ON UNBOUNDED SOLUTIONS FOR DIFFERENTIAL EQUATIONS WITH MEAN CURVATURE OPERATOR

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Dedicated to the memory of Jaroslav Kurzweil

Abstract. We present necessary and sufficient conditions for the existence of unbounded increasing solutions to ordinary differential equations with mean curvature operator. The results illustrate the asymptotic proximity of such solutions with those of an auxiliary linear equation on the threshold of oscillation. A new oscillation criterion for equations with mean curvature operator, extending Leighton criterion for linear Sturm-Liouville equation, is also derived.

Keywords: nonlinear differential equation; curvatore operator; boundary value problem on the half line; fixed point theorem; unbounded solution

MSC 2020: 34B16, 34C25

1. Introduction

Consider the nonlinear equation

$$(a(t)\Phi_E(x'))' + b(t)F(x) = 0, \tag{1.1}$$

where Φ_E denotes the Euclidean mean curvature operator

$$\Phi_E(u) = \frac{u}{\sqrt{1+u^2}},$$

the functions a, b are positive continuous on $[t_0,\infty)$ and the function F is continuous on $\mathbb{R}$ with $F(u)u>0$ for $u\neq 0$.

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Denote by J_a, J_b, J_{ba} the following integrals:

$$J_a = \int_{t_0}^{\infty} \frac{1}{a(t)}\,\mathrm{d}t, \quad J_b = \int_{t_0}^{\infty} b(t)\,\mathrm{d}t, \quad J_{ba} = \int_{t_0}^{\infty} b(t)\left(\int_{t_0}^{t} \frac{1}{a(s)}\,\mathrm{d}s\right)\mathrm{d}t.$$

In some results troughout the paper we need the following additional assumptions:

$$\text{(H1)} \qquad J_b < \infty, \quad J_{ba} = \infty,$$

$$\text{(H2)} \qquad \liminf_{u\to\infty} \frac{F(u)}{u} > 0, \quad \limsup_{u\to\infty} \frac{F(u)}{u} < \infty,$$

$$\text{(H3)} \qquad \liminf_{t\to\infty} a(t) > 0.$$

Observe that (H1) implies $J_a = \infty$.

By a solution of (1.1) we mean a differentiable function x on the interval $I_x = [t_x, \infty)$, $t_x \geqslant t_0$ such that $a(\cdot)\Phi_E(x'(\cdot))$ is continuously differentiable and satisfies (1.1) on I_x. As usual, a solution x of (1.1) is said to be *oscillatory* if it has a sequence of zeros tending to infinity, otherwise it is said to be *nonoscillatory.* Equation (1.1) is said to be *oscillatory* if any of its solutions is oscillatory.

Boundary value problems (BVPs) associated to equations with the Sturm-Liouville operator

$$(a(t)y')' + b(t)F(y) = 0, \quad t \in I, \tag{1.2}$$

where I is an interval, not necessarily compact, have been extensively studied in the literature, starting from the famous Armellini-Sansone-Tonelli BVP (see [2]), and its generalizations, see, e.g., [11], [15]. Later on, the study was extended in various directions, for instance without assuming Carathéodory conditions on F and requiring weaker conditions on the solutions, see, e.g., the theory of generalized ordinary differential equations, introduced by Kurzweil, see [16]. Another direction was that of considering equations with nonlinear differential operators, such as the curvature operator Φ_E. In particular, BVPs in compact intervals for equations involving Φ_E have been considered in many recent papers, see, e.g., [4], [5], [6], [17] and the references therein. Similarly, asymptotic properties for solutions of equations with Φ_E, especially BVPs on the whole half-line $[t_0, \infty)$, have been widely studied, see, e.g., [3], [7], [10], [13].

The aim of this paper is to study when (1.1) has a solution x satisfying

$$\lim_{t\to\infty} x(t) = \infty, \quad \lim_{t\to\infty} x'(t) = 0. \tag{1.3}$$

Moreover, our results illustrate the asymptotic proximity of these solutions with the solutions of the linear equation

$$(a(t)y')' + b(t)y = 0, \tag{1.4}$$

satisfying

$$\lim_{t\to\infty} y(t) = \infty, \quad \lim_{t\to\infty} a(t)y'(t) = 0. \tag{1.5}$$

Recall that if (H1) is satisfied (so $J_a = \infty$), then (1.4) has a solution y satisfying (1.5) if and only if it is nonoscillatory, see, e.g., Proposition 2.2 below. Assumption (H1) means, roughly speaking, that (1.4) is on the threshold of oscillation, see, e.g., [19], Chapter 2. The existence of solutions y satisfying (1.5) for equation (1.2) is a well-known problem, which has a long history, started seventy years ago by Moore and Nehari in [18]. We refer to [12] for a detailed summary on this topic. The focal point in that problem depends on the difficulty to find sharp upper and lower bounds for solutions y satisfying (1.5) as, for instance, it is illustrated in [1], page 241, [20], page 2. Here, for equation (1.1) we overcome this difficulty by giving a condition that links the solutions of the nonlinear BVP to the ones of a corresponding linear BVP, for which a necessary and sufficient condition is known. A crucial role in this approach is played by condition (H2), which, roughly speaking, means that the nonlinearity has a linear growth at infinity. When the nonlinearity has a nonlinear growth at infinity, then the existence problem is still open for Emden-Fowler type differential equations, see, e.g., [12].

Our results complement the ones in [7], [10], [13]. In particular, in [10] a similar BVP for (1.1) is considered under the assumption

$$\liminf_{t\to\infty} a(t) = 0.$$

In [13], the existence of the so-called Kneser solutions for (1.1) is studied by assuming $J_a < \infty$ and $J_{ba} < \infty$. Finally, Theorem 4.1 below extends (1.1) to a Leighton type oscillation criterion obtained in Theorem 2.1 of [7].

Some auxiliary results for the second order linear differential equations are given in Section 2. Sufficient and necessary existence conditions are proved in Section 3 and in Section 4, respectively. Moreover, in Section 5 some comments, examples and an extension to the relativistic operator are presented.

