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The McKay conjecture with group automorphisms and the Okuyama-Wajima argument ☆

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ABSTRACT

Let N be normal subgroup of a finite group G, p be a prime, P be a Sylow p-subgroup of G and θ be a P-invariant irreducible character of N. Suppose that G/N is a p-solvable group. In this note we show that, whenever a finite group A acts on G stabilizing P, there exists an A-equivariant McKay bijection between irreducible characters lying over θ of degree prime to p of G and $\mathbf{N}_G(P)N$. This is a consequence of a recent result of D. Rossi. Our approach here is independent from Rossi's and follows the original idea of the proof of the McKay conjecture for p-solvable groups. In particular, we rely on the so-called Okuyama-Wajima argument to deal with characters above Glauberman correspondents. For this purpose, we generalize a classical result of P. X. Gallagher on the number of irreducible characters of G lying over θ.

0. Introduction

Let G be a finite group, p be a prime and $P \in \mathrm{Syl}_p(G)$ be a Sylow p-subgroup of G. The McKay conjecture asserts that there exists a bijection

$$\mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\mathbf{N}_G(P))$$

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where $\mathrm{Irr}_{p'}(H)$ denotes the set of irreducible characters of degree coprime to p of H, for every finite group H.

The McKay conjecture has been proven via a reduction to finite simple groups [7] and the subsequent verification of the inductive McKay conditions [9,16,1] derived from such a reduction theorem. This approach, in fact, leads to stronger conclusions. The validity of the inductive McKay conditions for every finite non-abelian simple group at every prime implies stronger statements for all finite groups; for instance, the following.

Theorem *(The relative McKay conjecture). Let G be a finite group, p be a prime and $P \in \mathrm{Syl}_p(G)$ be a Sylow p-subgroup of G. Let N be a normal subgroup of G and θ be a P-invariant irreducible character of N. There exists a bijection*

$$\mathrm{Irr}_{p'}(G|\theta) \to \mathrm{Irr}_{p'}(\mathbf{N}_G(P)N|\theta),$$

where $\mathrm{Irr}_{p'}(H|\theta)$ denotes the subset of $\mathrm{Irr}_{p'}(H)$ consisting of characters ψ whose restriction to N has θ as a constituent, for every $N \subseteq H \leq G$.

Theorem *(The McKay conjecture with group automorphisms). Let G be a finite group, p be a prime and $P \in \mathrm{Syl}_p(G)$ be a Sylow p-subgroup of G. Let A be a group that acts on G by automorphisms stabilizing P, then there exists an A-equivariant McKay bijection*

$$\mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\mathbf{N}_G(P)).$$

The relative McKay conjecture was reduced to the inductive McKay conditions in [7, Thm. B]. The McKay conjecture with group automorphisms was proposed by E. C. Dade and has been recently shown to also be a consequence of the inductive McKay conditions in [15, Thm. B].

Our interest in this note is on p-solvable groups. While the p-solvable case of the relative McKay conjecture relies on the so-called Okuyama-Wajima argument (see Theorem 2.1), the only available proof of the p-solvable case of the McKay conjecture with group automorphisms (that is, the one derived from [15, Thm. B]) relies on much deeper results on the character theory above Glauberman correspondents. In particular, the proof of [15, Thm. B] depends on results contained in [12, §5], that follow from work of F. Ladisch [8, Corollary 11.3] and ultimately relies on Dade's classification of the endo-p-permutation modules of p-groups. In this note, we remove this dependence. We prove the relative version of the McKay conjecture with group automorphisms for p-solvable groups relying solely on the Okuyama–Wajima argument. We care to mention that similar results to those of Ladisch were previously obtained by Dade [2], L. Puig [14] and A. Turull [17].

Theorem A. *Let G be a finite group, p be a prime and $P \in \mathrm{Syl}_p(G)$ be a Sylow p-subgroup of G. Let N be a normal subgroup of G and θ be a P-invariant irreducible character of N. Let A be a group acting by automorphisms on G stabilizing N and P. Suppose that G/N is p-solvable, then there exists an A-equivariant bijection*

$$\mathrm{Irr}_{p'}(G|\theta) \to \mathrm{Irr}_{p'}(\mathbf{N}_G(P)N|\theta).$$

In order to prove Theorem A, we generalize a classical result of P. X. Gallagher on the number of irreducible characters of G lying above a fixed character of some normal subgroup of G [3], which could be of independent interest. Recall that when $N \triangleleft A$ and $\theta \in \mathrm{Irr}(N)$ is A-invariant, we say that an element $a \in A$ is θ-good in A if the extensions of θ to $N\langle a\rangle$ are D-invariant, where $D/N = \mathbf{C}_{A/N}(Na)$.

Notice that if $a \in A$ is a θ-good, then every conjugate of a, as well as every $b \in A$ such that $N\langle a\rangle = N\langle b\rangle$, is also θ-good. In particular, if $a \in A$ is θ-good, then every element in the coset Na is θ-good. Keeping this in mind, we can refer to θ-good conjugacy classes of A/N as those consisting of θ-good elements, and we will not distinguish between calling $a \in A$ or its coset $Na \in A/N$ θ-good.

Theorem B. *Let N and G be normal subgroups of a group A with $N \subseteq G$. Let $\theta \in \mathrm{Irr}(N)$ be A-invariant. Then the number $|\mathrm{Irr}_A(G|\theta)|$ of irreducible A-invariant characters of G lying over θ equals the number of conjugacy classes of G/N which consist of θ-good elements in A.*

If we do not assume θ to be A-invariant, but we assume that some irreducible character in $\mathrm{Irr}(G|\theta)$ is A-invariant, we obtain that $|\mathrm{Irr}_A(G|\theta)|$ equals the number of conjugacy classes of G_θ/N which consist of θ-good elements in A_θ, where A_θ and G_θ are the corresponding inertia subgroups of θ in A and G. We shall comment more on this in Corollary 1.4.

We note that recently, a different generalization of Gallagher's counting theorem has been established by J. Murray. In [10, Thm. A], it is shown that Gallagher's ideas can be extended to count the number of real characters of a group G lying over a fixed character of a normal subgroup.

This paper is structured as follows. In Section 1 we prove Theorem B. In Section 2, we use Theorem B to obtain some consequences of the Okuyama–Wajima argument, specifically Theorem 2.4. Finally, in Section 3, we prove Theorem A. We follow the notation [5,11], with [11] serving as a guiding reference for our proofs.

