Zero-convergent solutions for equations with generalized relativistic operator: a fixed point approach.

ZERO-CONVERGENT SOLUTIONS FOR EQUATIONS WITH GENERALIZED RELATIVISTIC OPERATOR: A FIXED POINT APPROACH

ZUZANA DOŠLÁ, MAURO MARINI, AND SERENA MATUCCI

Abstract. A new abstract fixed point theorem is presented and applied to the solvability of a boundary value problem on the half-line for a differential equation with the generalized relativistic operator. The method does not require the explicit form of the fixed point map and can be applied also for solving boundary value problems associated to equations with various types of non-homogeneous operators. The concept of principal solutions for half-linear equations is also used for finding suitable *a-priori* bounds for solutions.

Cordially dedicated to Ravi P. Agarwal

1. Introduction

In this paper we study the existence of zero-convergent solutions for the second order equation

$$(a(t)\Phi_R(x'))' + b(t)F(x) = 0, \quad t \in I = [t_0, \infty), \tag{1.1}$$

satisfying the boundary conditions

$$x(t_0) = c > 0, \; x(t) > 0 \text{ and } x'(t) < 0 \text{ on } I, \; \lim_{t\to\infty} x(t) = 0, \tag{1.2}$$

where $\Phi_R : (-1, 1) \to \mathbb{R}$ is the homeomorphismus

$$\Phi_R(u) = \left(1 - |u|^{1+\alpha}\right)^{-\alpha/(1+\alpha)} |u|^{\alpha} \operatorname{sgn} u, \quad \alpha > 0.$$

Throughout the paper we assume that the functions a, b are continuous and positive on $[t_0, \infty)$, $t_0 \geq 0$, and the function F is a continuous function on $\mathbb{R}$ such that $uF(u) > 0$ for $u \neq 0$.

The operator Φ_R is the so-called *generalized relativistic operator* and occurs in studying some nonlinear elasticity problems, see [20, 25, 26]. It originates from the Minkowski mean curvature operator $\Phi_M : (-1, 1) \to \mathbb{R}$

$$\Phi_M(u) = \frac{u}{\sqrt{1 - u^2}},$$

which corresponds to Φ_R when $\alpha = 1$. In this case, equation (1.1) reads as

$$(a(t)\Phi_M(x'))' + b(t)F(x) = 0, \tag{1.3}$$

and arises in studying certain extrinsic properties of the mean curvature of hypersurfaces in the relativity theory. For this reason Φ_M is called also the *relativity*

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operator. Moreover, Φ_M can be found also in the theory of electromagnetism, where it is referred to as *Born–Infeld operator*, see [5, 21].

Boundary value problems [BVPs] associated to (1.1) have been investigated by many authors. We refer to [25, 26] for (1.1) and to [5, 6, 19, 23] for its special case (1.3), and references therein.

As usual, a solution x of (1.1), is said to be *nonoscillatory* if it is defined in a neighborhood of infinity and $x(t) \neq 0$ for large t. Moreover, eventually positive decreasing solutions x of (1.1) are called *Kneser solutions* and a Kneser solution x is said to be a *global Kneser solution,* if it is defined on the whole interval $I = [t_0, \infty)$ and $x(t)x'(t) < 0$ for $t \in I$, see, e.g., [24, Sections 13.1, 13.2, 16.1] and references therein.

Under the condition

$$\int_1^\infty \frac{1}{a(t)} dt < \infty, \tag{1.4}$$

the existence of global Kneser solutions to (1.3) has been discussed by the authors in [14]. In particular, in [14] an asymptotic proximity between Kneser solutions of (1.3) and the corresponding ones of the linear equation

$$(a(t)y')' + b(t)y = 0 \tag{1.5}$$

has been investigated. This qualitative similarity between (1.3) and (1.5) continues to hold also in other different contexts, say, for instance, in the search of periodic solutions [6] or when (1.5) is oscillatory [10, Theorem 2.1].

