi$*), if for every three-valued model* $\mathcal{M}$*, every* $e_{\mathcal{M}}$ *induced by* $\mathcal{M}$ *that makes all the sentence in* Γ M*-true, also makes* φ N*-true.*

2 Naïve Truth in Three-Valued Logics

We now use strong Kleene semantics and logics SS, TT, TS, and ST to formulate theories of truth. In order to include a treatment of truth-predications, we move from a starting acceptable three-valued model $\mathcal{M} = \langle M, f\rangle$ to a triple $\langle M, f, S\rangle$, where S is called a Kripke model, the *extension* of the truth predicate, i.e. the elements of M to which Tr applies. The main model-theoretic technique to construct such an extension was articulated by [6], and we refer the reader to it.

Let's now explicitly associate a strong Kleene transparent evaluation to a Kripke model.

Definition 6. *For every Kripke model* $\mathcal{M} = \langle M, f, S\rangle$ *for* $\mathcal{L}_{\mathrm{t,v}}$*, the* Kripke (strong Kleene) evaluation *induced by* $\mathcal{M}$ *is the function* e *from sentences to* $\{0, 1/2, 1\}$ *s.t.:*

$$e_{\mathcal{M}}(\varphi) := \begin{cases} 1, & \textit{if } \varphi \in S \\ 0, & \textit{if } \neg\varphi \in S \\ 1/2, & \textit{otherwise} \end{cases}$$

Lemma 1. *For every Kripke model* $\mathcal{M}$*, the evaluation* $e_{\mathcal{M}}$ *is a strong Kleene evaluation, and it validates a form of naïveté, i.e. for every* $\varphi \in \mathcal{L}_{\mathrm{t,v}}$ *and every truth-theoretic substitution* φ^{t}*:*

$$e_{\mathcal{M}}(\varphi) = e_{\mathcal{M}}(\varphi^{\mathsf{t}})$$

Let's call the above evaluations 'Kripke-Kleene'. Finally, we associate theories of transparent truth (tt) proper to the above models and evaluations.

Definition 7. *For* $\mathsf{M}, \mathsf{N} \in \{\mathsf{S}, \mathsf{T}\}$*, for every* $\Gamma \subseteq \mathcal{L}_{\mathrm{t,v}}$ *and for every* $\varphi \in \mathcal{L}_{\mathrm{t,v}}$*, we say that* Γ MNtt*-entails* φ *(*$\Gamma \models_{\mathsf{MNtt}} \varphi$*) if for every Kripke model* $\mathcal{M}$*, if the Kripke-Kleene evaluation* $e_{\mathcal{M}}$ *makes all the sentence in* Γ M*-true, it also makes* φ N*-true.*

SStt, TTtt, TStt, and STtt share the same Kripke models, but their logical differences has an impact on the versions of naïveté they recover, as detailed in the next result.

Proposition 1

- *For every* $\varphi \in \mathcal{L}_{\mathsf{t,v}}$*,* $\varphi \models_{\mathsf{SStt}} \varphi^{\mathsf{t}}$*,* $\varphi \models_{\mathsf{TTtt}} \varphi^{\mathsf{t}}$*, and* $\varphi \models_{\mathsf{STtt}} \varphi^{\mathsf{t}}$*.*
- *For some* $\varphi \in \mathcal{L}_{\mathsf{t,v}}$*,* $\varphi \not\models_{\mathsf{TStt}} \varphi^{\mathsf{t}}$*. However, for every* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_{\mathsf{t,v}}$*:*

$$\frac{\Gamma \models_{\mathsf{TStt}} \varphi}{\Gamma \models_{\mathsf{TStt}} \varphi^{\mathsf{t}}} \ \textsc{MSub}_{\mathsf{Tr}}$$

- *For every* $\varphi \in \mathcal{L}_{\mathsf{t,v}}$*:*

$$\models_{\mathsf{TTtt}} \varphi \leftrightarrow \mathsf{Tr}(\ulcorner \varphi \urcorner) \qquad \models_{\mathsf{STtt}} \varphi \leftrightarrow \mathsf{Tr}(\ulcorner \varphi \urcorner)$$

3 Vagueness in Three-Valued Logics

We now consider the applications of strong Kleene semantics and of the four resulting logics (SS, TT, TS, and ST) to vague predicates and to soritical paradoxes.