1. Counting A-invariant characters after Gallagher

Our aim in this section is to prove Theorem B from the Introduction. Let $N \triangleleft A$ and $\theta \in \mathrm{Irr}(N)$. We write $\mathrm{Irr}(A|\theta)$ to denote the subset of $\mathrm{Irr}(A)$ consisting of characters χ such that $[\chi_N, \theta] \neq 0$, and we often refer to characters in $\mathrm{Irr}(A|\theta)$ as characters lying over θ. If θ is A-invariant, a classical result from Gallagher asserts that the number $|\mathrm{Irr}(A|\theta)|$ equals the number of A/N conjugacy classes consisting of θ-good elements, where an element $a \in A$ is said to be θ-good if all the extensions of θ to $N\langle a\rangle$ are D-invariant for $D/N = \mathbf{C}_{A/N}(Na)$. The original proof of Gallagher follows by adapting a computation of I. Schur and can be found in [3, §2]. As often happens with classical results, there are several proofs of Gallagher's theorem from different perspectives. We will build upon the one given by G. Navarro in [11, Thm. 5.16].

We recall the following property of θ-good elements.

Lemma 1.1. *Let $N \triangleleft G$. Let $\theta \in \mathrm{Irr}(N)$ be G-invariant. If $g \in G$ is not θ-good in G, then $\chi(g) = 0$ for every $\chi \in \mathrm{Irr}(G|\theta)$.*

Proof. Write $U = N\langle g\rangle$. By hypothesis, there is some $x \in G$ and some extension $\eta \in \mathrm{Irr}(U)$ of θ such that $\eta^x \neq \eta$ and xN centralizes U/N. By the Gallagher correspondence $\eta^x = \lambda\eta$ with $\lambda \in \mathrm{Irr}(U/N)$ not principal, and hence such that $\lambda(g) \neq 1$. Using again the Gallagher correspondence we can write $\chi_U = \eta\Delta$, where Δ is a character of U/N. Notice that $g, g^{x^{-1}} \in U$. We can compute $\eta(g)\Delta(g) = \chi(g) = \chi(g^{x^{-1}}) = \eta^x(g)\Delta(g) = \lambda(g)\eta(g)\Delta(g)$. The fact that $\lambda(g) \neq 1$ forces $\chi(g) = 0$, as wanted. □

For a non-empty set X, we denote by $\mathbb{C}^X$ the $\mathbb{C}$-vector space of $\mathbb{C}$-valued functions with domain X. Let $N \triangleleft A$ and $\theta \in \mathrm{Irr}(N)$. In the following, we denote by $\mathrm{cf}(A|\theta)$ the $\mathbb{C}$-vector space spanned by $\mathrm{Irr}(A|\theta)$. If $N \subseteq G \triangleleft A$, then we denote by $\mathrm{cf}_A(G|\theta)$ the $\mathbb{C}$-vector space spanned by $\mathrm{Irr}_A(G|\theta)$, the subset of $\mathrm{Irr}(G|\theta)$ consisting of A-invariant characters. Next is [11, Thm. 5.14].

Theorem 1.2. *Let $N \triangleleft A$ and $\theta \in \mathrm{Irr}(N)$ be A-invariant. Suppose that $N \subseteq \mathbf{Z}(A)$ and θ is faithful. Let $Nx_1, \dots, Nx_s$ be a complete set of representatives of the different conjugacy classes of A/N consisting of θ-good elements in A. Write $X = \{x_1, \dots, x_s\}$. The map*

$$\begin{aligned}\mathrm{cf}(A|\theta) &\to \mathbb{C}^X \\ \varphi &\mapsto \varphi|_X\,,\end{aligned}$$

is an isomorphisms of $\mathbb{C}$*-vector spaces, where* $\varphi|_X$ *denotes the restriction of* φ *to* X*.*

Proof. A proof can be found in [11, Thm. 5.14]. We care to highlight the construction of a preimage for a given map $\alpha \in \mathbb{C}^X$, as it will be a key construction in our next proof. If $a \in A$ is not θ-good, we define $\varphi(a) = 0$. Otherwise, $a^b = nx_i$ for some $b \in A$, some $n \in N$ and $x_i \in X$. We define then $\varphi(a) = \alpha(x_i)\theta(a^b x_i^{-1})$. It is clear that $\varphi|_X = \alpha$. It can be shown that $\varphi \in \mathrm{cf}(A|\theta)$. □

We are ready to prove Theorem B from the Introduction, that we restate here for future reference within this text. We will use properties of strong isomorphism of character triples from [11, Def. 5.7 and Prob. 5.4] (see also [5, Def. 11.23 and Prob. 11.13]).

Theorem 1.3. *Let* N *and* G *be normal subgroups of a group* A *with* $N \subseteq G$*. Let* $\theta \in \mathrm{Irr}(N)$ *be* A*-invariant. Then the number* $|\mathrm{Irr}_A(G|\theta)|$ *equals the number of conjugacy classes of* G/N *which consist of* θ*-good elements in* A*.*

Proof. We first prove the case where $N \subseteq \mathbf{Z}(A)$ and θ is faithful. Let $Nx_1, \ldots, Nx_k$ be a complete set of representatives of the conjugacy classes of G/N consisting of θ-good elements in A, and write $Y = \{x_1, \ldots, x_k\}$. We want to prove that

$$|\mathrm{Irr}_A(G|\theta)| = \dim(\mathrm{cf}_A(G|\theta)) = \dim(\mathbb{C}^Y)\,.$$

In order to do so, we will construct $\mathbb{C}$-isomorphisms between $\mathrm{cf}_A(G|\theta)$ and Ω, and between $\mathbb{C}^Y$ and Ω, where Ω is the subspace of $\mathrm{cf}(A|\theta)$ generated by the class functions of A that vanish outside G.

We claim that

$$\begin{aligned}f\colon \Omega &\to \mathrm{cf}_A(G|\theta) \\ \varphi &\mapsto \varphi|_G\end{aligned}$$

is an isomorphism of $\mathbb{C}$-vector spaces. Notice that f is well-defined as the restriction of every $\chi \in \mathrm{Irr}(A|\theta)$ to G is an A-invariant character of G lying over θ. It is clear that f is injective. We prove now that f is also surjective. Given $\alpha \in \mathrm{cf}_A(G|\theta)$, we define $\hat\alpha\colon A \to \mathbb{C}$ as $\hat\alpha(a) = 0$ if $a \in A \setminus G$, an $\hat\alpha(a) = \alpha(a)$ otherwise. Notice that $\hat\alpha$ is a class function of A that vanishes outside G. Moreover $\hat\alpha|_N = \alpha_N$, so $\hat\alpha \in \Omega$.