2. Auxiliary results

Consider the Sturm-Liouville differential equation

$$(r(t)y')' + q(t)y = 0, \tag{2.1}$$

where r, q are continuous functions for $t \geqslant T$ and $r(t) > 0$. When $J_r = J_q = \infty$, where

$$J_r = \int_T^\infty \frac{1}{r(t)}\,\mathrm{d}t, \quad J_q = \int_T^\infty q(t)\,\mathrm{d}t,$$

then the Leighton criterion states that (2.1) is oscillatory, see, e.g., [19], Theorem 2.24. Recall that, in view of the Sturm comparison theorem, the oscillation of (2.1) is equivalent to the existence of at least one oscillatory solution, and, equivalently, the nonoscillation of (2.1) is equivalent to the existence of at least one nonoscillatory solution.

When at least one of the integrals J_r, J_q converges, then (2.1) can be either oscillatory or nonoscillatory. In the nonoscillatory case, Leighton and Morse introduced the notion of the principal solution for (2.1), see, e.g., [14], Chapter XI, Section 6. It reads as follows. If (2.1) is nonoscillatory, then there exists a solution y_0 of (2.1), called *principal solution*, which is uniquely determined up to a constant factor by the condition

$$\lim_{t\to\infty} \frac{y_0(t)}{y(t)} = 0, \tag{2.2}$$

where y denotes an arbitrary solution of (2.1), linearly independent of y_0. Since the principal solution of (2.1) is uniquely determined up to a constant factor, in the following by *the principal solution* of (2.1) we mean a solution of (2.1) satisfying (2.2) and positive for any large t. Roughly speaking, property (2.2) means that principal solutions of (2.1) are the smallest solutions in a neighbourhood of infinity. Moreover, the following holds, see [14], Chapter XI, Section 6.

Proposition 2.1. *Assume that* (2.1) *is nonoscillatory and let y_0 be a solution of* (2.1), *positive for large t. Then the following statements are equivalent.*

(i) *The solution y_0 is the principal solution.*

(ii) *For any arbitrary solution y of* (2.1), *linearly independent of y_0, we have for any large t*

$$\frac{y_0'(t)}{y_0(t)} < \frac{y'(t)}{y(t)}.$$

(iii)

$$\int_T^\infty \frac{1}{r(t)y_0^2(t)}\,\mathrm{d}t = \infty.$$

Claim (ii) of Proposition 2.1 is connected with the Riccati associated equation

$$\xi' + \frac{\xi^2}{r(t)} - q(t) = 0,$$

and Claim (iii) is related to the Lagrange identity, see [14], Chapter XI, Sections 2 and 6 for more details.

When q is positive, some results on the asymptotic behavior of solutions of (2.1), which are needed in the following, are listed below.

Proposition 2.2. *Assume that* (2.1) *is nonoscillatory and* $q(t) > 0$. *Then the following hold.*

(i) *Any nontrivial solution* y *of* (2.1) *satisfies*

$$y(t)\, y'(t) > 0 \quad \text{for large } t$$

if and only if $J_r = \infty$.

(ii) *If* $J_r = \infty$, $J_q < \infty$ *and*

$$\int_T^\infty q(t)\Big(\int_T^t \frac{1}{r(s)}\,\mathrm{d}s\Big)\,\mathrm{d}t = \infty,$$

then every nontrivial solution y *of* (2.1) *satisfies*

$$\lim_{t\to\infty} y(t) = \infty, \quad \lim_{t\to\infty} y^{[1]}(t) = \lim_{t\to\infty} r(t)y'(t) = 0. \tag{2.3}$$

(iii) *If*

$$J_r < \infty \quad \text{or} \quad J_q = \infty \quad \text{or} \quad \int_T^\infty q(t)\Big(\int_T^t \frac{1}{r(s)}\,\mathrm{d}s\Big)\,\mathrm{d}t < \infty,$$

then (2.1) *does not have solutions* y *satisfying* (2.3).

P r o o f. Claim (i) follows, e.g., from Theorem 1 of [9] or Theorem 2.39 of [19]. Claims (ii) and (iii) are a direct consequence of Theorem 1 and Theorem 2 of [9]. $\square$

We close this section with a comparison result between the principal solutions of the linear equations

$$(r_1(t)y')' + q_1(t)y = 0, \tag{2.4}$$

$$(r_2(t)y')' + q_2(t)y = 0, \tag{2.5}$$

where r_i, q_i are continuous functions for $t \geqslant T$, $r_i(t) > 0$, $q_i(t) \geqslant 0$, $i = 1, 2$, and for any large t

$$r_1(t) \geqslant r_2(t), \quad q_1(t) \leqslant q_2(t). \tag{2.6}$$

In virtue of (2.6), equation (2.5) is said to be a *Sturm majorant* of (2.4) and, clearly, equation (2.4) is said to be a *Sturm minorant* of (2.5).

The following holds, see also [14], Chapter XI, Section 6.

Lemma 2.1. *Assume that* (2.5) *is nonoscillatory and let the principal solution* y_2 *of* (2.5) *be positive nondecreasing on* $[\tilde t, \infty)$, $\tilde t \geqslant T$. *Then the principal solution* y_1 *of* (2.4) *is positive on* $[\tilde t, \infty)$ *and satisfies*

$$\frac{y_1'(t)}{y_1(t)} \leqslant \frac{y_2'(t)}{y_2(t)} \quad \text{for } t \geqslant \tilde t. \tag{2.7}$$

Moreover, if $J_{r_1} = \infty$, *then* y_1 *is nondecreasing on the same interval* $[\tilde t, \infty)$.

