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Nonrationality Degree of Toric Quasifolds

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Abstract

Toric quasifolds are nonrational generalizations of toric manifolds and orbifolds. We introduce the notion of nonrationality degree, an invariant that measures the extent to which a toric quasifold fails to be Hausdorff. We do so by first defining analogous notions for the underlying quasilattice and the corresponding quasitorus. We conclude by applying the nonrationality degree to reframe Gordan's lemma in the nonrational setting.

Keywords: toric quasifold; quasilattice; quasitorus; nonrational fan; nonrational polytope

MSC: 14M25; 52B20

1. Introduction

Toric quasifolds are a natural generalization of toric manifolds and orbifolds in the nonrational case. They were first introduced by the second author, in the symplectic setting, in [1]. Their complex and algebraic–geometric counterparts were introduced by both authors in [2] and [3], respectively. These spaces are locally modeled by $\mathbb{C}^n \cong \mathbb{R}^{2n}$ modulo the action of countable groups and are generally not Hausdorff.

The aim of this article is to introduce the notion of *nonrationality degree* of toric quasifolds, which measures the extent to which they fail to be Hausdorff.

We recall that the starting data for a toric quasifold is provided by the *fundamental triple*

$$(\Sigma, Q, \{v_1, \dots, v_d\}),$$

where Σ is a complete simplicial fan in $\mathbb{R}^n$, Q is a quasilattice, namely the $\mathbb{Z}$–span of a set of $\mathbb{R}$–spanning vectors in $\mathbb{R}^n$, and the vectors $v_1, \dots, v_d$ are in Q and generate the rays of Σ. The corresponding toric quasifold, which we denote by X_Σ, is acted upon by the real quasitorus $\mathbb{R}^n/Q$ in the symplectic setting, and by the complex quasitorus $\mathbb{C}^n/Q$ in the complex and algebraic–geometric settings.

In order to define the nonrationality degree of X_Σ, we first investigate the structure of the underlying quasilattice Q.

As a consequence of the structure theorem for closed groups of $\mathbb{R}^n$, we prove that Q admits a canonical decomposition

$$Q = (Q \cap V) \oplus (Q \cap W),$$

where V and W are complementary subspaces of $\mathbb{R}^n$, $Q \cap V$ is dense in V, and $Q \cap W$ is a lattice in W (see Theorem 2). The subspace V is unique and maximal with this property. We define the nonrationality degree of Q to be the dimension of V. This integer will ultimately define the nonrationality degree of the real and complex quasitori and of the toric quasifold.

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In fact, the decomposition of Q naturally yields structure theorems for the corresponding quasitori. For example, in the complex case, we get

$$\mathbb{C}^n/Q \cong D^h_{\mathbb{C}} \times (\mathbb{C}^*)^{n-h},$$

where h is the nonrationality degree of Q and $D^h_{\mathbb{C}}$ is a totally nonrational complex quasitorus of dimension h (see Theorem 4). Accordingly, we define h to be the nonrationality degree of $\mathbb{C}^n/Q$. The same applies to $\mathbb{R}^n/Q$. Observe that Q is a lattice and both quasitori are tori if and only if their nonrationality degree h is zero. Increasing values of h correspond to the presence of higher–dimensional totally nonrational directions.

As in the rational case, the action of the complex quasitorus has a dense open orbit. This naturally leads to defining the notion of nonrationality degree of the toric quasifold to coincide with that of its residing quasitorus.

A given complete simplicial fan may be viewed in the context of infinitely many different triples, and the degree of the corresponding quasifolds will vary accordingly. We take the minimum possible nonrationality degree of these quasifolds to define the intrinsic nonrationality degree of the fan. The fan is rational if and only if its nonrationality degree is zero. This also applies to simple polytopes, by considering their normal fan. This invariant is purely convex–geometric. It has the interesting feature of providing a lower bound for the nonrationality degree of the corresponding toric quasifolds.

The motivation for introducing these notions of nonrationality degree is twofold. On the one hand, it comes from the framework of LVMB manifolds, a special class of complex foliated manifolds [4–6], and their measures of nonrationality (see ([7], Section 2.2.3)). The relation between these measures and the nonrationality degree of quasifolds is currently being investigated by F. Thiella. On the other hand, the motivation also stems from the study of nonrational toric geometry in the algebraic–geometric setting (see [3]). In this framework, replacing a lattice with a quasilattice has significant repercussions, such as the different behavior of duals (see Remarks 1 and 3) and the failure of Gordan's lemma, a fundamental result in classical toric geometry. In fact, we recast the lemma by proving that it holds if and only if the nonrationality degree of the quasilattice is zero (see Theorem 5). In other words, any positive nonrationality degree precludes the classical finite generation property.

We conclude by remarking that quasilattices are key objects in several different areas of mathematics. For example, they form the basis for the study of nonperiodic tilings that are related to quasicrystals (see, for example, [8,9]). They also appear, explicitly or implicitly, in most other works addressing nonrational toric geometry (see [7,10–15]). Moreover, they arise naturally in the framework of diffeology (see, for example, [16,17]).

We expect these notions of nonrationality degree to provide a conceptual basis for future developments in nonrational toric and convex geometry, as well as in other mathematical settings in which quasilattices arise naturally.

The remainder of this paper is organized as follows. In Section 2, we investigate the structure of quasilattices and define their nonrationality degree. In Section 3, we establish the structure theorems for quasitori and introduce their nonrationality degree. In Section 4, we extend this notion to toric quasifolds. In Section 5, we discuss the case of fans and polytopes. Finally, in Section 6, we reframe Gordan's lemma in the nonrational setting.