By Theorem 1.2, we know that $\dim(\mathrm{cf}(A|\theta))$ is the number of conjugacy classes of A/N consisting of θ-good elements in A. We can consider a complete set of representatives of those classes of the form $Nx_1, \ldots, Nx_k, Nx_{k+1}, \ldots, Nx_s$, where $k \le s$ and $x_j \notin G$ for every $k < j \le s$. Notice that if $a \in A \setminus G$ is θ-good in A, then there are $b \in A$, $n \in N$ and $j > k$ for which $a^b = nx_j$. Write $X = \{x_1, \ldots, x_s\}$, so that $\dim(\mathrm{cf}(A|\theta)) = s$.

We define

$$\begin{aligned}h\colon \Omega &\to \mathbb{C}^Y \\ \varphi &\mapsto \varphi|_Y\,.\end{aligned}$$

We work to prove that h is an isomorphism of $\mathbb{C}$-vector spaces. The map h is clearly $\mathbb{C}$-linear. Let $\varphi \in \Omega$ vanish on Y. We want to prove that $\varphi(g) = 0$ for every $g \in G$. Suppose that g is not θ-good in A. By Lemma 1.1 we have that $\chi(g) = 0$ for every $\chi \in \mathrm{Irr}(A|\theta)$. In particular, $\varphi(g) = 0$. Hence, we may assume without loss of generality that g is θ-good in A. Then there exist $a \in A$, $n \in N$ and $i \in \{1, \ldots, k\}$ such that $g^a = nx_i$. Notice that if $\chi \in \mathrm{Irr}(A|\theta)$ and ρ is a representation affording χ then $\rho(n)$ is a scalar matrix, since we are assuming $N \subseteq \mathbf{Z}(A)$. Therefore $\chi(mb) = \chi(m)\chi(b)$ for all $m \in N$ and $b \in A$. In particular $\varphi(nx_i) = \theta(n)\varphi(x_i) = 0$ as φ is a linear combination of characters in $\mathrm{Irr}(A|\theta)$. Then $\varphi(g) = \varphi(g^a) = \varphi(nx_i) = 0$ as wanted.

Now, given $\alpha \in \mathbb{C}^Y$ we define $\hat{\alpha} \in \mathbb{C}^X$ by $\hat{\alpha}(x_i) = \alpha(x_i)$ if $i \leq k$ and $\hat{\alpha}(x_j) = 0$ for every $k+1 \leq j \leq s$. By Theorem 1.2, there exists a unique $\varphi \in \mathrm{cf}(A|\theta)$ such that $\varphi|_X = \hat{\alpha}$. In particular, $\varphi|_Y = \alpha$. We claim that $\varphi \in \Omega$, and that would complete the proof in this case. In order to prove that $\varphi \in \Omega$ it suffices to prove that φ vanishes outside G. Let $a \in A \setminus G$. If a is not θ-good in A, then $\varphi(a) = 0$ by Lemma 1.1. Otherwise there exist $b \in A$, $j \in \{k+1, \ldots, s\}$ and $n \in N$ such that $a^b = nx_j$. Recalling the definition of φ on θ-good elements from the proof of Theorem 1.2, we get that

$$\varphi(a) = \hat{\alpha}(x_j)\theta(n) = 0\,.$$

In the general case, we note the character triple (A, N, θ) is strongly isomorphic (via $(^*, \sigma)$) to another character triple (A^*, N^*, φ); where $N^* \subseteq \mathbf{Z}(A^*)$ and φ is faithful by [11, Prob. 5.4]. In particular

$$|\mathrm{Irr}_A(G|\theta)| = |\mathrm{Irr}_{A^*}(G^*|\theta^*)|$$

where G^*/N^* is the image of G/N under the above-mentioned strong character triple isomorphism. By [11, Lem. 5.15], an element $g \in G$ is θ-good in A if, and only if, g^* is φ-good in A^*, whenever gN corresponds to g^*N^* under the strong character triple isomorphism. Then the statement follows from the first part of this proof. □

We can derive the following statement that allows us to count $|\mathrm{Irr}(A)G|\theta|$ when θ is no longer assumed to be A-invariant but we know that the set $\mathrm{Irr}_A(G|\theta)$ is non-empty.

Corollary 1.4. *Let N and G be normal subgroups of a group A with $N \subseteq G$. Let $\theta \in \mathrm{Irr}(N)$. If $\in \mathrm{Irr}_A(G|\theta)$ is non-empty, then the number $|\mathrm{Irr}_A(G|\theta)|$ equals the number of conjugacy classes of G_θ/N which consist of θ-good elements in A_θ.*

Proof. By the Clifford correspondence, we know that character induction induces a bijection

$$\mathrm{Irr}(G_\theta|\theta) \to \mathrm{Irr}(G|\theta)\,.$$

Notice that the above bijection commutes with the action of A_θ on characters. In particular,

$$|\mathrm{Irr}_{A_\theta}(G_\theta|\theta)| = |\mathrm{Irr}_{A_\theta}(G|\theta)|\,.$$

Let $\chi \in \mathrm{Irr}_A(G|\theta)$. We show that $A = GA_\theta$. In fact, for every $a \in A$, we have that θ and θ^a lie under χ. Hence there is some $g \in G$ such that $\theta^g = \theta^a$. In particular $ag^{-1} \in A_\theta$. We can therefore conclude that $\mathrm{Irr}_A(G|\theta) = \mathrm{Irr}_{A_\theta}(G|\theta)$ and apply Theorem 1.3. □

We care to mention that the equality $|\mathrm{Irr}_A(G|\theta)| = |\mathrm{Irr}_{A_\theta}(G_\theta|\theta)|$ does not always hold as shown by the example $A = \mathsf{D}_8 \times \mathsf{C}_2$ with $G = \mathsf{C}_4 \times \mathsf{C}_2$, $N = \mathsf{C}_4$ and $\theta \in \mathrm{Irr}(N)$ faithful. In this case $|\mathrm{Irr}_A(G|\theta)| = 0$ while $|\mathrm{Irr}_{A_\theta}(G_\theta|\theta)| = 2$.

As we have already mentioned in the Introduction, in recent work, Murray shows that Gallagher's ideas can also be generalized to count the number of $\mathrm{Irr}_{\mathbb{R}}(G|\theta)$ real-valued irreducible characters lying over θ. In fact, Murray suggests that one might be able to prove his statement using Navarro's approach as above. It would be interesting to understand whether Murray's or Navarro's ideas could be used to more generally count the number of $\mathrm{Irr}_{\mathcal{H}}(G|\theta)$, where $\mathcal{H}$ is a subgroup of the absolute Galois of $\mathbb{Q}$, in terms of θ-good conjugacy classes of G/N.