P r o o f. Since y_2 is positive for $t \geqslant \tilde{t}$, from [14], Chapter XI, Corollary 6.1, equation (2.5) is disconjugate on $[\tilde{t}, \infty)$. Hence, by using [14], Chapter XI, Corollary 6.5 we obtain that $y_1(t) > 0$ for $t \geqslant \tilde{t}$, and

$$\frac{r_1(t)y_1'(t)}{y_1(t)} \leqslant \frac{r_2(t)y_2'(t)}{y_2(t)}, \quad t \geqslant \tilde{t}.$$

Since $y_2'(t)/y_2(t)$ is positive on $[\tilde{t}, \infty)$, in view of (2.6) we obtain

$$\frac{r_1(t)y_1'(t)}{y_1(t)} \leqslant \frac{r_2(t)y_2'(t)}{y_2(t)} \leqslant \frac{r_1(t)y_2'(t)}{y_2(t)},$$

i.e., (2.7) is satisfied. Now, consider the linear equation

$$(r_1(t)y')' = 0. \tag{2.8}$$

Equation (2.8) is a minorant of (2.4). Since $J_{r_1} = \infty$, it is easy to show that the principal solution of (2.8) is the constant function $y_3(t) \equiv c > 0$. Thus, applying again [14], Chapter XI, Corollary 6.5 to equations (2.4) and (2.8) we obtain $r_1(t)y_1'(t)/y_1(t) \geqslant 0$ on the whole interval $[\tilde{t}, \infty)$ and the assertion follows. $\square$

3. The existence result

In virtue of (H2), set

$$F_m = \inf_{u\geqslant 1} \frac{F(u)}{u} > 0, \quad F_M = \sup_{u\geqslant 1} \frac{F(u)}{u} < \infty. \tag{3.1}$$

Our main result here is the following.

Theorem 3.1. *Assume that* (H1), (H2) *and* (H3) *hold. If the linear equation*

$$\Big(\frac{\sqrt{3}}{2}a(t)w'\Big)' + F_M b(t)w = 0 \tag{3.2}$$

is nonoscillatory, then (1.1) *has a solution satifying* (1.3). *Moreover,*

$$\lim_{t\to\infty} a(t)\Phi_E(x'(t)) = \lim_{t\to\infty} a(t)x'(t) = 0 \tag{3.3}$$

and

$$x(t) = o\left(\int_{t_0}^{t} \frac{1}{a(s)}\,\mathrm{d}s\right) \quad \textit{if } t \to \infty. \tag{3.4}$$

To prove Theorem 3.1, we use a fixed point result for operators defined in a Fréchet space, which originates from [8]. We present this result in the form that is needed in the sequel, by using an argument which is partially similar to the one in Theorem 1.1

of [8]. Roughly speaking, this result reduces the solvability of a boundary value problem for nonlinear equations to the solvability of an associated boundary value problem for a linear equation.

Let I be the interval $[t_1, \infty)$, $t_1 \geqslant t_0$, and denote by $C(I, \mathbb{R}^2)$ the Fréchet space of all continuous vector-functions (f_1, f_2), defined on I, with the topology of the uniform convergence on compact subintervals of I. Similarly, denote by $C^1(I, \mathbb{R})$ the Fréchet space of all continuously differentiable functions on I. Recall that a subset Ω of $C(I, \mathbb{R}^2)$ is bounded if and only if there exists a positive continuous function φ such that for all $t \in I$ and $(f_1, f_2) \in \Omega$ it holds $|f_1(t)| + |f_2(t)| \leqslant \varphi(t)$. Since $C^1(I, \mathbb{R})$ can be embedded into $C(I, \mathbb{R}^2)$ via the map $h \to (h, h')$, the following BVP is stated by considering boundary conditions as a subset of $C(I, \mathbb{R}^2)$. This result complements Theorem 3.2 of [13].

Lemma 3.1. *Let S_0 be a subset of $C(I, \mathbb{R}^2)$. Assume that there exists a nonempty closed convex bounded subset $\Omega \subset C(I, \mathbb{R}^2)$ such that*

$$u(t) \neq 0, \quad |v(t)| < a(t) \quad \text{for all } (u, v) \in \Omega \text{ and } t \in I, \tag{3.5}$$

and a nonempty closed subset S_1 of $S_0 \cap \Omega$ such that for each $(u, v) \in \Omega$ the linear equation

$$\frac{\mathrm{d}}{\mathrm{dt}}\big(\sqrt{a^2(t) - v^2(t)}\,y'\big) + b(t)\frac{F(u(t))}{u(t)}y = 0 \tag{3.6}$$

has a unique solution y_{uv} with $(y_{uv}, y_{uv}^{[1]}) \in S_1$, where

$$y_{uv}^{[1]} = \sqrt{a^2(t) - v^2(t)}\,y'_{uv}$$

denotes the quasiderivative of y_{uv}. Then (1.1) *admits a solution x such that $(x, a\Phi_E(x')) \in S_0$.*

P r o o f. For every $(u, v) \in \Omega$ fixed, the change of variables

$$y_1(t) = y(t), \quad y_2(t) = \sqrt{a^2(t) - v^2(t)}\,y'(t)$$

transforms the linear equation (3.6) into the system

$$\begin{cases} y_1' - \dfrac{1}{\sqrt{a^2(t) - v^2(t)}}y_2 = 0, \\[2ex] y_2' + b(t)\dfrac{F(u(t))}{u(t)}y_1 = 0. \end{cases} \tag{3.7}$$

Let $\mathcal{T}$ be the operator $\Omega \to C(I, \mathbb{R}^2)$, given by

$$\mathcal{T}(u, v) = (y_{uv}, y_{uv}^{[1]}).$$

Since $y_2 = y^{[1]} = \sqrt{a^2(t) - v^2(t)}\; y'(t)$, the operator $\mathcal{T}$ maps (u, v) into the unique solution (y_1, y_2) of (3.7) in S_1. From [8], Theorem 1.1 and Remark at page 175, the operator $\mathcal{T}$ is continuous and relatively compact in Ω. Since $(y_1, y_2) \in S_1$ and $S_1 \subset \Omega$, we get $\mathcal{T}(\Omega) \subset \Omega$. Thus, the Tychonoff fixed point theorem gives the existence of $(\widehat{y_1}, \widehat{y_2}) \in \Omega$ such that,

$$(\widehat{y_1}, \widehat{y_2}) = \mathcal{T}(\widehat{y_1}, \widehat{y_2}). \tag{3.8}$$