2. Nonrationality Degree of a Quasilattice

We begin by recalling the following

Definition 1 (Quasilattice)**.** *A quasilattice Q in $\mathbb{R}^n$ is the $\mathbb{Z}$–span of a set of $\mathbb{R}$–spanning vectors of $\mathbb{R}^n$.*

A quasilattice is, in particular, a free $\mathbb{Z}$–module of finite rank and a subgroup of $\mathbb{R}^n$. The *rank* of Q is the number of vectors in a set of minimal generators. We recall that a lattice L in $\mathbb{R}^n$ is the $\mathbb{Z}$–span of a basis of $\mathbb{R}^n$. It is therefore a quasilattice of rank equal to n. On the other hand, the classical definition of lattice ([18], Chapter 7, Section 2.2) is that of a subgroup L of $\mathbb{R}^n$ that satisfies the following equivalent conditions:

- L is discrete and $\mathbb{R}^n/L$ is compact;
- L is discrete and spans $\mathbb{R}^n$;
- There exists an $\mathbb{R}$–basis of $\mathbb{R}^n$, which is a $\mathbb{Z}$–basis of L.

Therefore, a quasilattice Q in $\mathbb{R}^n$ is a lattice if and only if it is a discrete subset of $\mathbb{R}^n$.

The aim of this section is to introduce the notion of nonrationality degree of a quasilattice. We will be making use of the classification theorem of closed subgroups of $\mathbb{R}^n$, which we recall from Bourbaki ([19], Chapter VII, Section 1.2, Theorem 2):

Theorem 1. *Let G be a closed subgroup of $\mathbb{R}^n$, and let $k = \dim(\mathrm{span}_{\mathbb{R}}(G))$. Then, there exists a maximal vector subspace $V \subset \mathbb{R}^n$, $\dim(V) \leq k$, such that, for every complement W of V, $\Lambda = W \cap G$ is a discrete subgroup of rank $r = k - \dim V$ and $G = V \oplus \Lambda$.*

Given a quasilattice Q in $\mathbb{R}^n$, we apply this classification theorem to the closed subgroup $\overline{Q}$ and obtain the following

Theorem 2. *Let Q be a quasilattice of $\mathbb{R}^n$. Then, there exists a unique maximal vector subspace $V \subset \mathbb{R}^n$ and a complement W of V such that*

$$Q = (Q \cap V) \oplus (Q \cap W),$$

where $Q \cap V$ is dense in V and $Q \cap W$ is a lattice in W.

Proof. Since Q is a subgroup of $\mathbb{R}^n$, $\overline{Q}$ is a closed subgroup of $\mathbb{R}^n$. Moreover, since Q spans $\mathbb{R}^n$, so does $\overline{Q}$. Applying Theorem 1 to $\overline{Q}$, we obtain a maximal vector subspace $V \subset \mathbb{R}^n$ such that, for every complement W of V, we have $\overline{Q} = V \oplus \Lambda$, where $\Lambda = \overline{Q} \cap W$ is a discrete subgroup of rank $r = n - \dim(V)$, thus a lattice in W. Let

$$\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^n/V$$

be the quotient map. Since Q is dense in $\overline{Q}$ and π is continuous, $\pi(Q)$ is dense in $\pi(\overline{Q})$. Since

$$\pi(\overline{Q}) \cong \overline{Q}/V \cong \Lambda,$$

the group $\pi(\overline{Q})$ is a lattice in $\mathbb{R}^n/V$; in particular, it is discrete.

Let us show that $\overline{\pi(Q)} = \pi(\overline{Q})$. Since $Q \subset \overline{Q}$, we have $\overline{\pi(Q)} \subset \overline{\pi(\overline{Q})}$. However, $\pi(\overline{Q})$ is discrete and, hence, closed. Therefore, $\overline{\pi(Q)} \subset \pi(\overline{Q})$. Conversely, consider an element $\pi(h)$, $h \in \overline{Q}$ and take a sequence $(q_m) \subset Q$ converging to h. By continuity, the sequence $(\pi(q_m)) \subset \pi(Q)$ converges to $\pi(h)$. Therefore, $\pi(\overline{Q}) \subset \overline{\pi(Q)}$.

Since $\pi(\overline{Q})$ is discrete and $\pi(Q)$ is dense in $\pi(\overline{Q})$, so we obtain $\pi(Q) = \pi(\overline{Q})$.

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Now choose a $\mathbb{Z}$–basis of $\pi(Q)$ and write it as $\{\pi(w_1), \dots, \pi(w_r)\}$, with $w_1, \dots, w_r \in Q$. Take the subspace

$$W = \mathrm{span}_{\mathbb{R}}(w_1, \dots, w_r).$$

Since the vectors $\pi(w_1), \dots, \pi(w_r)$ form a basis of $\mathbb{R}^n/V$, the subspace W is a complement of V.

We claim that

$$Q \cap W = \mathrm{span}_{\mathbb{Z}}(w_1, \dots, w_r).$$

Let $w \in Q \cap W$. Since $w \in W$, there exist real numbers $a_1, \dots, a_r$ such that $w = \sum_{i=1}^{r} a_i w_i$. Thus, $\pi(w) = \sum_{i=1}^{r} a_i \pi(w_i)$. Since $w \in Q$, its image via π belongs to $\pi(Q)$, and therefore, $\pi(w) = \sum_{i=1}^{r} m_i \, \pi(w_i)$, for uniquely determined integers $m_1, \dots, m_r$. Thus $a_i = m_i$ for all i. Therefore $w \in \mathrm{span}_{\mathbb{Z}}(w_1, \dots, w_r)$. The reverse inclusion is obvious, proving the claim. It follows that $Q \cap W$ is a discrete subgroup of W of rank $r = n - \dim(V)$, and is thus a lattice in W.