2. The Okuyama–Wajima argument revisited

The aim of this section is to apply Theorem 1.3 to derive a new consequence of the Okuyama-Wajima argument, namely Theorem 2.4.

The Okuyama-Wajima argument concerns the character theory above Glauberman correspondents. Recall that when a p-group Q acts by automorphisms on a group K of order coprime to p, then the Glauberman correspondence is a canonical bijection ${}^*\colon \mathrm{Irr}_Q(K) \to \mathrm{Irr}(\mathbf{C}_K(Q))$ between the set $\mathrm{Irr}_Q(K)$ of Q-invariant irreducible characters of K and the set $\mathrm{Irr}(\mathbf{C}_K(Q))$ of irreducible characters of the fixed-point subgroup under the action. In fact, given $\theta \in \mathrm{Irr}_Q(K)$, its Glauberman correspondent θ^* is the unique irreducible constituent of the restriction $\theta|_{\mathbf{C}_K(Q)}$ of θ to $\mathbf{C}_K(Q)$ appearing with multiplicity coprime to p (see, for instance, [11, Thm. 2.9]).

Theorem 2.1 *(The Okuyama–Wajima Argument). Let A be a group, let p be any prime. Suppose $K \lhd A$ has order not divisible by p. Let Q be a p-subgroup of A such that $KQ \lhd A$. Let $B = \mathbf{N}_A(Q)$ and $C = \mathbf{C}_K(Q)$. Let $\theta \in \mathrm{Irr}(K)$ be Q-invariant and let $\theta^* \in \mathrm{Irr}(C)$ be the Glauberman correspondent of θ. Suppose that $C \le U \le B$ is such that U/C is abelian. Then θ extends to KU if and only if θ^* extends to U.*

Proof. See [13, Thm. 2]. Notice that in the statement of [13, Thm. 2], Q is a Sylow p-subgroup of G and $U = B$, but these stronger hypotheses are not necessary (using [11, Thm. 5.10 and Cor. 6.2], for instance). A more detailed proof can be found in [11, Thm. 8.16]. □

The statement of Theorem 2.1 remains valid even when the quotient U/C is not assumed to be abelian; however, the proof of this seemingly innocuous generalization requires completely different ideas and a fairly great amount of work, see for instance [8, Cor. 11.3 and Thm. 4.3] or [17, Thm. 6.5] combined with [18, Thm. 7.12]. When the prime $p = 2$, T. Wolf found a character-theoretical proof of this fact [19, Thm. 1.5], relying on properties of the Isaacs correspondence [4].

As mentioned in the Introduction and for the purpose of this note, the Okuyama-Wajima argument will suffice. We need to derive some consequences from it using the results contained in the previous section. Before doing so, we need to prove an easy, yet useful, lemma. Given groups $K, G \lhd A$ with $K \subseteq G$, $P \in \mathrm{Syl}_p(G)$, and a P-invariant $\theta \in \mathrm{Irr}(K)$, we will be interested in counting the number

$$|\mathrm{Irr}_{p',A}(G|\theta)|$$

of irreducible A-invariant characters of G of degree coprime to p that lie over θ.

Lemma 2.2. *Let A be a group, and let $G \lhd A$. Suppose that $B \le A$ is such that $A = GB$, and write $H = G \cap B$. Let $K \subseteq G$, with $K \lhd A$, and let $C \subseteq H$, with $C \lhd B$. Let $\theta \in \mathrm{Irr}(K)$ and let $\varphi \in \mathrm{Irr}(C)$. Assume that $B_\theta = B_\varphi$. If the following holds*

$$|\mathrm{Irr}_{A_\theta}(G_\theta|\theta)| = |\mathrm{Irr}_{B_\theta}(H_\varphi|\varphi)|,$$

then we have that

$$|\mathrm{Irr}_A(G|\theta)| = |\mathrm{Irr}_B(H|\varphi)|.$$

Let p be a prime and $P \in \mathrm{Syl}_p(G)$. Assume further that θ and φ are P-invariant characters. If the following holds

$$|\mathrm{Irr}_{p',A_\theta}(G_\theta|\theta)| = |\mathrm{Irr}_{p',B_\theta}(H_\varphi|\varphi)|,$$

then we have that

$$|\mathrm{Irr}_{p',A}(G|\theta)| = |\mathrm{Irr}_{p',B}(H|\varphi)|.$$

Proof. Following the proof of Corollary 1.4, in order to complete the proof of the first statement, it will be enough to show that $\mathrm{Irr}_A(G|\theta)$ is non-empty if, and only if, $\mathrm{Irr}_B(H|\varphi)$ is non-empty. Assume that $\mathrm{Irr}_A(G|\theta)$ is empty while $\mathrm{Irr}_B(H|\varphi)$ is not empty. Arguing as in Corollary 1.4, we have that $B = HB_\theta$. Hence $A = GB = G(HB_\theta) = GA_\theta$. Then $0 = |\mathrm{Irr}_{A_\theta}(G|\theta)| = |\mathrm{Irr}_{A_\theta}(G_\theta|\theta)|$ and $|\mathrm{Irr}_{B_\theta}(H_\theta|\varphi)| = |\mathrm{Irr}_B(H|\theta)| \neq 0$, which contradicts our hypothesis. We obtain a similar contradiction assuming that $\mathrm{Irr}_A(G|\theta)$ is non-empty while $\mathrm{Irr}_B(H|\varphi)$ is empty.

The second statement easily follows from previous ideas by noticing that under the hypotheses, character induction actually yields bijections

$$\mathrm{Irr}_{p'}(G_\theta|\theta) \to \mathrm{Irr}_{p'}(G|\theta),$$
$$\mathrm{Irr}_{p'}(H_\varphi|\varphi) \to \mathrm{Irr}_{p'}(H|\varphi), \quad \square$$

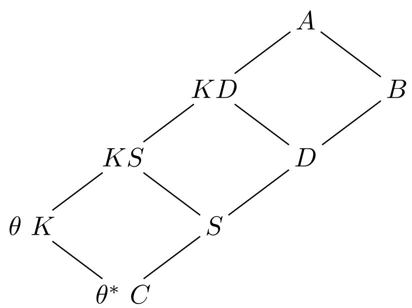
since θ and φ are P-invariant.