Now, let us show that $\widehat{y_1}$ is a solution of (1.1). From (3.8) the couple $(\widehat{y_1}, \widehat{y_2})$ is a solution of (3.7) with $u = \widehat{y_1}$ and $v = \widehat{y_2}$, that is,

$$\frac{\mathrm{d}}{\mathrm{dt}}\widehat{y_1} = \frac{\widehat{y_2}}{\sqrt{a^2(t) - (\widehat{y_2}(t))^2}} = \frac{\widehat{y_2}/a(t)}{\sqrt{1 - (\widehat{y_2}(t)/a(t))^2}}, \tag{3.9}$$

$$\frac{\mathrm{d}}{\mathrm{dt}}\widehat{y_2} + b(t)F(\widehat{y_1}) = 0. \tag{3.10}$$

Denote by Φ_R the inverse of the operator Φ_E, i.e.,

$$\Phi_R(v) = \frac{v}{\sqrt{1 - v^2}}. \tag{3.11}$$

The operator Φ_R is called the *Minkowski mean curvature operator*, or sometimes the relativistic operator, see [4], [13]. Since from (3.9),

$$\frac{\mathrm{d}}{\mathrm{dt}}\widehat{y_1} = \Phi_R\Big(\frac{\widehat{y_2}(t)}{a(t)}\Big),$$

we get

$$\widehat{y_2}(t) = a(t)\Phi_E\Big(\frac{\mathrm{d}}{\mathrm{dt}}\widehat{y_1}\Big). \tag{3.12}$$

Hence, from (3.10) the function $\widehat{y_1}$ is a solution of (1.1). Since $(\widehat{y_1}, \widehat{y_2}) \in S_1 \subset S_0$, from (3.12) we get $(\widehat{y_1}, a\Phi_E(\widehat{y_1}')) \in S_0$ and the proof is complete. $\square$

P r o o f of Theorem 3.1. Let w_0 be the principal solution of equation (3.2). Using assumptions (H1), (H3) and applying Proposition 2.2, the solution w_0 satisfies

$$\lim_{t\to\infty} w_0(t) = \infty, \quad \lim_{t\to\infty} a(t)w_0'(t) = 0. \tag{3.13}$$

Without loss of generality, assume that $w_0(t) \geqslant 1$, $w_0'(t) > 0$ for $t \geqslant t_1 \geqslant t_0$. In view of (H3), there exists $A > 0$ such that

$$a(t) \geqslant A \quad \text{for } t \geqslant t_0. \tag{3.14}$$

Since $\lim_{t\to\infty} a(t)w_0'(t) = 0$, we can also suppose that for $t \geqslant t_1$

$$a(t)w_0'(t) \leqslant \frac{1}{2}A. \tag{3.15}$$

Consider the linear equation

$$(a(t)z'(t))' + F_m b(t)z(t) = 0. \tag{3.16}$$

Since (3.16) is a minorant of (3.2), equation (3.16) is nonoscillatory. Denote by z_0 the principal solution of (3.16) such that $z_0(t_1) = w_0(t_1)$. From Lemma 2.1, the solution z_0 is positive nondecreasing on $[t_1, \infty)$. Moreover, from Proposition 2.2, the solution z_0 satisfies

$$\lim_{t\to\infty} z_0(t) = \infty, \quad \lim_{t\to\infty} a(t)z_0'(t) = 0. \tag{3.17}$$

Let us prove that (1.1) admits a solution x satisfying (1.3). Let Ω and S_1 be the subsets of $C(I,\mathbb{R}^2)$, $I = [t_1, \infty)$, given by

$$\Omega = \Big\{(u,v) \in C(I,\mathbb{R}^2)\colon\ z_0(t) \leqslant u(t) \leqslant w_0(t),\, 0 \leqslant v(t) \leqslant \frac{1}{2}a(t)\Big\},$$

and

$$S_1 = \{(u,v) \in \Omega\colon\ 0 \leqslant v(t) \leqslant a(t)w_0'(t)\}.$$

Denote by S_0 the set

$$S_0 = \Big\{(u,v) \in C(I,\mathbb{R}^2)\colon\ \lim_{t\to\infty} u(t) = \infty,\, \lim_{t\to 0} v(t) = 0\Big\}.$$

Obviously $S_1 \subset S_0$. Indeed, if (u,v) is an element of S_1, we have for $t \in I$

$$z_0(t) \leqslant u(t) \leqslant w_0(t), \quad 0 \leqslant v(t) \leqslant a(t)w_0'(t)$$

and from (H3), (3.13) and (3.17) we obtain $(u,v) \in S_0$.

For applying Lemma 3.1, we have to show that for each $(u,v) \in \Omega$ the linear equation (3.6) has a unique solution y_{uv} with $(y_{uv}, y_{uv}^{[1]}) \in S_1 \subset (\Omega \cap S_0)$.

First, we show that equation (3.6) has at least one solution y_{uv} such that $(y_{uv}, y_{uv}^{[1]}) \in S_1$. Let $(u,v) \in \Omega$. We have for $t \in I$

$$a(t) \geqslant \sqrt{a^2(t) - v^2(t)} \geqslant \frac{\sqrt{3}}{2}a(t). \tag{3.18}$$

Since $F_m \leqslant F(u(t))/u(t) \leqslant F_M$, equations (3.2) and (3.16) are a majorant and a minorant of (3.6), respectively. Hence, (3.6) is nonoscillatory. Denote by y_{uv}

the principal solution of (3.6) such that $y_{uv}(t_1) = w_0(t_1)$. From Lemma 2.1 the solution y_{uv} is positive nondecreasing on $[t_1, \infty)$ and for $t \in I$

$$\frac{y'_{uv}(t)}{y_{uv}(t)} \leqslant \frac{w'_0(t)}{w_0(t)}. \tag{3.19}$$

Integrating on I and taking into account that $w_0(t_1) = y_{uv}(t_1)$, we get $y_{uv}(t) \leqslant w_0(t)$ for $t \in I$. Comparing (3.6) with the minorant (3.16) and reasoning in a similar way, we have also $z_0(t) \leqslant y_{uv}(t)$ on the same interval I. Hence,

$$z_0(t) \leqslant y_{uv}(t) \leqslant w_0(t) \quad \text{for } t \in I. \tag{3.20}$$

Moreover, from (3.14), (3.15), (3.18), (3.19), (3.20) and $y_{uv} \leqslant w_0$ we obtain for $t \in I$

$$y_{uv}^{[1]}(t) \leqslant a(t)w'_0(t)\frac{y_{uv}(t)}{w_0(t)} \leqslant a(t)w'_0(t) \leqslant \frac{1}{2}A \leqslant \frac{1}{2}a(t). \tag{3.21}$$

Taking into account that y_{uv} is nondecreasing, we get also $y_{uv}^{[1]}(t) \geqslant 0$ and so $(y_{uv}, y_{uv}^{[1]}) \in S_1 \subset \Omega$.