Consider a vague predicate P (such as 'tall'), and a countable set $C = \{c_0, c_1, \ldots\}$. Assume that c_0 is a clear case of P, and that c_0, c_1, … are progressively ordered as far as the application of P goes: c_0 is the clearest case of P, c_1 is the clearest case of P after c_0, and so on. Finally, assume that there is a c_j which is a borderline case of P, and that there is an n such that c_n is a clear case of not-P. We now encode these assumptions in a three-valued model $\mathcal{M} = \langle M, f \rangle$ and the evaluation $e_{\mathcal{M}}$ based on it.

(a) $e_{\mathcal{M}}(\mathsf{P}(c_0)) = 1$.
(b) There is an individual c_j s.t. $e_{\mathcal{M}}(\mathsf{P}(c_j)) = 1/2$.
(c) There is an individual c_n s.t. $e_{\mathcal{M}}(\mathsf{P}(c_n)) = 0$.
(d) For every q, $e_{\mathcal{M}}(c_q \sim_{\mathsf{P}} c_{q+1}) = 1$.
(e) $e_{\mathcal{M}}(\mathsf{P}(c_q)) \geq e_{\mathcal{M}}(\mathsf{P}(c_r))$ in case $q \leq r$.

Call a three-valued model and evaluation that respects all of (a)-(e) *soritical*[3]. We now use soritical models and evaluations to specify theories of vagueness, which employ our four three-valued logics.

Definition 8. *For* $\mathsf{M}, \mathsf{N} \in \{\mathsf{S}, \mathsf{T}\}$*, for every* $\Gamma \subseteq \mathcal{L}_{\mathrm{t,v}}$ *and for every* $\varphi \in \mathcal{L}_{\mathrm{t,v}}$*, we say that* Γ MNv*-entails* φ *(*$\Gamma \models_{\mathsf{MNv}} \varphi$*) if for every for every soritical model* $\mathcal{M}$ *and every induced evaluation* $e_{\mathcal{M}}$*, if* $e_{\mathcal{M}}$ *makes all the sentences in* Γ M*-true, it also makes* φ N*-true.*

SSv, TTv, TSv, and STv share the same soritical models, but their logical differences induce differences in the principles they satisfy about vagueness, as the next result shows.

Proposition 2

- TTv *and* STv *are tolerant logics. For every vague predicate* P*:*

$$\models_{\mathsf{TTv}} \forall x \forall y (\mathsf{P}(x) \land x \sim_{\mathsf{P}} y \rightarrow \mathsf{P}(y)) \qquad \models_{\mathsf{STv}} \forall x \forall y (\mathsf{P}(x) \land x \sim_{\mathsf{P}} y \rightarrow \mathsf{P}(y))$$

- SSv *and* TSv *are intolerant logics. For every vague predicate* P*:*

$$\not\models_{\mathsf{SSv}} \forall x \forall y (\mathsf{P}(x) \land x \sim_{\mathsf{P}} y \rightarrow \mathsf{P}(y)) \qquad \not\models_{\mathsf{TSv}} \forall x \forall y (\mathsf{P}(x) \land x \sim_{\mathsf{P}} y \rightarrow \mathsf{P}(y))$$

[3] Unlike the T-models of [2], soritical models do not impose reflexivity or symmetry on $\sim_{\mathsf{P}}$.

4 Unifying the Paradoxes

We now analyze semantic and soritical paradoxes in a unified setting. We start by providing a semantics for $\mathcal{L}_{\mathrm{t,v}}$-sentences that is explicitly devised to analyze (potentially problematic) truth-predications and vague predications. Such semantics is *equational*, in that it interprets sentences via systems of equations, and the properties of their solutions (i.e. whether there are solutions at all, and whether solutions are unique). We then develop our equational semantics into a full-fledged notion of equational consequence, tailored to analyze arguments that lead to contradiction via uses of naïveté and tolerance. The upshot is that arguments to contradictions employing naïveté on the one hand, and tolerance on the other correspond to specific kinds of arguments in our equational consequence.