The following is a consequence of the Okuyama–Wajima argument and Theorem 1.3. Recall that, if $N \lhd A$ and $\theta \in \mathrm{Irr}(N)$ is A-invariant, then an element $a \in A$ is θ-good if, and only if, θ extends to $N\langle a, x\rangle$ for every $x \in A$ such that $N\langle a, x\rangle/N$ is abelian; by [11, Lem. 5.13].

Corollary 2.3. *Let A be a group and let p be a prime. Suppose $K \lhd A$ has order not divisible by p, let P be a p-subgroup of A such that $KP \lhd A$. Let $B = \mathbf{N}_A(P)$ and $C = \mathbf{C}_K(P)$. Let $\theta \in \mathrm{Irr}(K)$ be P-invariant and let $\theta^* \in \mathrm{Irr}(C)$ be the Glauberman correspondent of θ. Let $C \leq D \leq B$. If $C \leq S \leq D$, with $S \lhd D$, then*

$$|\mathrm{Irr}_{KD}(KS|\theta)| = |\mathrm{Irr}_D(S|\theta^*)|.$$

Proof. By the Frattini argument, $A = KB$. Our set-up can be visualized in the following diagram.



To simplify the notation, we can assume that $D = B$.

We claim that we may assume both θ and θ^* to be B-invariant. Note that $B_\theta = B_{\theta^*}$ and $S_\theta = S_{\theta^*}$ by [11, Lem. 2.1]. Since $S \lhd B$, $K \lhd A$, $A = (KS)B$, and $S = KS \cap B$, the claim follows by applying Lemma 2.2.

Notice that θ is, in particular, A-invariant. We have that (A, K, θ) is a character triple with $K \leq KS \lhd A$, and (B, C, θ^*) is a character triple with $C \leq S \lhd B$. Then, by Theorem 1.3, we have that $|\mathrm{Irr}_A(KS|\theta)|$ is the number of conjugacy classes of KS/K which consist of θ-good elements in A, while $|\mathrm{Irr}_B(S|\theta^*)|$ is the number of conjugacy classes of S/C which consist of θ^*-good elements in B. Notice that the map

$$\begin{aligned} S/C &\to KS/K, \\ Cs &\mapsto Ks \end{aligned}$$

is an isomorphism. Therefore, in order to prove the statement it suffices to show that for all $s \in S$, the element Cs is θ^*-good in B if, and only if, Ks is θ-good in A.

Recall that $A = KB$. Fix $s \in S$. We prove that θ^* extends to $\langle C, s, b\rangle$, for every $b \in B$ such that $\langle C, s, b\rangle/C$ is abelian if, and only if, θ extends to $\langle K, s, b\rangle$, for every $b \in B$ such that $\langle K, s, b\rangle/K$ is abelian. Let $b \in B$ be such that $\langle C, s, b\rangle/C$ is abelian and write $U = \langle C, s, b\rangle$, then $C \leq U \leq B$. We need to prove that θ extends to KU if, and only if, θ^* extends to U. This is indeed true by Theorem 2.1. We conclude our proof with an application of [11, Lem. 5.13.(c)]. $\square$

The statement of Theorem 2.1 holds more generally under the weaker assumption that K and Q have coprime orders, by work of Wolf [19, Thm. 1.5] (for the case where Q is solvable see also [6, Thm. 6.10]). Consequently, the statement of Corollary 2.3 also holds in this more general context.

Finally, we can generalize [11, Thm. 8.11] taking into account the action of group automorphisms. This is the main result of this section. Recall that, when $N \triangleleft G$ and $(|G : N|, |N|) = 1$, if $\theta \in \mathrm{Irr}(N)$ is G-invariant, then θ admits a canonical extension $\widehat{\theta} \in \mathrm{Irr}(G)$. The extension $\widehat{\theta}$ is canonical, in the sense that it is the only extension with the same determinantal order as θ (see [11, Cor 6.2], for instance).

Theorem 2.4. *Let A be a group, let $G \triangleleft A$ and let p be a prime. Let $K \subseteq G$ be such that $K \triangleleft A$ and p does not divide the order of K. Let $P \in \mathrm{Syl}_p(G)$ be such that $KP \triangleleft A$. Let Z be a p-subgroup of G such that $Z \triangleleft A$. Let $\lambda \in \mathrm{Irr}(Z)$ be A-invariant. Write $B = \mathbf{N}_A(P)$, $H = \mathbf{N}_G(P)$, and $C = \mathbf{C}_K(P)$. Let $\theta \in \mathrm{Irr}(K)$ be P-invariant and let $\theta^* \in \mathrm{Irr}(C)$ be the Glauberman correspondent of θ. Then*

$$|\mathrm{Irr}_{p',A}(G|\theta \times \lambda)| = |\mathrm{Irr}_{p',B}(H|\theta^* \times \lambda)|.$$

Proof. By the Frattini argument, $A = (KP)B = KB$ and $G = (KP)H = KH$. Notice that $Z \subseteq P$. The following diagram depicts the relevant character and group-theoretic situation.

$$\begin{tikzcd}[column sep=small]
 & & & & A \arrow[dl, dash] \arrow[dr, dash] & \\
 & & & G \arrow[dl, dash] \arrow[dr, dash] & & B \arrow[dl, dash] \\
 & & KP \arrow[dl, dash] \arrow[dr, dash] & & H \arrow[dl, dash] & \\
 & K \times Z \arrow[dl, dash] \arrow[dr, dash] & & C \times P \arrow[dl, dash] \arrow[dr, dash] & & \\
\theta\ K \arrow[dr, dash] & & C \times Z \arrow[dl, dash] \arrow[dr, dash] & & P \arrow[dl, dash] & \\
 & \theta^*\ C \arrow[dr, dash] & & \lambda\ Z \arrow[dl, dash] & & \\
 & & 1 & & &
\end{tikzcd}$$

We show that we may assume that both θ and θ^* are B-invariant (and hence that θ is A-invariant). By [11, Lem. 2.1], the Glauberman correspondence is B-equivariant. Hence $B_\theta = B_{\theta^*}$, and $H_\theta = H_{\theta^*}$. Notice that $A_{\theta\times\lambda} = A_\theta \cap A_\lambda = A_\theta$, and $B_{\theta^*\times\lambda} = B_{\theta^*}$, hence $B_{\theta\times\lambda} = B_\theta = B_{\theta^*} = B_{\theta^*\times\lambda}$. Since $A = GB$, $H = G \cap B$, $KZ \triangleleft A$ and $CZ \triangleleft B$, we can apply the second part of Lemma 2.2. Therefore it will suffice to show that

$$|\mathrm{Irr}_{p',A_\theta}(G_\theta|\theta \times \lambda)| = |\mathrm{Irr}_{p',B_\theta}(H_\theta|\theta^* \times \lambda)|.$$

This proves our first claim.