Now, let $u \in Q$. Since the classes $\pi(w_1), \dots, \pi(w_r)$ generate $\pi(Q)$, there exist integers $h_1, \dots, h_r$ such that $\pi(u) = \sum_{i=1}^{r} h_i \, \pi(w_i)$. Therefore,

$$u - \sum_{i=1}^{r} h_i w_i \in V.$$

Since both terms belong to Q, we obtain

$$u - \sum_{i=1}^{r} h_i w_i \in Q \cap V.$$

Thus, every element of Q can be written as the sum of an element of $Q \cap V$ and an element of $Q \cap W$. Therefore, $Q = (Q \cap V) + (Q \cap W)$. Since $V \cap W = \{0\}$, we have $(Q \cap V) \cap (Q \cap W) = \{0\}$, and therefore,

$$Q = (Q \cap V) \oplus (Q \cap W).$$

Let us now show that $Q \cap V$ is dense in V. Let $v \in V \subset \overline{Q}$, and take a sequence $(q_m) \subset Q$ converging to v. Since $Q = (Q \cap V) \oplus (Q \cap W)$, we can write $q_m = v_m + w_m$, with $v_m \in Q \cap V$ and $w_m \in Q \cap W$. Now, since $\mathbb{R}^n = V \oplus W$, we can consider the linear projection $p_V \colon \mathbb{R}^n \to V$. Since it is continuous, the sequence $(p_V(q_m)) = (v_m)$ converges to $p_V(v) = v$. Therefore, $Q \cap V$ is dense in V.

To conclude, we show that V is unique. Suppose there are two maximal subspaces, V_1 and V_2, such that $Q \cap V_1$ is dense in V_1, and $Q \cap V_2$ is dense in V_2. Note that $Q \cap (V_1 + V_2)$ is dense in $V_1 + V_2$. Indeed, take $v_1 + v_2 \in V_1 + V_2$. We know that there exists a sequence $(v_m^1) \subset Q \cap V_1$ converging to v_1 and a sequence $(v_m^2) \subset Q \cap V_2$ converging to v_2. Then, the sequence $(v_m^1 + v_m^2) \subset Q \cap (V_1 + V_2)$ converges to $v_1 + v_2$. Therefore, by maximality, $\dim(V_1 + V_2) \leq \dim(V_1) = \dim(V_2)$, which implies $V_1 = V_2$. $\square$

Definition 2 (Maximal totally nonrational subspace)**.** *Given a quasilattice Q in $\mathbb{R}^n$, we call the unique subspace V of Theorem 2 maximal totally nonrational subspace with respect to Q.*

We are now finally ready for the following

Definition 3 (Nonrationality degree of a quasilattice)**.** *Let Q be a quasilattice in $\mathbb{R}^n$. The nonrationality degree of Q, denoted $\deg_{\mathrm{NR}}(Q)$, is the dimension of the maximal totally nonrational subspace V.*

We clearly have

$$0 \leq \deg_{\mathrm{NR}}(Q) \leq n.$$

As a first, straightforward, consequence of Theorem 2, we get

Corollary 1. *Let Q be a quasilattice in $\mathbb{R}^n$. Then, $\deg_{\mathrm{NR}}(Q) = 0$ if and only if Q is a lattice.*

One of the crucial differences between lattices and quasilattices is in the notion of dual. Let us first recall from [3] the following:

Definition 4 (Weak dual of a quasilattice)**.** *Given a quasilattice Q in $\mathbb{R}^n$, we define the weak dual of Q to be the $\mathbb{Z}$–module*

$$Q^\vee = \{\lambda \in (\mathbb{R}^n)^* \mid \lambda(Q) \subset \mathbb{Z}\}.$$

As a consequence of Theorem 2, we see that the elements of $Q^\vee$ only keep track of rational directions:

Corollary 2. *Let Q be a quasilattice of $\mathbb{R}^n$ and consider the subspaces V, W of Theorem 2. Then, we have*

1. *$Q^\vee \cong (Q \cap W)^*$ and is thus a discrete subgroup of $(\mathbb{R}^n)^*$ of rank $r = n - \deg_{\mathrm{NR}}(Q)$;*
2. *$V = \mathrm{Ann}(Q^\vee)$.*

Proof.

1. The direct sum decomposition $\mathbb{R}^n = V \oplus W$ allows us to write $\lambda = \lambda|_V + \lambda|_W$ for every $\lambda \in (\mathbb{R}^n)^*$. Now note that, if $\lambda \in Q^\vee$, by continuity, $\lambda(\overline{Q}) \subset \mathbb{Z}$. Thus, since $V \subset \overline{Q}$ and V is connected, we have $\lambda|_V = 0$. Therefore, $\lambda = \lambda|_W$ and

$$Q^\vee = \{\lambda \in (\mathbb{R}^n)^* \mid \lambda(Q) \subset \mathbb{Z}\} \cong \{\mu \in W^* \mid \mu(Q \cap W) \subset \mathbb{Z}\} = (Q \cap W)^*.$$

2. We have already noticed that $\lambda|_V = 0$, for all $\lambda \in Q^\vee$. Thus, $V \subset \mathrm{Ann}(Q^\vee)$. However, by 1. we have

$$\dim(\mathrm{Ann}(Q^\vee)) = n - \mathrm{rank}(Q^\vee) = n - (n - \deg_{\mathrm{NR}}(Q)) = \dim(V).$$

So, the two subspaces must coincide.

□

Remark 1. *Consider now the actual dual, $\mathrm{Hom}(Q, \mathbb{Z})$; it is a free $\mathbb{Z}$–module having the same rank as Q. When Q is a lattice, $\mathrm{Hom}(Q, \mathbb{Z})$ is naturally identified with $Q^\vee$ and can be thus thought of as a lattice in $(\mathbb{R}^n)^*$. However, if $\deg_{\mathrm{NR}}(Q) \neq 0$, by Corollary 2, this is no longer the case. This fact has some notable consequences in nonrational toric geometry (see [3] and Remark 3 below).*

It will be useful, for what is to follow, to have Theorem 2 reframed as

Corollary 3. *Let Q be a quasilattice in $\mathbb{R}^n$ and let $h = \deg_{\mathrm{NR}}(Q)$. Then, there is an isomorphism*

$$f \colon \mathbb{R}^n \longrightarrow \mathbb{R}^n = \mathbb{R}^h \oplus \mathbb{R}^{n-h}$$

such that

$$Q \cong f(Q) = (f(Q) \cap \mathbb{R}^h) \oplus \mathbb{Z}^{n-h},$$

where $f(Q) \cap \mathbb{R}^h$ is a dense quasilattice in $\mathbb{R}^h$.