The equational semantics we employ was developed in [9], and we now extend it to soritical paradoxes, and adapting it to our target three-valued logics. We will now explain the basic idea, mostly via examples. Let's consider an arbitrary sentence $\varphi \in \mathcal{L}_{\mathrm{t,v}}$. If φ is an atomic, non-semantic sentence (i.e. an atomic sentence which is not a truth-predication), then its semantic value is determined by the (acceptable) base model $\mathcal{M}$ we are considering: it receives value 1, or $1/2$, or 0 in it[4]. If, instead, φ is a complex sentence (a negation, a conjunction, or a universally quantified formula), its value depends on the value of its subformulae. As per Definition 3, in a strong Kleene evaluation:

- the value of a negation $\neg\psi$ is $1-$ the value of the negand ψ,
- the value of a conjunction $\psi \wedge \chi$ is the minimum of the values of the conjuncts ψ and χ,
- the value of a universally quantified sentence $\forall x \psi(x)$ is the infimum of the values of its instances $\psi(t)$.

Now, the above clauses can be used to define individual functions—the strong Kleene evaluation functions of Definition 3—but also to define *equation systems*. That is, we can write them as

$$v = 1 - v_1; v = \min(v_1, v_2); v = \inf(w_1, w_2, \ldots)$$

respectively, for v the value of φ, v_1 the value of ψ, v_2 the value of χ, and $w_1, w_2, \ldots$ the values of $\psi(t_1), \psi(t_2), \ldots$ Since strong Kleene semantics is compositional, this process goes on: when we have associated an equation with φ, we associate another equation to its sub-formulae $\psi_1, \ldots, \psi_n, \ldots$, and then we associate and equation with each of the latter, and so on.

So far so good: up to now, we have just a re-writing of a strong Kleene evaluation induced by a base model. But we have neglected the truth predicate and the vague vocabulary. Let's start with truth. As we have done above, we can re-write the semantics for truth-predication, provided by Kripke-Kleene evaluations (Definition 6, Lemma 1) in equational terms. More specifically:

- the value of a truth-predication $\mathsf{Tr}(\ulcorner\psi\urcorner)$ is the value of ψ,

[4] This of course include vague atomic sentences of the form $\mathsf{P}(c)$.

Therefore, we now extend [9]'s semantics (in order to cover the vague vocabulary as well), and then extend it from sentences to *arguments*. Let N_3 be our three-valued numerical value space, i.e., $\mathsf{N}_3 = \{0, 1/2, 1\}$. We now need to fix the language we will use to assign equations to formulas of $\mathcal{L}_{\mathrm{t,v}}$, as sketched above.

Definition 9. *Let $\mathcal{L}_3$ be the language whose alphabet includes:*

- *a countable set Var_3 of variables $\{v_{\varphi_1}, \ldots, v_{\varphi_n}, \ldots\}$, where each φ_k is the k-th element in a non-redundant enumeration of sentences of $\mathcal{L}_{\mathrm{t,v}}$,*
- *a set of constants Con_3 containing an individual constant for every element in N_3,*
- *a binary relation $=$ for equality.*

Definition 10. *Let the set of terms and the set of formulas of $\mathcal{L}_3$ be defined by the following clauses:*

- *the set of $\mathcal{L}_3$-terms is defined by recursively closing Var_3 and Con_3 under the following operations: $(1 - x)$, $\min(x, y)$, $\inf\{x_1, x_2, \ldots, x_n, \ldots\}$[5].*
- *atomic formulas of $\mathcal{L}_3$ are just expressions of the form $s = t$ where s and t are $\mathcal{L}_3$-terms.*

Let $\mathbb{E}_3$ denote the set of atomic $\mathcal{L}_3$-formulas, $\mathbf{e}$, possibly with indices, vary over elements of $\mathbb{E}_3$, while $\mathbf{E}$, possibly with indices, is used to refer to sets of such formulas (that is, to elements of $\mathcal{P}(\mathbb{E}_3)$). Finally, for $\mathbf{E} \subseteq \mathbb{E}_3$, we let $\mathsf{Var}(\mathbf{E})$ indicate the collection of $\mathcal{L}_3$-variables of formulas in $\mathbf{E}$.