Now, we show that the solution y_{uv} is the unique solution of (3.6) such that $(y_{uv}, y_{uv}^{[1]}) \in S_1$. By contradiction, let ξ_{uv} be another solution of (3.6) such that $(\xi_{uv}, \xi_{uv}^{[1]}) \in S_1$. Since $z_0(t_1) = w_0(t_1)$, we have $\xi_{uv}(t_1) = y_{uv}(t_1)$ and $\xi_{uv} \not\equiv y_{uv}$. Hence, ξ_{uv} is a nonprincipal solution of (3.6) and so, in view of Proposition 2.1, we have

$$\int_{t_1}^{\infty} \frac{1}{\sqrt{a^2(t) - v^2(t)}} \frac{1}{\xi_{uv}^2(t)} \, \mathrm{d}t < \infty. \tag{3.22}$$

Further, since $\xi_{uv}(t) \leqslant w_0(t)$ for $t \geqslant t_1$, using (3.18) we obtain for $t \geqslant t_1$

$$\frac{1}{a(t)w_0^2(t)} \leqslant \frac{1}{\sqrt{a^2(t) - v^2(t)}} \frac{1}{\xi_{uv}^2(t)}.$$

Since w_0 is the principal solution of (3.2), using Proposition 2.1 we get

$$\int_{t_1}^{\infty} \frac{1}{\sqrt{a^2(t) - v^2(t)}} \frac{1}{\xi_{uv}^2(t)} \, \mathrm{d}t = \infty,$$

which is a contradiction with (3.22).

Hence, using Lemma 3.1, equation (1.1) has a solution x such that $(x, a\Phi_E(x')) \in S_0$, that is

$$\lim_{t\to\infty} x(t) = \infty, \quad \lim_{t\to 0} a(t)\Phi_E(x'(t)) = 0. \tag{3.23}$$

From this, in virtue of (H3) we get that x satisfies (1.3). Further, $\lim_{t\to\infty} x'(t) = 0$ gives

$$\lim_{t\to\infty} \frac{a(t)\Phi_E(x'(t))}{a(t)x'(t)} = \lim_{t\to\infty} \frac{1}{\sqrt{1+(x'(t))^2}} = 1.$$

Thus, from (3.23) we obtain (3.3) and the l'Hopital rule gives (3.4). The proof is now complete. □

A closer examination of the proof of Theorem 3.1 yields that (1.1) has infinitely many solutions which satisfy (1.3).

4. Necessary conditions

In this section we study the necessity of conditions (H1) for the existence of solutions x to (1.1) satisfying

$$\lim_{t\to\infty} x(t) = \infty, \quad \lim_{t\to\infty} a(t)\Phi_E(x'(t)) = 0. \tag{4.1}$$

If x is a nonoscillatory solution of (1.1) such that $x(t) \neq 0$ for $t \geqslant T \geqslant t_0$, then $a\Phi_E(x')$ is strictly monotone for $t \geqslant T$ and thus it has at most one zero greater than T, i.e., x' has no zeros for large t.

Hence, nonoscillatory solutions of (1.1), if any, may by divided into the following two classes:

$$\begin{aligned}
\mathbb{M}^+ &= \{x \text{ solution of (1.1)}\colon \exists\, t_x \geqslant t_0\colon\ x(t)x'(t) > 0 \text{ for } t > t_x\},\\
\mathbb{M}^- &= \{x \text{ solution of (1.1)}\colon \exists\, t_x \geqslant t_0\colon\ x(t)x'(t) < 0 \text{ for } t > t_x\}.
\end{aligned}$$

Using this a-priori classification, we obtain the following oscillation criterion. It can be viewed as an extension to (1.1) of the claimed Leighton result for (2.1).

Theorem 4.1.

(i) *If $J_a = \infty$, then equation* (1.1) *does not have solutions in the class $\mathbb{M}^-$, that is, $\mathbb{M}^- = \emptyset$ for* (1.1).

(ii) *If $J_b = \infty$ and* (H2) *is satisfied, then equation* (1.1) *does not have solutions in the class $\mathbb{M}^+$, that is, $\mathbb{M}^+ = \emptyset$ for* (1.1).

(iii) *Consequently, if* (H2) *is satisfied and $J_a = J_b = \infty$, then any continuable solution at infinity of* (1.1) *is oscillatory.*

P r o o f. Claim (i). By contradiction, let x be a solution of (1.1) in the class $\mathbb{M}^-$. Without loss of generality, assume that $x(t) > 0$, $x'(t) < 0$ for $t \geqslant T \geqslant t_0$. From (1.1), the function $a\Phi_E(x')$ is negative decreasing on $[T, \infty)$. Since x is also decreasing, we have for $t \geqslant T$

$$0 > a(T)\Phi_E(x'(T)) \geqslant a(t)\Phi_E(x'(t)) = \frac{a(t)x'(t)}{\sqrt{1+|x'(t)|^2}} \geqslant a(t)x'(t) \tag{4.2}$$

or $x'(t) \leqslant a(T)\Phi_E(x'(T))a^{-1}(t)$. Integrating this inequality on $[T, t]$, t large, we obtain

$$x(t) \leqslant x(T) + a(T)\Phi_E(x'(T)) \int_T^t a^{-1}(s)\,\mathrm{d}s,$$

which, in view of $J_a = \infty$, contradicts the positiveness of x as $t \to \infty$.