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Proof. Consider the subspaces V and W of Theorem 2. Choose a basis $\{v_1, \ldots, v_h, w_1, \ldots, w_{n-h}\}$ of $\mathbb{R}^n$ such that $\{v_1, \ldots, v_h\}$ is a basis of V and $\{w_1, \ldots, w_{n-h}\}$ is a $\mathbb{Z}$–basis of $Q \cap W$. Let f be the isomorphism that sends this basis to the standard basis of $\mathbb{R}^n$. $\square$

We conclude this section with some examples.

Example 1. *Let us consider the quasilattice in $\mathbb{R}$ given by $Q = \mathbb{Z} + a\mathbb{Z}$, with a a positive real number. The nonrationality degree obviously depends on a. In fact, if $a \in \mathbb{Z}$, we have $Q = \overline{Q} = \mathbb{Z}$, $V = \{0\}$, $Q^\vee = \mathbb{Z}$. If $a \in \mathbb{Q} \setminus \mathbb{Z}$, and we write $a = \frac{p}{q}$, with $p, q \in \mathbb{Z}_{>0}$ coprime, then $Q = \overline{Q} = \frac{1}{q}\mathbb{Z}$, $V = \{0\}$, $Q^\vee = q\mathbb{Z}$. Finally, if $a \notin \mathbb{Q}$, then $\overline{Q} = \mathbb{R}$, $V = \mathbb{R}$, $Q^\vee = \{0\}$. In conclusion,*

$$\deg_{\mathrm{NR}}(Q) = \begin{cases} 0 & a \in \mathbb{Q} \\ 1 & a \notin \mathbb{Q}. \end{cases}$$

Example 2. *Consider the quasilattice $Q = \mathbb{Z} \times (\mathbb{Z} + a\mathbb{Z})$ in $\mathbb{R}^2$, where a is a positive real number. Then, if $a \in \mathbb{Z}$, we have $Q = \overline{Q} = \mathbb{Z}^2$, $V = \{0\}$, $Q^\vee = \mathbb{Z}^2$. If $a \in \mathbb{Q} \setminus \mathbb{Z}$, and we write $a = \frac{p}{q}$, with $p, q \in \mathbb{Z}_{>0}$ coprime, then $Q = \overline{Q} = \mathbb{Z} \times \frac{1}{q}\mathbb{Z}$, $V = \{0\}$, $Q^\vee = \mathbb{Z} \times q\mathbb{Z}$. Finally, if $a \notin \mathbb{Q}$, then $\overline{Q} = \mathbb{Z} \times \mathbb{R}$, $V = \{0\} \times \mathbb{R}$, $Q^\vee = \mathbb{Z} \times \{0\}$. Here, we have, again,*

$$\deg_{\mathrm{NR}}(Q) = \begin{cases} 0 & a \in \mathbb{Q} \\ 1 & a \notin \mathbb{Q}. \end{cases}$$

Example 3. *Consider the fifth roots of unity in $\mathbb{C} \cong \mathbb{R}^2$*

$$\begin{aligned} u_0 &= (1, 0) \\ u_1 &= (\cos\tfrac{2\pi}{5}, \sin\tfrac{2\pi}{5}) \\ u_2 &= (\cos\tfrac{4\pi}{5}, \sin\tfrac{4\pi}{5}) \\ u_3 &= (\cos\tfrac{6\pi}{5}, \sin\tfrac{6\pi}{5}) \\ u_4 &= (\cos\tfrac{8\pi}{5}, \sin\tfrac{8\pi}{5}). \end{aligned}$$

Their $\mathbb{Z}$–span, Q_5, is a dense quasilattice in $\mathbb{R}^2$. To see this, notice that the fifth roots of unity generate a four-dimensional $\mathbb{Q}$–vector space, since the fifth cyclotomic polynomial has degree 4. Hence, rank$(Q_5) = 4$. By Theorem 2 and Corollary 1, there exists a unique maximal linear subspace V of $\mathbb{R}^2$ such that $Q_5 \cap V$ is dense in V, and dim$(V) \geq 1$. Since Q_5 is invariant under rotation by an angle $2\pi/5$, by uniqueness, so is V. As no real one–dimensional subspace is invariant under such a rotation, it follows that $\dim V = 2$; hence, $V = \mathbb{R}^2$. This quasilattice turns out to be the crucial ingredient in the study of Penrose rhombus tilings and of the Penrose kite from the nonrational symplectic toric viewpoint (see [9] and references therein, and also Example 10 below). Here, we have $\overline{Q} = V = \mathbb{R}^2$, $Q^\vee = 0$ and, finally, $\deg_{\mathrm{NR}}(Q) = 2$.

3. Structure Theorems for Quasitori

We begin by recalling the following

Definition 5 (Real and complex quasitori)**.** *Let Q be a quasilattice in $\mathbb{R}^n$, the group $\mathbb{R}^n/Q$ is called a real quasitorus of dimension n and the group $\mathbb{C}^n/Q$ is called a complex quasitorus of dimension n.*

Notice that when Q is a lattice, these groups are, respectively, a real torus and a complex torus. At the other extreme, we have

Definition 6 (Totally nonrational quasitori)**.** *Consider a quasilattice Q in $\mathbb{R}^n$. We say that the quasitori $\mathbb{R}^n/Q$ and $\mathbb{C}^n/Q$ are totally nonrational if Q is dense in $\mathbb{R}^n$.*

The general case is a bit of both, as described in the following structure theorems.