The elements of $\mathbb{E}_3$ are the equations definable from the strong Kleene evaluation clauses from Definition 3. As in [9], these equations are assigned to $\mathcal{L}_{\mathrm{t,v}}$-formulae, in a way that 'mimics' the strong Kleene schema. This justifies the following definition, which provides a semantics for $\mathcal{L}_{\mathrm{t,v}}$-sentences, which gives them both numerical values (as usual) *and* equations.

Definition 11. *A* equational structure *for $\mathcal{L}_{\mathrm{t,v}}$ is a structure S_3 given by*

$$S_3 = \langle \mathcal{M}, \mathsf{Con}_3, \mathbb{E}_3, e, A \rangle$$

with $\mathcal{M}$ a soritical, acceptable $\mathcal{L}_{\mathrm{t,v}}$-structure, Con_3 and $\mathbb{E}_3$ as above, and s.t.:

- *e is an* evaluation function *$e : \mathsf{Sent}_{\mathcal{L}_{\mathrm{t,v}}} \longmapsto \mathcal{P}(\mathbb{E}_3)$ from $\mathcal{L}_{\mathrm{t,v}}$-sentences to equations, obeying the clauses of Definition 3 and naïveté for truth;*
- *A is a (possibly infinite) set of partial functions α which are* assignments of values *in Con_3 to variables in any set $\mathsf{Var}(\{\mathbf{e}\})$ for $\mathbf{e} \in \mathbb{E}_3$. That is, $\alpha : \{\mathsf{Var}(\{\mathbf{e}\}) \,|\, \mathbf{e} \in \mathbb{E}_3\} \longmapsto \mathsf{Con}_3$.*

For the sake of readability, we refrain from giving the exact construction of the evaluation e, and of the set A (we refer the reader to [9, §§3-4] for details). The following result (that comes from [9]) guarantees that the above Definition 11 is not vacuous, and that the informal description and examples accurately represent how equational structures work.

[5] This entails that terms of $\mathcal{L}_3$ may end up being possibly infinite strings of symbols.

Proposition 3. *For every acceptable $\mathcal{L}_{t,v}$-structure $\mathcal{M}$:*

(i) *There exists a non-empty set of equational structures,*
(ii) *If $S_3' = \langle \mathcal{M}, \mathsf{Con}_3, \mathbb{E}_3, e', A' \rangle$ and $S_3'' = \langle \mathcal{M}, \mathsf{Con}_3, \mathbb{E}_3, e'', A'' \rangle$ are two equational structures generated by $\mathcal{M}$, then $e' = e''$.*
(iii) *The set of equational structures generated by $\mathcal{M}$ has a $\subseteq$-least element.*

We now employ equational structures to define equational notions of *consequence*, and model arguments involving naïve truth or tolerance. Let us fix some more notation. For every assignment α and for every $\mathbf{e}$ in $\mathbb{E}_3$, let $\models^\alpha \mathbf{e}$ indicate that $\mathbf{e}$ is a true arithmetical equation under the assignment α of values in N_3 to its variables. So, $\models^\alpha \mathbf{e}$ holds if $\alpha(\mathbf{e})$ is a true arithmetical identity. Let us also put, for every assignment α and for every $\mathbf{E} \subseteq \mathbb{E}_3$, $\alpha(\mathbf{E}) = \{\alpha(\mathbf{e}) \,|\, \mathbf{e} \in \mathbf{E}\}$, and put $\models^\alpha \mathbf{E}$ if and only if $\models^\alpha \mathbf{e}$ for every $\mathbf{e} \in \mathbf{E}$. We can now use the existence of solutions to $\mathcal{L}_3$-equations to provide a generalized notion of satisfiability, which we will use to model paradoxical arguments.