Claim (ii). By contradiction, let x be a solution of (1.1) in the class $\mathbb{M}^+$. Without loss of generality, assume that $x(t) > 0$, $x'(t) > 0$ for $t \geqslant T \geqslant t_0$. In virtue of (H2), we have

$$\inf_{u \geqslant x(T)} F(u) = F_0 > 0.$$

Hence, from (1.1) we get

$$a(t)\Phi_E(x'(t)) = a(T)\Phi_E(x'(T)) - \int_T^t b(s)F(x(s))\,\mathrm{d}s \leqslant a(T)\Phi_E(x'(T)) - F_0 \int_T^t b(s)\,\mathrm{d}s,$$

which, for large t, contradicts the positiveness of $a(t)\Phi_E(x'(t))$, being $J_b = \infty$.

Claim (iii) follows from the classification of nonoscillatory solutions and from claims (i) and (ii). $\square$

Now, we study when condition (H1) becomes necessary for the validity of Theorem 3.1. Observe that from the quoted Leighton criterion and Proposition 2.2 (iii), linear equation (1.4) does not have solutions y satisfying (1.5) if $J_b = \infty$ or $J_a < \infty$ or $J_{ba} < \infty$. Theorem 4.1 (ii) illustrates that the same is true for (1.1) when $J_b = \infty$. A more sophisticated argument proves that this property continues to hold for (1.1) also in the remaining cases, as the following results show.

Theorem 4.2. *Let* (H2) *and* (H3) *be verified. If $J_a < \infty$ or $J_{ba} < \infty$, then equation* (1.1) *does not have solutions x satisfying* (4.1). *In addition, if*

$$\limsup_{t\to\infty} a(t) < \infty, \tag{4.3}$$

then (1.1) *does not have solutions x satisfying* (1.3).

P r o o f. First observe that if $J_a < \infty$ and x is a solution of (1.1) satisfying (4.1), then in virtue of Theorem 4.1, we have $J_b < \infty$ and thus $J_{ba} < \infty$.

Let $J_{ba} < \infty$. By contradiction, let x be a solution of (1.1) satisfying (4.1). Without loss of generality, suppose that $x(t) \geqslant 1$, $x'(t) > 0$ for any $t \geqslant t_1 \geqslant t_0$. Integrating (1.1) on $[t, \infty)$, $t \geqslant t_1$, we get

$$a(t)\Phi_E(x'(t)) = \int_t^\infty b(s)F(x(s))\,\mathrm{d}s. \tag{4.4}$$

From this and (H3) we obtain

$$\lim_{t\to\infty} \frac{1}{a(t)} \int_t^\infty b(s)F(x(s))\,\mathrm{d}s = 0. \tag{4.5}$$

Let Φ_E^* be the inverse of the operator Φ_E, see (3.11). Since $\lim_{u\to 0} \Phi_E^*(u)/u = 1$, in the right neigborhood of zero we have

$$\Phi^*(u) \leqslant \frac{3}{2}u.$$

Using this inequality, in view of (4.5) we can suppose for large t, say $t \geqslant T \geqslant t_1$,

$$\Phi^*\Big(\frac{1}{a(t)} \int_t^\infty b(s)F(x(s))\,\mathrm{d}s\Big) \leqslant \frac{3}{2}\frac{1}{a(t)} \int_t^\infty b(s)F(x(s))\,\mathrm{d}s. \tag{4.6}$$

Moreover, since $J_{ba} < \infty$, we can suppose also that

$$\int_T^\infty b(r) \int_T^r \frac{1}{a(s)}\,\mathrm{d}s\,\mathrm{d}r \leqslant \frac{1}{5F_M}. \tag{4.7}$$

From (4.4) and (4.6) we have

$$x'(t) = \Phi^*\Big(\frac{1}{a(t)} \int_t^\infty b(s)F(x(s))\,\mathrm{d}s\Big) \leqslant \frac{3}{2}\frac{1}{a(t)} \int_t^\infty b(s)F(x(s))\,\mathrm{d}s.$$

Hence, we get for $t \geqslant T$

$$\begin{aligned} x(t) - x(T) &\leqslant \frac{3}{2}\int_T^t \frac{1}{a(r)} \int_r^\infty b(s)F(x(s))\,\mathrm{d}s\,\mathrm{d}r \\ &= \frac{3}{2}\int_T^t \frac{1}{a(r)} \int_r^t b(s)F(x(s))\,\mathrm{d}s\,\mathrm{d}r \\ &\quad + \frac{3}{2}\Big(\int_T^t \frac{1}{a(r)}\,\mathrm{d}r\Big)\Big(\int_t^\infty b(s)F(x(s))\,\mathrm{d}s\Big). \end{aligned} \tag{4.8}$$

Using (4.7) and the monotonicity of x, we have

$$\begin{aligned}\int_T^t \frac{1}{a(r)} \int_r^t b(s)F(x(s))\,\mathrm{d}s\,\mathrm{d}r &= \int_T^t b(r)F(x(r)) \int_T^r \frac{1}{a(s)}\,\mathrm{d}s\,\mathrm{d}r \\ &\leqslant F_M x(t) \int_T^\infty b(r) \int_T^r \frac{1}{a(s)}\,\mathrm{d}s\,\mathrm{d}r \leqslant \frac{1}{5}x(t).\end{aligned}$$

Using this inequality, (4.8) yields for $t \geqslant T$

$$x(t) - x(T) \leqslant \frac{3}{10}x(t) + \frac{3}{2}\left(\int_T^t \frac{1}{a(r)}\,\mathrm{d}r\right)\left(\int_t^\infty b(s)F(x(s))\,\mathrm{d}s\right)$$

or, in view of (4.4),

$$\left(\int_T^t \frac{1}{a(r)}\,\mathrm{d}r\right)\frac{a(t)\Phi_E(x'(t))}{x(t)} \geqslant \frac{2}{3}\left(1 - \frac{x(T)}{x(t)}\right) - \frac{1}{5} = \frac{7}{15} - \frac{2x(T)}{3x(t)}. \tag{4.9}$$