Theorem 3 (Structure theorem for a real quasitorus)**.** *Let Q be a quasilattice in $\mathbb{R}^n$, and let $h = \deg_{\mathrm{NR}}(Q)$. Then,*

$$\mathbb{R}^n/Q \cong D^h \times \mathbb{T}^{n-h},$$

where D^h is a totally nonrational real quasitorus of dimension h and $\mathbb{T}^{n-h}$ is the standard torus of dimension $n-h$.

Proof. By Corollary 3, we have

$$Q \cong (f(Q) \cap \mathbb{R}^h) \oplus \mathbb{Z}^{n-h},$$

where $f(Q) \cap \mathbb{R}^h$ is dense in $\mathbb{R}^h$. Thus,

$$\mathbb{R}^n/Q \cong \left(\mathbb{R}^h/(f(Q) \cap \mathbb{R}^h)\right) \times \left(\mathbb{R}^{n-h}/\mathbb{Z}^{n-h}\right).$$

□

Similarly, in the complex case, we have

Theorem 4 (Structure theorem for a complex quasitorus)**.** *Let Q be a quasilattice in $\mathbb{R}^n$, and let $h = \deg_{\mathrm{NR}}(Q)$. Then,*

$$\mathbb{C}^n/Q \cong D^h_{\mathbb{C}} \times (\mathbb{C}^*)^{n-h},$$

where $D^h_{\mathbb{C}}$ is a totally nonrational complex quasitorus of dimension h.

Motivated by this, we have the following

Definition 7 (Nonrationality degree of a quasitorus)**.** *Let Q be a quasilattice in $\mathbb{R}^n$. We call nonrationality degree of the quasitorus $\mathbb{R}^n/Q$ (respectively $\mathbb{C}^n/Q$), denoted $\deg_{\mathrm{NR}}(\mathbb{R}^n/Q)$ (respectively $\deg_{\mathrm{NR}}(\mathbb{C}^n/Q)$), the nonrationality degree of Q.*

Let us see some examples. To avoid unnecessary repetition, we will be focusing on the real case.

Example 4. *Let us consider the quasilattice $\mathbb{Z} + a\mathbb{Z}$ in $\mathbb{R}$, where a is a positive real number (see Example 1) and take the quasitorus*

$$\mathbb{R}/(\mathbb{Z} + a\mathbb{Z}).$$

If $a \in \mathbb{Z}$, we obtain $\mathbb{R}/\mathbb{Z} \cong S^1$. If $a \in \mathbb{Q} \setminus \mathbb{Z}$, and we write $a = \frac{p}{q}$, with $p, q \in \mathbb{Z}_{>0}$ coprime, we get $\mathbb{R}/\left(\frac{1}{q}\mathbb{Z}\right) \cong S^1$. Finally, if $a \notin \mathbb{Q}$, we get the irrational torus $\mathbf{T}^a$, which was first introduced within the framework of diffeology by Donato–Iglesias [16]. In conclusion, we have

$$\deg_{\mathrm{NR}}(\mathbb{R}/(\mathbb{Z} + a\mathbb{Z})) = \begin{cases} 0 & a \in \mathbb{Q} \\ 1 & a \notin \mathbb{Q}. \end{cases}$$

Example 5. *Consider the quasilattice $Q = \mathbb{Z} \times (\mathbb{Z} + a\mathbb{Z})$ in $\mathbb{R}^2$, where a is a positive real number (see Example 2) and take the quasitorus*

$$\mathbb{R}^2/(\mathbb{Z} \times (\mathbb{Z} + a\mathbb{Z})).$$

Then, if $a \in \mathbb{Z}$, we obtain $\mathbb{R}^2/\mathbb{Z}^2$. If $a \in \mathbb{Q} \setminus \mathbb{Z}$, and we write $a = \frac{p}{q}$, with $p, q \in \mathbb{Z}_{>0}$ coprime, then we get $(\mathbb{R}/\mathbb{Z}) \times \left(\mathbb{R}/\frac{1}{q}\mathbb{Z}\right) \cong \mathbb{R}^2/\mathbb{Z}^2$. Finally, if $a \notin \mathbb{Q}$ we get $(\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/(\mathbb{Z} + a\mathbb{Z})) \cong S^1 \times \mathbf{T}^a$. In particular,

$$\deg_{\mathrm{NR}}(\mathbb{R}^2/(\mathbb{Z} \times (\mathbb{Z} + a\mathbb{Z}))) = \begin{cases} 0 & a \in \mathbb{Q} \\ 1 & a \notin \mathbb{Q}. \end{cases}$$

Example 6. *Take the $\mathbb{Z}$–span, Q_5, of the fifth roots of unity (see Example 3). Since Q_5 is a dense quasilattice in $\mathbb{R}^2$, the quasitorus $\mathbb{R}^2/Q_5$ is totally nonrational and its nonrationality degree is* 2.

4. Nonrationality Degree of a Toric Quasifold

Let Σ be a complete simplicial fan in $\mathbb{R}^n$ (we refer the reader to [20] for the necessary background on fans).

We recall that a *fundamental triple* for Σ is given by

$$(\Sigma, Q, \{v_1, \ldots, v_d\}),$$

where Q is a quasilattice in $\mathbb{R}^n$, and the vectors $v_1, \ldots, v_d \in Q$ generate the rays of Σ. One way of obtaining a fundamental triple for *any* Σ is to first choose a set of ray generators and then take Q to be their $\mathbb{Z}$–span.

We recall that the fan Σ is *rational* if all of its rays have nonzero intersection with a lattice L. Notice that, in this case, there is an underlying canonical triple, obtained by taking Q to be equal to L and the ray generators to be primitive in L.

To any fundamental triple, there corresponds a toric quasifold, which we will denote by X_Σ. For the symplectic construction of X_Σ, which holds when Σ is also polytopal, see [1]; its complex and algebraic–geometric counterparts are treated in [2] and [3], respectively. The space X_Σ is a toric manifold or orbifold if and only if the fan Σ is rational and Q is a lattice, if and only if $\deg_{\mathrm{NR}}(Q) = 0$.