Let $\widehat{\theta}$ be the canonical extension of θ to KP and write $\widehat{\theta^*} = \theta^* \times 1_P \in \mathrm{Irr}(CP)$. It is clear that $\widehat{\theta^*}$ is B-invariant. Moreover, θ is A-invariant and hence so is $\widehat{\theta}$. Since the groups KP/K, CP/C, and P are isomorphic, we can identify their characters by inflation and deflation.

Let $\mathcal{A}$ be a complete set of representatives of the orbits of the action of H on the linear characters of KP/K which lie over $1_K \times \lambda \in \mathrm{Irr}(KZ/K)$. We claim that

$$\mathrm{Irr}_{p',A}(G|\theta \times \lambda) = \dot{\bigcup_{\mu \in \mathcal{A}}} \mathrm{Irr}_A(G|\widehat{\theta}\mu),$$

and

$$\mathrm{Irr}_{p',B}(H|\theta^* \times \lambda) = \dot{\bigcup_{\mu \in \mathcal{A}}} \mathrm{Irr}_B(H|\widehat{\theta^*}\mu),$$

where unions above are disjoint. We prove the first claim, the second will follow by using the same kind of arguments. Notice that every character in $\mathrm{Irr}_A(G|\widehat{\theta}\mu)$ has degree not divisible by p by [11, Thm. 5.12]. Let $\chi \in \mathrm{Irr}_{p',A}(G|\theta)$ and let ρ be an irreducible constituent of χ_{KP}. Since $KP \lhd G$, we have that ρ has degree coprime to p. Furthermore, since θ is the only irreducible constituent of χ_K, we have that θ is a constituent of ρ_K. Since ρ covers θ, we can write $\rho = \widehat{\theta}\mu$, for some $\mu \in \mathrm{Irr}(KP/K)$, by using the Gallagher correspondence. But ρ and $\widehat{\theta}$ are p'-degree, hence μ must be linear, since KP/K is a p-group. Replacing ρ by some H-conjugate, we may assume $\mu \in \mathcal{A}$, hence $\chi \in \bigcup_{\mu\in\mathcal{A}} \mathrm{Irr}_A(G|\widehat{\theta}\mu)$. Conversely, let $\mu \in \mathcal{A}$ and let $\chi \in \mathrm{Irr}_A(G|\widehat{\theta}\mu)$. We have that χ has degree coprime to p, again by [11, Thm. 5.12], and

$$[\chi_K, \theta] \geq [(\widehat{\theta}\mu)_K, \theta] = [\theta, \theta] = 1,$$

so $\chi \in \mathrm{Irr}_{p',A}(G|\theta)$. Lastly, notice that if $\chi \in \mathrm{Irr}_A(G|\widehat{\theta}\mu) \cap \mathrm{Irr}_A(G|\widehat{\theta}\eta)$ for $\mu, \eta \in \mathcal{A}$, then $\widehat{\theta}\mu$ and $\widehat{\theta}\eta$ are G-conjugate. Since $\widehat{\theta}$ is G-invariant, this implies that μ and η are G-conjugate, by the Gallagher correspondence. In particular, μ and η are H-conjugate, because $G = (KP)H$, and therefore $\mu = \eta$.

Thus, in order to prove the statement, it suffices to show that

$$|\mathrm{Irr}_A(G|\widehat{\theta}\mu)| = |\mathrm{Irr}_B(H|\widehat{\theta^*}\mu)|, \tag{2.1}$$

for all $\mu \in \mathcal{A}$.

We next show that in order to prove Equation (2.1) for $\mu \in \mathcal{A}$, we may assume μ to be A-invariant. Note that $A_{\widehat{\theta}\mu} = A_\mu$ and $B_{\widehat{\theta}\mu} = B_{\widehat{\theta^*}\mu}$ by using again the Gallagher correspondence. Since $KP \lhd A$ and $CP \lhd B$, our claim follows from the first part of Lemma 2.2. Notice that $(|G : KP|, |KP : K|) = 1$, therefore μ admits a canonical extension $\widehat{\mu} \in \mathrm{Irr}(G)$. We have that $\widehat{\mu}$ is A-invariant. Then [5, Thm. 6.16] (or [11, Lem. 5.8]) tells us that the map

$$\begin{aligned} \mathrm{Irr}(G|\widehat{\theta}) &\to \mathrm{Irr}(G|\widehat{\theta}\mu) \\ \beta &\mapsto \beta\widehat{\mu} \end{aligned}$$

is a bijection. Since $\widehat{\mu}$ is A-invariant, this bijection is also A-equivariant. Hence

$$|\mathrm{Irr}_A(G|\widehat{\theta}\mu)| = |\mathrm{Irr}_A(G|\widehat{\theta})|.$$

Similarly, we obtain that

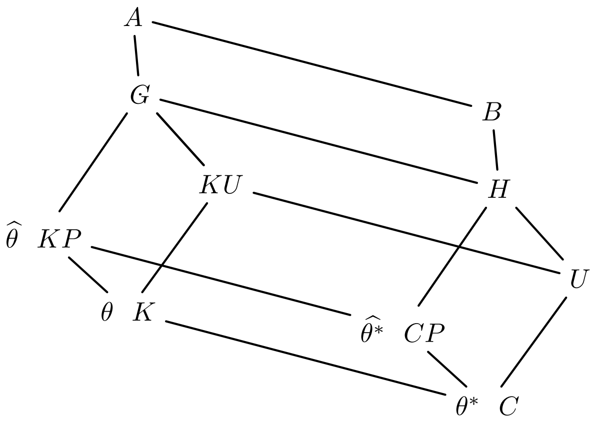
$$|\mathrm{Irr}_B(H|\widehat{\theta^*}\mu)| = |\mathrm{Irr}_B(H|\widehat{\theta^*})|.$$

Therefore, in order to prove Equation (2.1) for every $\mu \in \mathcal{A}$, it will be enough to show that

$$|\mathrm{Irr}_A(G|\widehat{\theta})| = |\mathrm{Irr}_B(H|\widehat{\theta^*})|. \tag{2.2}$$

We work to prove Equation (2.2). We will show that the proof reduces to the situation described by Corollary 2.3. We begin by describing the group theory that arises in this context.