Definition 12. *Let $S_3 = \langle \mathcal{M}, \mathsf{Con}_3, \mathbb{E}_3, e, A \rangle$ be an equational structure.*

- *A set $\mathbf{E} \subseteq \mathbb{E}_3$ is* solvable in S_3 *if and only if there exists an assignment $\alpha \in A$ such that $\models^\alpha \mathbf{E}$.*
- *An $\mathcal{L}_{t,v}$-sentence φ is* satisfiable in S_3 *if and only if $e(\varphi)$ is solvable.*
- *A set Γ of $\mathcal{L}_{t,v}$-sentences of is* satisfiable in S_3 *if and only if all sentences φ of Γ are.*

Definition 13. *Let $S_3 = \langle \mathcal{M}, \mathsf{Con}_3, \mathbb{E}_3, e, A \rangle$ be an equational structure.*

- *An $\mathcal{L}_{t,v}$-sentence φ is* strictly (tolerantly) true in S_3 *(S(T)-true), if there exists $\alpha \in A$ such that $\alpha(v_\varphi) = 1$ ($\alpha(v_\varphi) \geq 1/2$) and $\models^\alpha e(\varphi)$.*
- *A set Γ of $\mathcal{L}_{t,v}$-sentences is* strictly (tolerantly) true in S_3 *(S(T)-true), if for every $\varphi \in \Gamma$, $\alpha(v_\varphi) = 1$ ($\alpha(v_\varphi) \geq 1/2$).*

We can now use the above definition to specify notions of NM-*satisfiability in* S_3 for arguments, where $\mathsf{N}, \mathsf{M} \in \{\mathsf{S}, \mathsf{T}\}$, as follows.

Definition 14. *Let $\Gamma \cup \{\varphi\}$ be a set of $\mathcal{L}_{t,v}$-sentences. For $\mathsf{M}, \mathsf{N} \in \{\mathsf{S}, \mathsf{T}\}$, and for every $\varphi \in \mathcal{L}_{t,v}$, we say that the argument from Γ to φ is* MN-equationally valid *($\Gamma \models_{\mathsf{MNe}} \varphi$) if, for every equational structure S_3, every assignment α in S_3 that makes Γ M-true, makes also φ N-true.*

SSe, TTe, TSe, and STe are clearly patterned after SS, TT, TS, and ST (Definition 5). However, there are a few differences. First, the semantic values (1, 0, and 1/2) employed to determined whether φ equationally MN-follows from Γ are results of equations. As such, the possibility equations *not* admitting solutions is explicitly incorporated into the notion of consequence, thereby making it possible to reproduce paradoxical reasonings. Second, equational structures, by design, are defined over soritical models and already incorporate naïveté for the evaluation of truth-predications. Therefore, vague atomic sentences and truth-predications are not treated as arbitrary atomic sentences, and arbitrarily interpreted by any given semantic structure[6]. So, SSe, TTe, TSe, and STe can perhaps be seen as incorporating the notions of consequence defined by both (Definition 7) and (Definition 8), combining them in an equational framework, where paradoxical arguments that cannot be modelled in the standard (non-equational) frameworks can be fully represented here instead.

We now put SSe, TTe, TSe, and STe at work, and see how they provide a unifying analysis of semantic and soritical paradoxes. Let's start from the Liar case. First of all, notice that the 'inferential constituents' of the Liar reasoning, namely the transparent stance on truth, as formalized by the metainference $\text{MS}\textsc{ub}_{\mathsf{Tr}}$ from Proposition 1 which is the common form holding in all of our reference theories MNtt (with $\mathsf{M}, \mathsf{N} \in \{\mathsf{S}, \mathsf{T}\}$), as well as diagonalization, in the form ($\text{MD}\textsc{iag}_\lambda$), have been incorporated in the notion of equational structure (by the clauses defining e on the one hand, and by the ground model $\mathcal{M}$ being acceptable on the other). Let then Γ_λ be $\{\lambda\}$. It easily follows that:

Proposition 4. $\Gamma_\lambda \models_{\mathsf{MNe}} \bot$ *only vacuously (i.e. the argument from* Γ_λ *to* $\bot$ *is vacuously MN-equationally valid) if* $\mathsf{M}{=}\mathsf{S}$ *and for* $\mathsf{N} \in \{\mathsf{S}, \mathsf{T}\}$[7]*, and* $\Gamma_\lambda \not\models_{\mathsf{MNe}} \bot$ *(i.e. the argument from* Γ_λ *to* $\bot$ *is MN-equationally invalid) if* $\mathsf{M}{=}\mathsf{T}$ *and for* $\mathsf{N} \in \{\mathsf{S}, \mathsf{T}\}$.