Analogously to what happens in the rational case, the quasitorus acts on X_Σ and, in the complex setting, there is a dense open orbit

$$\mathcal{O}_o \cong \mathbb{C}^n/Q$$

which corresponds to the zero cone; all other points of X_Σ belong to the lower dimensional orbits corresponding to the nonzero cones of Σ. It is natural to define the nonrationality degree of X_Σ to be the same as that of the quasitorus, namely

Definition 8 (Nonrationality degree of a toric quasifold)**.** *We define the nonrationality degree of X_Σ to be the nonrationality degree of Q, denoted* $\deg_{\mathrm{NR}}(X_\Sigma)$.

Definition 9 (Totally nonrational quasifold)**.** *We say that the quasifold X_Σ is totally nonrational when the corresponding quasilattice Q is dense.*

Let us compute the nonrationality degree of some notable examples of toric quasifolds.

Example 7 (Quasisphere)**.** *Consider the quasisphere X_a, namely the toric quasifold corresponding to the triple*

$$(\Sigma, \mathbb{Z} + a\mathbb{Z}, \{a, -1\}),$$

where a is a positive real number. Σ is the complete simplicial fan in $\mathbb{R}$ given by the cones $\mathbb{R}_{\geq 0}$, $\mathbb{R}_{\leq 0}$, and $\{0\}$. We refer the reader to ([1], Example 1.13) for the symplectic construction, to ([2], Examples 2.6, 3.8), ([21], Example 3.1) for the complex, and to [3] for the algebraic–geometric.

Since the underlying quasilattice is $\mathbb{Z} + a\mathbb{Z}$, we have, by Example 1,

$$\deg_{\mathrm{NR}}(X_a) = \begin{cases} 0 & a \in \mathbb{Q} \\ 1 & a \notin \mathbb{Q}. \end{cases}$$

The quasisphere is totally nonrational if and only if $a \notin \mathbb{Q}$.

Example 8 (Generalized weighted projective space)**.** *We consider now generalized weighted projective space $\mathbb{CP}^2_{(1,1,a)}$. It is the toric quasifold corresponding to the triple*

$$(\Sigma, \mathbb{Z} \times (\mathbb{Z} + a\mathbb{Z}), \{(1,0), (-1,-a), (0,1)\}),$$

where a is any positive real number. Σ is the complete simplicial fan in $\mathbb{R}^2$ whose rays are generated by the vectors $(1,0)$, $(-1,-a)$, and $(0,1)$. See ([1], Example 3.6), ([2], Example 2.8), ([21], Example 3.2), and [3] for descriptions of this toric quasifold in the symplectic, complex, and algebraic–geometric settings, respectively.

Since the underlying quasilattice here is $\mathbb{Z} \times (\mathbb{Z} + a\mathbb{Z})$, we have, by Example 2,

$$\deg_{\mathrm{NR}}(\mathbb{CP}^2_{(1,1,a)}) = \begin{cases} 0 & a \in \mathbb{Q} \\ 1 & a \notin \mathbb{Q}. \end{cases}$$

Example 9 (Generalized Hirzebruch surface)**.** *Take now the generalized Hirzebruch surface $\mathbb{H}_a$. This is the toric quasifold corresponding to the triple*

$$(\Sigma, \mathbb{Z} \times (\mathbb{Z} + a\mathbb{Z}), \{(1,0), (-1,-a), (0,1), (0,-1)\}),$$

where a is again any positive real number. Σ is the complete simplicial fan in $\mathbb{R}^2$ whose rays are generated by the vectors $(1,0)$, $(-1,-a)$, $(0,1)$, and $(0,-1)$. The toric quasifold $\mathbb{H}_a$ was first introduced in both the symplectic and complex setting in [22]; see [3] for the algebraic–geometric version. The corresponding quasilattice is again $\mathbb{Z} \times (\mathbb{Z} + a\mathbb{Z})$, so we have

$$\deg_{\mathrm{NR}}(\mathbb{H}_a) = \begin{cases} 0 & a \in \mathbb{Q} \\ 1 & a \notin \mathbb{Q}. \end{cases}$$

Example 10 (The Penrose kite quasifold)**.** *Let us consider the toric quasifold X_K corresponding to the triple*

$$(\Sigma, Q_5, \{v_1, v_2, v_3, v_4\}),$$

where $v_1 = -u_1$, $v_2 = -u_3$, $v_3 = u_2$, $v_4 = u_4$ (see Example 3). Σ is the complete simplicial fan whose rays are generated by these vectors. It is the normal fan of the Penrose kite. This toric quasifold was introduced and studied from the symplectic viewpoint in [9]; see ([21], Example 3.2) for the complex case and [3] for the algebraic–geometric. Since the underlying quasilattice is the pentagonal quasilattice Q_5, by Example 3, we have that $\deg_{\mathrm{NR}}(X_K) = 2$. *The toric quasifold X_K is totally nonrational.*

5. Nonrationality Degree of Fans and Polytopes

Let Σ be a complete simplicial fan in $\mathbb{R}^n$. Take the set $\mathcal{Q}_\Sigma$ of all quasilattices $Q \subset \mathbb{R}^n$ that have nonzero intersection with all rays of Σ.

Definition 10 (Nonrationality degree of a fan)**.** *We define the nonrationality degree of Σ to be the nonnegative integer*

$$\deg_{\mathrm{NR}}(\Sigma) = \min_{Q \in \mathcal{Q}_\Sigma} \deg_{\mathrm{NR}}(Q).$$

Note that the minimum is taken over a nonempty set of nonnegative integers; hence, it is attained.

Let us consider now an n–dimensional simple polytope $\Delta \subset \mathbb{R}^n$ (a standard reference for polytopes is [23]). Let Σ_Δ be its normal fan; it is a complete simplicial fan. We recall that the polytope is rational if and only if its normal fan is.