Since $P \lhd H$, we have that CP/C is a normal Sylow p-subgroup of H/C. By the Schur-Zassenhaus theorem, the group H/C has a p-complement U/C, that is $U \leq H$ with $U(CP) = H$ and $U \cap CP = C$. In particular, U has order coprime to p. Since $K \lhd G$ and $PU = UP \leq G$, we have that $(KP)(KU) \leq G$. The group $(KP)(KU)$ contains both $U(CP) = H$ and K, hence we have that $(KP)(KU) = G$. Furthermore, we have that $KU \cap KP = K(U \cap P) = K$ and $KU \cap H = U(K \cap H) = UC = U$, by Dedekind's lemma. By the conjugation part in the Schur-Zassenhaus theorem, we know that the B-conjugates of U are actually H-conjugates. Another application of the Frattini argument yields $B = H\mathbf{N}_B(U)$. Moreover, in this situation we have that $K\mathbf{N}_B(U) = \mathbf{N}_A(KU)$. Indeed, since $K \lhd A$, it is clear that $K\mathbf{N}_B(U) \subseteq \mathbf{N}_A(KU)$. For the reverse inclusion, note that $A = (KU)B$ and $KU \cap B = U$. As $U \lhd \mathbf{N}_A(KU) \cap B$, we obtain $\mathbf{N}_A(KU) \cap B \subseteq \mathbf{N}_B(U)$. Hence $\mathbf{N}_A(KU) \subseteq K\mathbf{N}_B(U)$, which establishes the equality. We conclude that $A = KB = KH\mathbf{N}_B(U) = H\mathbf{N}_A(KU)$. The following figure depicts most of the information gathered above.



Recall that $\widehat{\theta}_K = \theta$ and $(\widehat{\theta^*})|_C = \theta^*$. Taking into account that $G = (KP)(KU)$ with $KP \lhd G$ and $KP \cap KU = K$, as well as that $H = (CP)U$ with $CP \lhd H$ and $CP \cap U = C$, and using [4, Lem. 10.5] (that appears as [11, Lem. 6.8]), we have that character restriction yields bijections

$$\mathrm{Irr}(G|\widehat{\theta}) \to \mathrm{Irr}(KU|\theta)$$

and

$$\mathrm{Irr}(H|\widehat{\theta^*}) \to \mathrm{Irr}(U|\theta^*)\,.$$

These bijections do preserve $\mathbf{N}_A(KU)$ and $\mathbf{N}_B(U)$ fixed characters, respectively. In particular, as $A = G\mathbf{N}_A(KU)$ and $B = H\mathbf{N}_B(U)$, we have that

$$|\mathrm{Irr}_A(G|\widehat{\theta})| = |\mathrm{Irr}_{\mathbf{N}_A(KU)}(G|\widehat{\theta})| = |\mathrm{Irr}_{\mathbf{N}_A(KU)}(KU|\theta)|$$

and

$$|\mathrm{Irr}_B(H|\widehat{\theta^*})| = |\mathrm{Irr}_{\mathbf{N}_B(U)}(H|\widehat{\theta^*})| = |\mathrm{Irr}_{\mathbf{N}_B(U)}(U|\theta^*)|.$$

Recall that $\mathbf{N}_A(KU) = K\mathbf{N}_B(U)$ and $A = \mathbf{N}_A(KU)B$. Moreover, by Dedekind's lemma, we have that $KU \cap \mathbf{N}_B(U) = U(K \cap \mathbf{N}_B(U)) = U$. Therefore, the proof can be reduced to studying the situation described below.

$$\begin{tikzcd}
 & & & A \arrow[dl, dash] \arrow[dr, dash] & \\
 & & \mathbf{N}_A(KU) \arrow[dl, dash] \arrow[dr, dash] & & B \arrow[dl, dash] \\
 & KU \arrow[dl, dash] \arrow[dr, dash] & & \mathbf{N}_B(U) \arrow[dl, dash] & \\
\theta\ K \arrow[dr, dash] & & U \arrow[dl, dash] & & \\
 & \theta^*\ C & & &
\end{tikzcd}$$

Notice that we can apply Corollary 2.3 with $S := U \lhd D := \mathbf{N}_B(U) \leq B$, yielding

$$|\mathrm{Irr}_{\mathbf{N}_A(KU))}(KU|\theta)| = |\mathrm{Irr}_{\mathbf{N}_B(U))}(U|\theta^*)|.$$

This proves Equation (2.2) and therefore concludes our proof. $\square$

3. The relative McKay conjecture with group automorphisms

The aim of this section is to prove Theorem A from the Introduction. We will prove the following statement, that can be shown to be equivalent to the statement of Theorem A by applying [5, Lem. 13.23]. In order to prove the following theorem, we will carefully follow the relevant parts of the proof of [11, 10.26].

Theorem 3.1. *Let A be a group and let p be a prime. Let $G \lhd A$ and $P \in \mathrm{Syl}_p(G)$. Let $N \lhd A$, with $N \subseteq G$, and let $\mu \in \mathrm{Irr}_{p'}(N)$ be P-invariant. Write $B = \mathbf{N}_A(P)$. Assume that G/N is p-solvable. Then*

$$|\mathrm{Irr}_{p',A}(G|\mu)| = |\mathrm{Irr}_{p',B}(\mathbf{N}_G(P)N|\mu)|.$$

Proof of Theorem 3.1. We argue by induction on $|A : N|$. Note that $A = GB$, by the Frattini argument. Let $H = \mathbf{N}_G(P)$. We first show that we can assume μ to be A-invariant. Since $A = G(BN)$, $HN = G \cap BN$, $N \lhd A$, and $N \lhd BN$, we can apply the second part of Lemma 2.2. Therefore it suffices to show that

$$|\mathrm{Irr}_{p',A_\mu}(G_\mu|\mu)| = |\mathrm{Irr}_{p',(BN)_\mu}(H_\mu N|\mu)| = |\mathrm{Irr}_{p',B_\mu}(H_\mu N|\mu)|.$$