Since $\lim\limits_{t\to\infty} x(T)/x(t) = 0$, we can suppose for $t \geqslant T$

$$\frac{x(T)}{x(t)} \leqslant \frac{1}{10}.$$

Thus, (4.9) yields for $t \geqslant T$

$$\left(\int_T^t \frac{1}{a(r)}\,\mathrm{d}r\right)\frac{a(t)\Phi_E(x'(t))}{x(t)} > \frac{7}{15} - \frac{2}{3}\frac{1}{10} = \frac{2}{5}$$

or

$$x(t) \leqslant \frac{5}{2}\left(\int_T^t \frac{1}{a(r)}\,\mathrm{d}r\right)a(t)\Phi_E(x'(t)). \tag{4.10}$$

From (H2) and (3.1) we have $F(u) \leqslant F_M\,u$ for $u \geqslant 1$. Integrating (1.1) on $[T,t]$ we obtain

$$a(T)\Phi_E(x'(T)) - a(t)\Phi_E(x'(t)) = \int_T^t b(r)F(x(r))\,\mathrm{d}r \leqslant F_M \int_T^t b(r)x(r)\,\mathrm{d}r.$$

From this and (4.10) we get

$$a(T)\Phi_E(x'(T)) - a(t)\Phi_E(x'(t)) \leqslant \frac{5}{2}F_M \int_T^t b(r)\left(\int_T^r \frac{1}{a(s)}\,\mathrm{d}s\right)a(r)\Phi_E(x'(r))\,\mathrm{d}r.$$

Since $a(r)\Phi_E(x'(r))$ is decreasing, using (4.7) we obtain

$$\begin{aligned} a(T)\Phi_E(x'(T)) - a(t)\Phi_E(x'(t)) &\leqslant \frac{5}{2}F_M a(T)\Phi_E(x'(T)) \int_T^t b(r)\left(\int_T^r \frac{1}{a(s)}\,\mathrm{d}s\right)\mathrm{d}r \\ &\leqslant \frac{1}{2}a(T)\Phi_E(x'(T)) \end{aligned}$$

or

$$a(T)\Phi_E(x'(T)) \leqslant 2a(t)\Phi_E(x'(t)),$$

which gives a contradiction as $t \to \infty$ tends to infinity.

Finally, let x be a solution of (1.1). If (H3) and (4.3) hold, then we have

$$\lim_{t\to\infty} a(t)\Phi_E(x'(t)) = 0 \Leftrightarrow \lim_{t\to\infty} x'(t) = 0.$$

From this, using the above argument, (1.1) does not have solutions x satisfying (1.3). The proof is now complete. □

From Theorems 3.1, 4.1 and 4.2 we obtain the following necessary and sufficient criteria.

Corollary 4.1. *Assume that* (H2), (H3) *hold and linear equation* (3.2) *is nonoscillatory. Equation* (1.1) *has solutions x satisfying* (4.1) *if and only if* (H1) *is satisfied.*

Corollary 4.2. *Assume that* (H2), (H3), (4.3) *hold and linear equation* (3.2) *is nonoscillatory. Equation* (1.1) *has solutions x satisfying* (1.3) *if and only if* (H1) *is satisfied.*

5. Examples and concluding remarks

Theorem 3.1 requires the existence of a suitable nonoscillatory linear equation, which, roughly speaking, can be viewed with respect to (1.1), as a dominant equation. Indeed, this equation has to be a majorant of the linearized auxiliary equation (3.6). Useful examples of comparison nonoscillatory linear equations are the Riemann-Weber equation

$$w'' + \frac{1}{4t^2} + \frac{\gamma}{t^2\log^2 t}w = 0, \tag{5.1}$$

or the Euler equation

$$w'' + \gamma t^{-2} w = 0, \tag{5.2}$$

where $\gamma \leqslant 4^{-1}$ is a real constant. Both equations (5.1) and (5.2) are nonoscillatory and have been widely discussed in the literature, see, e.g., [19], pages 45 and 86.

Clearly, any other nonoscillatory linear equation which satisfies (H1) can be used as a majorant equation. For instance, the Euler equation and the Riemann-Weber equation are merely the first instance of an infinite sequence of linear differential equations, which form a kind of logarithmic scale. All these equations can be used as a comparison equations via the Sturm theory for obtaining oscillatory or nonoscillatory criteria, see [19], Theorem 2.43. Moreover, other oscillatory criteria can be obtained from (5.1) or (5.2) by using a suitable substitution. For instance, the substitution

$$z(t) = t^{-\lambda} w \tag{5.3}$$

transforms (5.2) with $\gamma = 4^{-1}$ into the linear equation

$$(t^{2\lambda} z')' + \Big(\lambda - \frac{1}{2}\Big)^2 t^{2(\lambda-1)}\, z = 0. \tag{5.4}$$

Moreover, (5.4) is nonoscillatory because transformation (5.3) maintains the oscillatory character. If $\lambda < 2^{-1}$, then (H1) is satisfied for (5.4) for any $t_0 > 0$. Hence, from Theorem 3.1 we get the following.

Corollary 5.1. *Let* (H2) *be verified. Assume that there exists* $\lambda \in (0, 2^{-1})$ *such that for large* t

$$a(t) \geqslant \frac{2}{\sqrt{3}} t^{2\lambda}, \quad b(t) \leqslant \frac{(\lambda - 2^{-1})^2}{F_M} t^{2(\lambda-1)}, \tag{5.5}$$

where F_M *is given in* (3.1)*. Then* (1.1) *has a solution* x *satisfying* (1.3).

P r o o f. From (5.5), conditions (H1) and (H3) are satisfied. Moreover, again in virtue of (5.5), linear equation (3.2) is a minorant of (5.4). Hence, the Sturm comparison theorem yields the nonoscillation of (3.2) and the assertion follows by Theorem 3.1. □

Clearly, analogous criteria can be obtained using the nonoscillation of (5.1) when $\gamma \leqslant 4^{-1}$. The following examples illustrate Theorem 3.1 and Corollary 5.1.