Definition 11 (Nonrationality degree of a polytope)**.** *We define the nonrationality degree of Δ to be the nonrationality degree of Σ_Δ.*

Remark 2. *Clearly, the fan Σ is rational if and only if its nonrationality degree is 0. Notice also that*

$$\deg_{\mathrm{NR}}(\Sigma) \leq \deg_{\mathrm{NR}}(X_\Sigma)$$

for any associated quasifold X_Σ. In other words, the nonrationality degree of the fan provides a lower bound for the nonrationality degree of the toric quasifold. A similar discussion applies to the polytope Δ.

Let us now compute the nonrationality degree of the fans that we introduced in the examples of Section 4.

Example 11. *The fan of Example 7 is rational with respect to $\mathbb{Z}$, so its nonrationality degree is 0.*

Example 12. *The fans of Examples 8 and 9 are rational with respect to the lattice $\mathbb{Z} \times a\mathbb{Z}$, so the nonrationality degree of both is also 0.*

Example 13. *The Penrose kite (see Example 10), and its normal fan, are nonrational. In fact, fix arbitrary positive real multiples of the four fifth roots of unity that generate the rays of Σ: $\{r_1u_1, r_2u_2, r_3u_3, r_4u_4\}$ (see Example 3). We claim that no lattice contains all these vectors. Suppose that L is such a lattice, and let $\{w_1, w_2\}$ be a basis of L. Then, for all i, $r_iu_i = m_iw_1 + n_iw_2$, with $m_i, n_i \in \mathbb{Z}$. Denote by $a_{ij} = \det(r_iu_i, r_ju_j)$. We have*

$$a_{ij} = r_ir_j \det(u_i, u_j) = r_ir_j \sin\left(\frac{2(j-i)\pi}{5}\right).$$

However, we also have

$$a_{ij} = (m_in_j - m_jn_i) \det(w_1, w_2).$$

Consider now the ratio

$$\frac{a_{12}a_{34}}{a_{41}a_{23}}$$

By the first identity, it is equal to the golden ratio $\frac{1+\sqrt{5}}{2}$, whereas the second identity shows that it is rational, a contradiction. The nonrationality degree is actually 1: normalize the fifth roots of unity so that each has x–coordinate equal to either 0 or 1, and then consider their $\mathbb{Z}$–span, $\widetilde{Q}_5$. By construction, $\deg_{\mathrm{NR}}(\widetilde{Q}_5) = 1$.

6. Gordan's Lemma Revisited

Let Q be a quasilattice in $\mathbb{R}^n$ and consider a maximal simplicial cone σ in $\mathbb{R}^n$ whose rays have nonzero intersection with Q. Gordan's lemma (see, for example, ([20], Proposition 1.2.17)) fails in this setting. As it turns out, this failure is measured precisely by the nonrationality degree of Q:

Theorem 5 (Gordan's Lemma revisited)**.** *Let Q be a quasilattice in $\mathbb{R}^n$ and let σ be a maximal simplicial cone whose rays all have nonzero intersection with Q. Then, the semigroup $\sigma \cap Q$ is finitely generated if and only if* $\deg_{\mathrm{NR}}(Q) = 0$.

Proof. If $\deg_{\mathrm{NR}}(Q) = 0$, then Q is a lattice, and the semigroup $\sigma \cap Q$ is finitely generated by Gordan's lemma. Conversely, suppose that the semigroup $\sigma \cap Q$ is finitely generated. We first prove that it is discrete. First, consider a finite set of generators of $\sigma \cap Q$. Since each ray of σ has nonzero intersection with Q, the above set of generators must contain at least one generator lying on each ray of σ. Indeed, in a simplicial cone, a ray cannot be generated unless one of the generators lies on that ray. Denote these n vectors by $\{v_1, \dots, v_n\}$ and let $\{v_1, \dots, v_n, v_{n+1}, \dots, v_{n+k}\}$ be the whole set of generators of the semigroup $\sigma \cap Q$. Then, for each $j = n+1, \dots, n+k$ we have $v_j = \sum_{l=1}^{n} a_{lj} v_l$, with a_{lj} nonnegative real numbers. Write each $a_{lj} = m_{lj} + b_{lj}$, with m_{lj} a nonnegative integer and $0 \leq b_{lj} < 1$. Let $u_j = \sum_{l=1}^{n} b_{lj} v_l$ and discard the vectors u_j that are equal to 0. After renumbering, we are left with $u_{n+1}, \dots, u_{n+h}$, with $h \leq k$. Then, it is straightforward to verify that the set $\{v_1, \dots, v_n, u_{n+1}, \dots, u_{n+h}\}$ still generates the semigroup $\sigma \cap Q$. Moreover, the vectors of this set all lie in the zonotope

$$\Delta = \left\{ \sum_{i=1}^{n} t_i v_i \mid 0 \leq t_i \leq 1 \right\}$$

defined by $v_1, \dots, v_n$. We now prove that $\Delta \cap Q = \Delta \cap (Q \cap \sigma)$ is finite. Indeed, let $u \in Q \cap \Delta$. Then, since $u \in Q \cap \sigma$, we have

$$u = \sum_{i=1}^{n} m_i v_i + \sum_{j=n+1}^{n+h} n_j u_j$$

with m_i, n_j nonnegative integers. Hence,

$$u = \sum_{i=1}^{n} \left(m_i + \sum_{j=n+1}^{n+h} n_j b_{ij} \right) v_i.$$

The matrix (b_{ij}) has entries $0 \leq b_{lj} < 1$ and all of its columns are nonzero, since the u_j's are nonzero and the v_i's are linearly independent. On the other hand, since $u \in Q \cap \Delta$ we must have

$$0 \leq m_i + \sum_{j=n+1}^{n+h} n_j b_{ij} \leq 1.$$

Each m_i can only be 0 or 1. Moreover, since every column of the matrix (b_{ij}) is nonzero, for every $j = n+1, \dots, n+h$ there exists an index i such that $b_{ij} > 0$. Therefore, $n_j b_{ij} \leq \sum_{l=n+1}^{n+h} n_l b_{il} \leq 1$, which implies that $n_j \leq \frac{1}{b_{ij}}$. Hence, each n_j is bounded above and there are only finitely many possible tuples $(m_1, \dots, m_n, n_{n+1}, \dots, n_{n+h})$. Therefore, $\Delta \cap (Q \cap \sigma)$ is finite. Moreover, the semigroup is the union of the translates $\Delta + \sum_{i=1}^{n} m_i v_i$ with m_i nonnegative integers. It is therefore discrete.