In other words, we can assume μ to be A-invariant and work to prove that

$$|\mathrm{Irr}_{p',A}(G|\mu)| = |\mathrm{Irr}_{p',B}(HN|\mu)|.$$

Next, we want to show that we can assume N to be a central subgroup of A. By [11, Problem 5.4.(b)] we have that the character triple (A, N, μ) is strongly isomorphic to a character triple (A^*, N^*, φ), where N^* is a central subgroup of A^* and $\varphi \in \mathrm{Irr}(N^*)$ (we denote a strong isomorphism $(^*, \sigma)$ as in [5, Def. 11.23]). If $N \subseteq K \leq A$ we will denote by K^* the subgroup of A^* containing N^* such that $(K/N)^* = K^*/N^*$ under the group isomorphism $^*\colon A/N \to A^*/N^*$. Notice that $N^* \subseteq G^*$. Since N^* is central in A^*, there exists a unique $P^* \in \mathrm{Syl}_p(G^*)$ such that $(PN)^*/N^* = P^*N^*/N^*$. Using that $\mathbf{N}_{G/N}(PN/N) = HN/N$ and $\mathbf{N}_{G^*/N^*}(P^*N^*/N^*) = \mathbf{N}_{G^*}(P^*)N^*/N^* = \mathbf{N}_{G^*}(P^*)/N^*$, we conclude that

$$\mathbf{N}_{G^*}(P^*) = (HN)^* \,.$$

Similarly, we can also prove that

$$\mathbf{N}_{A^*}(P^*) = (BN)^* \,.$$

The strong character triple isomorphism preserves character degree ratios and therefore induces bijections

$$\sigma_G \colon \mathrm{Irr}_{p'}(G|\mu) \to \mathrm{Irr}_{p'}(G^*|\varphi) \ \text{ and } \ \sigma_{(HN)} \colon \mathrm{Irr}_{p'}(HN|\mu) \to \mathrm{Irr}_{p'}(\mathbf{N}_{G^*}(P^*)|\varphi) \,.$$

Notice that σ_G is equivariant with respect to the action of $A/G \cong A^*/G^*$ and $\sigma_{(HN)}$ is equivariant with respect to the action of $BN/HN \cong \mathbf{N}_{A^*}(P^*)/\mathbf{N}_{G^*}(P^*)$. Therefore

$$|\mathrm{Irr}_{p',A}(G|\mu)| = |\mathrm{Irr}_{p',A^*}(G^*|\varphi)| \ \text{ and } \ |\mathrm{Irr}_{p',B}(HN|\mu)| = |\mathrm{Irr}_{p',B^*}(\mathbf{N}_{G^*}(P^*)|\varphi)| \,.$$

We may, without loss of generality, assume that $N \subseteq \mathbf{Z}(A)$, as wanted. In particular, $H = \mathbf{N}_G(P)N$.

Let L/N be a minimal normal subgroup of A/N contained in G/N. We have that $N < L$ and L/N is either a p-group or a p'-group because by hypothesis G/N is p-solvable.

Next, we show by induction that $|\mathrm{Irr}_{p',A}(G|\mu)| = |\mathrm{Irr}_{p',B}(LH|\mu)|$. Notice that H acts on set $\mathrm{Irr}_{p',P}(L|\mu)$. Let $\mathcal{A}$ be a complete set of representatives of the orbits of the action of H on $\mathrm{Irr}_{p',P}(L|\mu)$. Then, by using [11, 9.3], we have that

$$\mathrm{Irr}_{p',A}(G|\mu) = \dot{\bigcup_{\theta\in\mathcal{A}}} \mathrm{Irr}_{p',A}(G|\theta),$$
$$\mathrm{Irr}_{p',B}(LH|\mu) = \dot{\bigcup_{\theta\in\mathcal{A}}} \mathrm{Irr}_{p',B}(LH|\theta),$$

where unions above are disjoint. Since $|A : L| < |A : N|$, and $L \triangleleft A$, by induction we have that

$$|\mathrm{Irr}_{p',A}(G|\theta)| = |\mathrm{Irr}_{p',B}(LH|\theta)|,$$

for every $\theta \in \mathcal{A}$. Therefore $|\mathrm{Irr}_{p',A}(G|\mu)| = |\mathrm{Irr}_{p',B}(HL|\mu)|$, as claimed.

We claim that we may assume that $G = LH$, hence $A = LB$ and $PL \triangleleft A$, and that L/N is a p'-group. If $LH < G$, then $LB < A$. By induction, since $LH \triangleleft LB$ and $|LB : N| < |A : N|$, we would get that

$$|\mathrm{Irr}_{p',B}(LH|\mu)| = |\mathrm{Irr}_{p',LB}(LH|\mu)| = |\mathrm{Irr}_{p',B}(H|\mu)| \,.$$

Hence, we may assume that $LH = G$. If L/N is not a p'-group, then $L \subseteq PN$ and hence $G = LH = PNH = H$ and there is nothing to prove.

Now, write $Z = P \cap N$ and $\lambda = \mu_Z$. Notice that λ is A-invariant. Since Z is a central Sylow p-subgroup of L, we can write $L = K \times Z$, with K a p'-subgroup. Observe that K is characteristic in L, hence $K \triangleleft A$. Moreover $KP = LP \triangleleft A$.

We work to conclude this proof by applying Theorem 2.4. Write $M = N \cap K$ and $\xi = \mu_M$. We have that $N = M \times Z$. Let $\mathcal{B}$ be a complete set of representatives of the orbits of the action of H on $\mathrm{Irr}_P(K|\xi)$ and let $^* : \mathrm{Irr}_P(K) \to \mathrm{Irr}(\mathbf{C}_K(P))$ be the P-Glauberman correspondence. We have that * maps $\mathrm{Irr}_P(K|\xi)$ onto $\mathrm{Irr}(\mathbf{C}_K(P)|\xi)$ because for every $\theta \in \mathrm{Irr}_P(K|\xi)$, its Glauberman correspondent θ^* is an irreducible constituent of the restriction $\theta_{\mathbf{C}_K(P)}$, by [11, Thm. 2.9]. Since the Glauberman correspondent is equivariant with respect to the action of H on characters [11, Lem. 2.10], we have that $\mathcal{B}^* = \{\theta^* \mid \theta \in \mathcal{B}\}$ is a complete set of representatives of the orbits of the action of H on $\mathrm{Irr}(\mathbf{C}_K(P)|\xi)$. Another application of [11, 9.3], noticing that $\mu = \xi \times \lambda$, yields

$$\mathrm{Irr}_{p',A}(G|\mu) = \dot{\bigcup_{\theta\in\mathcal{B}}} \mathrm{Irr}_{p',A}(G|\theta \times \lambda),$$
$$\mathrm{Irr}_{p',B}(H|\mu) = \dot{\bigcup_{\theta\in\mathcal{B}}} \mathrm{Irr}_{p',B}(H|\theta^* \times \lambda).$$

Since $LP \triangleleft A$, we can apply Theorem 2.4 to conclude that

$$|\mathrm{Irr}_{p',A}(G|\theta \times \lambda)| = |\mathrm{Irr}_{p',B}(H|\theta^* \times \lambda)|,$$

for every $\theta \in \mathcal{B}$. This finishes the proof. $\square$

CRediT authorship contribution statement

Adele Maltempo: Writing – original draft, Investigation. **Carolina Vallejo:** Writing – original draft, Investigation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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