This is a welcome result as it perfectly matches what happens with SStt, TTtt, STtt, TStt in case the notion of consequences is formulated in terms of Kripke models (up to and including the vacuous case).

Let us now turn to the Sorites Paradox. The argument involves a vague predicate P of $\mathcal{L}_{\mathrm{t,v}}$, a clear-cut case in which it holds $\mathsf{P}(a_0)$, and a clear-cut case in which it does not hold $\mathsf{P}(a_n)$. Then, a contradiction arises by suitably applying all the instances of the tolerance principle involving the relation of P-similarity $\sim_{\mathsf{P}}$. We can formalize the argument in our framework as follows. Let Γ_σ be the following set of sentences of $\mathcal{L}_{\mathrm{t,v}}$:

$$\Gamma_\sigma = \left\{ \begin{array}{l} \mathsf{P}(a_0), \\ \mathsf{P}(a_0) \wedge a_0 \sim_{\mathsf{P}} a_1 \rightarrow \mathsf{P}(a_1), \\ \quad \vdots \\ \mathsf{P}(a_{n-1}) \wedge a_{n-1} \sim_{\mathsf{P}} a_n \rightarrow \mathsf{P}(a_n), \\ a_0 \sim_{\mathsf{P}} a_1, \\ \quad \vdots \\ a_{n-1} \sim_{\mathsf{P}} a_n \end{array} \right\}$$

6 One could argue that SSe, TTe, TSe, and STe are not, strictly speaking, logics or, conversely, that they treat truth-predication and vague atomic sentences as (quasi-)logical expressions. This matter is largely terminological, so we leave it aside here.

7 If $\mathsf{M}{=}\mathsf{S}$, then Definition 14 requires that all formulas in Γ_λ be S-true in the first place. In turn (see Definition 13), this requires that the set of equations associated with all of the formulas in Γ_λ be solvable - in the sense of Definition 12 - by setting the assignment to 1 (i.e., by putting $\alpha(v_\lambda) = 1$ in this case). However, this cannot happen due to the set of equations associated to λ being $\{v_\lambda = 1 - w, w = v_\lambda\}$.

Just as Γ_λ, also Γ_σ encodes the relevant assumptions at play in a soritical arguments. We can now prove the following:

Proposition 5. *$\Gamma_\sigma \models_{\mathsf{MNe}} \bot$ only vacuously (i.e. the argument from Γ_σ to $\bot$ is vacuously MN-equationally valid) if $\mathsf{M}{=}\mathsf{S}$ and for $\mathsf{N} \in \{\mathsf{S},\mathsf{T}\}$*[8]*, and $\Gamma_\sigma \not\models_{\mathsf{MNe}} \bot$ (i.e. the argument from Γ_σ to $\bot$ is MN-equationally invalid) if $\mathsf{M}{=}\mathsf{T}$ and for $\mathsf{N} \in \{\mathsf{S},\mathsf{T}\}$.*

As above, this is a welcome result: given our assumptions about clear-cut cases and similarity relations, codified in Γ_σ, none of our theories MNv with $\mathsf{M},\mathsf{N} \in \{\mathsf{S},\mathsf{T}\}$, allows to conclude $\mathsf{P}(a_n)$.

5 Conclusions

In this paper we have argued that the semantic paradoxes involving truth, and the soritical paradoxes are two sides of the same coin. To make our analysis more concrete, we focused on the Liar Paradox, specific versions of the Sorites, and a family of three-valued logics. After having reconstructed the paradoxes, we introduced a unified framework to formalize them and show that they are display a similar reasoning pattern across our four three-valued logics.

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[8] Like before, if $\mathsf{M}{=}\mathsf{S}$, then for the argument to be valid it is required that all of the sentences in Γ_σ be S-true, i.e. that the set of equations associated to them be solvable by setting the value of the principal variable to 1. In particular, this requires that value 1 is assigned to the variables corresponding to every $\mathsf{P}(a_i)$, which cannot happen in soritical models.