Finally, we observe that this implies that Q is, in fact, a lattice. Indeed, if it were a quasilattice of rank greater than n, then, by Corollary 1, $\deg_{\mathrm{NR}}(Q) > 0$. Hence, the vector subspace V such that $Q \cap V$ is dense in V would have positive dimension. Consequently, there would exist a sequence $(q_m) \subset Q \setminus \{0\}$ converging to 0. Now, consider the point $v = v_1 + \cdots + v_n$ in the interior of σ. Then, the sequence $(q_m + v)$ is contained in the interior of σ for all sufficiently large m. Hence $(q_m + v)$ is a sequence in $Q \cap \sigma$ converging to v, contradicting the discreteness of $Q \cap \sigma$. □

Remark 3. *We have already seen in Remark 1 that, if* $\deg_{\mathrm{NR}}(Q) \neq 0$*, the* $\mathbb{Z}$*–module* $\mathrm{Hom}(Q,\mathbb{Z})$ *can no longer be thought of as a quasilattice in* $(\mathbb{R}^n)^*$*. Even if that were the case, by Theorem 5, the semigroup algebra associated with the cone* σ *would not be finitely generated. Consequently, it cannot, in general, serve as the coordinate ring of an affine toric quasifold.*

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References

1. Prato , E. Simple non–rational convex polytopes via symplectic geometry. *Topology* **2001**, *40*, 961–975. [CrossRef]
2. Battaglia, F.; Prato, E. Generalized toric varieties for simple nonrational convex polytopes. *Int. Math. Res. Not.* **2001**, *24*, 1315–1337.
3. Battaglia, F.; Prato, E. Algebraic Toric Quasifolds. *arXiv* **2026**, arXiv:2604.15192 .
4. López de Medrano, S.; Verjovsky, A. A new family of complex, compact, non symplectic manifolds. *Bull. Braz. Math. Soc.* **1997**, *28*, 253–269. [CrossRef]
5. Meersseman, L. A new geometric construction of compact complex manifolds in any dimension. *Math. Ann.* **2000**, *317*, 79–115. [CrossRef]
6. Bosio, F. Variétés complexes compactes: Une généralisation de la construction de Meersseman et de López de Medrano–Verjovsky. *Ann. Inst. Fourier* **2001**, *51*, 1259–1297. [CrossRef]
7. Battaglia, F.; Zaffran, D. Foliations modeling nonrational simplicial toric varieties. *Int. Math. Res. Not.* **2015**, *2015*, 11785–11815. [CrossRef]
8. Senechal, M. *Quasicrystals and Geometry*; Cambridge University Press: Cambridge, UK, 1995.
9. Battaglia, F.; Prato, E. The symplectic Penrose kite. *Comm. Math. Phys.* **2010**, *299*, 577–601. [CrossRef]
10. Ratiu, T.; Zung, T.N. Presymplectic convexity and (ir)rational polytopes. *J. Symplectic Geom.* **2019**, *17*, 1479–1511. [CrossRef]
11. Hoffman, B.; Sjamaar, R. Stacky Hamiltonian actions and symplectic reduction. *Int. Math. Res. Not.* **2021**, *2021*, 15209–15300. [CrossRef]
12. Hoffman, B. Toric symplectic stacks. *Adv. Math.* **2020**, *368*, 107135. [CrossRef]
13. Katzarkov, L.; Lupercio, E.; Meersseman, L.; Verjovsky, A. Quantum (non–commutative) toric geometry: Foundations. *Adv. Math.* **2021**, *391*, 107945. [CrossRef]
14. Ishida, H.; Krutowski, R.; Panov, T. Basic cohomology of canonical holomorphic foliations on complex moment–angle manifolds. *Int. Math. Res. Not.* **2022**, *2022*, 5541–5563. [CrossRef]
15. Boivin, A. Non–simplicial quantum toric varieties. *Adv. Math.* **2024**, *441*, 109553. [CrossRef]
16. Donato, P.; Iglesias, P. Exemples de groupes différentiels: Flots irrationnels sur le tore. *C. R. Acad. Sci. Paris Sér. I Math.* **1985**, *301*, 127–130.
17. Iglesias-Zemmour, P.; Prato, E. Quasifolds, Diffeology and Noncommutative Geometry. *J. Noncommut. Geom.* **2021**, *15*, 735–759. [CrossRef]
18. Serre, J.-P. *Cours d'Arithmétique*, 3rd ed.; Presses Universitaires de France: Paris, France, 1988.
19. Bourbaki, N. *Éléments de Mathématique. Livre III: Topologie Générale. Chapitres V–VIII*; Hermann: Paris, France, 1963.
20. Cox, D.; Little, J.; Schenck, H. *Toric Varieties*; Graduate Studies in Mathematics; American Mathematical Society: Providence, RI, USA , 2011; Volume 124.
21. Battaglia, F.; Prato, E. Generalized Laurent monomials in nonrational toric geometry. *Contemp. Math.* **2024**, *794*, 179–193. [CrossRef]

22. Battaglia, F.; Prato, E.; Zaffran, D. Hirzebruch surfaces in a one–parameter family. *Boll. Unione Mat. Ital.* **2019**, *12*, 293–305. [CrossRef]
23. Ziegler, G. *Lectures on Polytopes*; Graduate Texts in Mathematics; Springer: New York, NY, USA, 1995; Volume 152.