Here, we study the existence of Kneser solutions of (1.1) and their convergence to zero, by extending and generalizing results in [14]. Our results illustrate also an asymptotic similarity concerning nonoscillatory solutions of (1.1) and those of the corresponding half-linear equation

$$\big(a(t)\Phi_\alpha(x')\big)' + b(t)\Phi_\alpha(x) = 0,$$

where $\Phi_\alpha : \mathbb{R} \to \mathbb{R}$ is the α-Laplacian operator

$$\Phi_\alpha(u) = |u|^\alpha \operatorname{sgn} u, \quad \alpha > 0. \tag{1.6}$$

Roughly speaking, as already noticed in [6], as the classical acceleration operator is an approximation of the operator Φ_M for small values of $|u|$, the $\alpha-$Laplacian operator Φ_α can be viewed as an approximation of the operator Φ_R for small values of $|u|$.

Our main tool is based on a fixed point result for an operator defined in a Fréchet space by a Schauder's half-linearization device. This approach is presented in Section 2. It does not require the explicit form of the fixed point operator and can be applied for solving BVP associated to equations with non-homogeneous operator. Some known properties of principal solutions in the half-linear case, which are needed in the sequel, are recalled in Section 3. In Section 4 the existence of Kneser solutions to (1.1) is studied. Finally, some comments, suggestions and open problems are in Section 5.

2. An abstract fixed point result

An approach for solving a BVP on the half-line $I = [t_0, \infty)$ is to reduce it to an abstract fixed point equation of the type

$$x = \mathcal{T}(x), \tag{2.1}$$

where $\mathcal{T}$ is a possible nonlinear operator defined in a subset of a suitable Banach or Fréchet space X, see, e.g., [4, 13] and references therein. However, the solvability of (2.1) requires an appropriate analysis, since one needs a suitable topology which makes $\mathcal{T}$ a continuous map with some additional properties, such as, for instance, the compactness. On the other hand, these properties are strongly related to the topological structure of X. Since I is a noncompact interval, the choice as X of the Frechét space $C(I, \mathbb{R}^n)$ of the continuous vectors defined on I, endowed with the topology of uniform convergence on compact subsets of I appears to be the most suitable for verifying the compactness of $\mathcal{T}$. However, in this case the operator $\mathcal{T}$ can be discontinuous, if it is considered in its whole domain and not merely in an appropriate bounded set, see, e.g., [12, Theorem 3.1]. On the other hand, by choosing a suitable Banach space as the space X, problems concerning the characterization of compact subsets may arise. This is due to the fact that each space has his own and not always easy to handle. For more details on this topic we refer to the monographs [1, 2, 3] and the paper [11].

Here, we present a fixed point result for an operator $\mathcal{T}$, which is defined by a Schauder's type half-linearization device. Roughly speaking this result reduces the solvability of the considered BVP to the solvability of a BVP for a suitable associated half-linear equation.

As claimed, let $C(I, \mathbb{R}^2)$ be the Frechét space of the continuous vector functions $\underline{\mathbf{u}} = (u_1, u_2)$ defined on I, endowed with the topology of uniform convergence on compact subsets of I. We recall that a subset Ω of $C(I, \mathbb{R}^2)$ is *bounded* if and only if there exists a positive continuous function φ from I into $\mathbb{R}$ such that $|\underline{\mathbf{u}}(t)| \le \varphi(t)$ for all $t \in I$ and $\underline{\mathbf{u}} \in \Omega$. Observe that the boundedness of φ on the whole interval I is not required. Further, a set Ω is *relatively compact* in $C(I, \mathbb{R}^2)$ if and only if it is bounded and the functions of Ω are equicontinuous on each compact subset of I.

Set $I_\rho = (-\rho, \rho)$, $I_\sigma = (-\sigma, \sigma), 0 < \rho \le \infty, 0 < \sigma \le \infty$ and let $\Phi : I_\rho \to I_\sigma$ be an increasing odd homeomorphismus. Since Φ is invertible, denote by Φ^* its inverse, that is $\Phi^* : I_\sigma \to I_\rho$ and $\Phi(\Phi^*(u)) = u$ for any $u \in I_\sigma$. Thus, we have $u\Phi^*(u) > 0$ for all $u \in I_\rho, u \ne 0$. Assume the following.