Example 5.1. Consider the equation with the Euclidean mean curvature operator

$$(\Phi_E(x'))' + kB(t)F(x) = 0, \quad t \geqslant \mathrm{e}, \tag{5.6}$$

where

$$B(t) = \frac{t}{(t^2+1)\sqrt{t^2+1}} \frac{1 + 8\log t}{\log^2 t} \quad \text{and} \quad F(u) = \frac{u^2}{8|u|+1} \operatorname{sgn} u.$$

Equation (3.2) for (5.6) reads as

$$w'' + \frac{k}{4\sqrt{3}} B(t) w = 0. \tag{5.7}$$

A standard calculation gives $kB(t) \leqslant t^{-2}$ for t sufficiently large and $k > 0$. Thus, linear equation (5.7) with $k > 1$ is a minorant of the Euler equation (5.2) with $\gamma = (4\sqrt{3})^{-1}$. Therefore, (5.7) is nonoscillatory for all $k > 0$. Moreover, assumptions (H1), (H2) and (H3) are satisfied for (5.6) and so, in view of Theorem 3.1, equation (5.6) has a solution x satisfying (1.3) for all $k > 0$. This fact can be directly verified by noticing that $x(t) = \log t$ is a solution of (5.6) with $k = 1$.

Example 5.2. Consider the equation with the Euclidean mean curvature operator

$$\Big(\frac{(t+\sqrt{3})\sqrt{4t+1}}{t\sqrt{3}}\Phi_E(x')\Big)' + \frac{k}{t^{3/2}(1+20t^{1/2})}F(x) = 0, \quad t \geqslant 1, \tag{5.8}$$

where $k > 0$ and

$$F(u) = \frac{u^2}{20|u|+1}\operatorname{sgn} u.$$

We have $a(t) \geqslant 2(3^{-1}t)^{1/2}$. With fixed $\lambda = 4^{-1}$, conditions (5.5) are verified for large t and any $k > 0$, since $F_M = 20^{-1}$ and

$$\frac{k}{t^{3/2}(1+20t^{1/2})} \leqslant \frac{5}{4}t^{-3/2}$$

for t sufficiently large. Moreover, F satisfies (H2) and so, by Corollary 5.1, equation (5.8) has a solution x which satisfies (1.3) for all $k > 0$. This fact can be directly verified in the case when $k = 1$ by noticing that $x(t) = \sqrt{t}$ is a solution of (5.8) with $k = 1$.

We end this section with some comments and open problems.

The method developed to prove the existence of unbounded solutions for (1.1), satisfying $\lim\limits_{t\to\infty} a(t)x'(t) = 0$, can be applied, with minor changes, to get similar results for the equation with the Minkowski mean curvature operator, sometimes called relativistic operator,

$$(a(t)\Phi_R(x'))' + b(t)F(x) = 0, \tag{5.9}$$

where Φ_R is defined in (3.11), see also [4], [13]. Recall that Φ_R is the inverse of the map Φ_E. The abstract existence result for solutions x of (5.9) satisfying $(x, a\Phi_R(x')) \in S_0$, given in Lemma 3.1 in the case of curvature operator Φ_E, continues to hold for Φ_R without *a-priori* estimates on v, that is, replacing (3.5) with

$$u(t) \neq 0 \quad \text{for all } (u, v) \in \Omega \text{ and } t \in I,$$

and considering, instead of (3.6), the linear equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\sqrt{a^2(t)+v^2(t)}\,y'\big) + b(t)\frac{F(u(t))}{u(t)}y = 0. \tag{5.10}$$

The fact that no a-priori estimates on v are needed in this case follows from the observation that the proof involves the inverse of the operator, and while Φ_R has a bounded domain, the operator Φ_E is defined on the whole real line.

This result leads to the following theorem, which is the analogue of Theorem 3.1.

Theorem 5.1. *Assume that* (H1), (H2) *and* (H3) *hold. If the linear equation*

$$\big(a(t)w'\big)' + F_M b(t)w = 0 \tag{5.11}$$

is nonoscillatory, then (5.9) *has a solution satifying* (1.3).

The proof of Theorem 5.1 is similar to the one of Theorem 3.1. More precisely, instead of (3.2), consider (5.11) as auxiliary linear equation, and replace (3.15) with a suitable bound, for instance $a(t)w_0'(t) \leqslant 5^{-1/2}A$. With this choice, a majorant and a minorant of (5.10) are equations (5.11) and

$$\Big(\frac{\sqrt{5}}{2}a(t)z'\Big)' + F_m b(t)z = 0,$$

respectively. Defining the same subsets Ω and S_1 of $C(I,\mathbb{R}^2)$, already considered in the proof of Theorem 3.1, the assertion follows using a similar argument, with minor changes.

The necessary conditions stated in Theorems 4.1 and 4.2 also continue to hold for (5.9), but Theorem 4.1 (i) requires now assumption (H3) and a slightly different proof. More precisely, instead of (4.2) we have for $t \geqslant T$

$$0 > -k = a(T)\Phi_R(x'(T) \geqslant a(t)\Phi_R(x'(t),$$

or

$$x'(t) \leqslant \Phi_E\Big(\frac{-k}{a(t)}\Big) = -\frac{k}{a(t)}\frac{a(t)}{\sqrt{a^2(t)+k^2}}. \tag{5.12}$$

Using (H3) we get that there exists $k_1 > 0$ such that

$$\frac{a(t)}{\sqrt{a^2(t)+k^2}} \geqslant k_1.$$

Thus, in view of (5.12), we obtain

$$x(t) \leqslant x(T) - k_2\int_T^t a^{-1}(s)\,\mathrm{d}s,$$

where $k_2 = kk_1$. This inequality gives the same contradiction as the one in the proof of Theorem 4.1 (i).

The above considerations naturally lead us to wonder if this approach can be used also to study the existence of unbounded solutions for more general nonlinear equation

$$(a(t)\Phi(x'))' + b(t)F(x) = 0,$$

where Φ is an odd homeomorphism. This is at the moment an open problem, and it will be studied in a forthcoming paper.

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