(i_0) There exists a continuous positive function Ψ, defined on I_σ, such that for any $z \in I_\sigma$

$$\Phi^*(z) = \Psi(z)\Phi_{1/\beta}(z) \tag{2.2}$$

where β is a positive real number and $\Phi_{1/\beta}(z) = |z|^{1/\beta} \operatorname{sgn} z$.

A prototype of Φ is the α-Laplacian operator Φ_α given by (1.6), where $\beta = \alpha$, which corresponds to case $\Psi(z) \equiv 1$ in (2.2). Indeed, the inverse of Φ_α is $\Phi_{1/\alpha}(z) = |z|^{1/\alpha} \operatorname{sgn} z$. Observe that Φ_α is a homogeneous operator. On the other hand, in the applications arise also some important operators satisfying (2.2) for which the homogeneity property fails. Some examples are given by the quoted operators Φ_M, Φ_R. Indeed, a standard calculation shows that the inverse of Φ_M is the operator $\Phi_E : \mathbb{R} \to (-1,1)$ given by

$$\Phi_E(z) = \frac{z}{\sqrt{1+z^2}}\,.$$

The operator Φ_E is called the *Euclidean mean curvature operator* and arises in the search for radial solutions to partial differential equations which model fluid mechanics problems, in particular capillarity-type phenomena for compressible and incompressible fluids, see, e.g., [6, 8].

Concerning the inverse operator Φ_R^* of Φ_R, from $z = \Phi_R(u)$ we obtain

$$1 - |u|^{\alpha+1} = \left(1 + |z|^{(\alpha+1)/\alpha}\right)^{-1}.$$

Thus, we get

$$|u|^{\alpha+1} = 1 - \left(1 + |z|^{(\alpha+1)/\alpha}\right)^{-1} = \left(1 + |z|^{(\alpha+1)/\alpha}\right)^{-1} |z|^{(\alpha+1)/\alpha},$$

which yields

$$u = \Phi_R^*(z) = \left(1 + |z|^{(\alpha+1)/\alpha}\right)^{-1/(\alpha+1)} \Phi_{1/\alpha}(z). \tag{2.3}$$

Thus, (2.2) is verified for Φ_R with $\beta = \alpha$ and

$$\Psi(z) = \left(1 + |z|^{(\alpha+1)/\alpha}\right)^{-1/(\alpha+1)}. \tag{2.4}$$

Other examples can be found in [7, page 3], where the plastic deformation of certain materials subject at high temperature is studied.

Now consider the differential equation

$$(a(t)\Phi(x'))' + F_1(t,x) = 0, \quad t \in I = [t_0, \infty), \tag{2.5}$$

where the function $F_1 : I \times \mathbb{R} \to \mathbb{R}$ is continuous and Φ satisfies (i_0). Set $G : I \times \mathbb{R} \to \mathbb{R}$ the continuous function

$$G(t,u)\Phi_\beta(u) = F_1(t,u). \tag{2.6}$$

The following fixed point result holds.

Theorem 2.1. *Assume (i_0) and let S_0 be a subset of $C(I,\mathbb{R}^2)$. Suppose that there exists a nonempty closed bounded convex subset $\Omega \subset C(I,\mathbb{R}^2)$ such that*

$$|v(t)| < \sigma\, a(t) \quad \textit{for all } (u,v) \in \Omega \textit{ and } t \in I,$$

and a nonempty closed subset S_1 of $S_0 \cap \Omega$ such that for each $(u,v) \in \Omega$ the half-linear equation

$$\frac{d}{dt}\left(H(t,v(t))\,\Phi_\beta(y')\right) + G(t,u(t))\,\Phi_\beta(y) = 0, \tag{2.7}$$

has a unique solution y_{uv} with $(y_{uv}, y_{uv}^{[1]}) \in S_1$, where $\Phi_\beta(z) = |z|^\beta \operatorname{sgn} z$, H is the function

$$H(t,v(t)) = a(t)\ \Psi^{-\beta}\left(\frac{v(t)}{a(t)}\right), \tag{2.8}$$

G is given in (2.6) and $y_{uv}^{[1]}$ is the quasiderivative of y_{uv}, that is

$$y_{uv}^{[1]} = H(t, v(t))\, \Phi_\beta(y_{uv}').$$

For each (u, v) denote by $\mathcal{T} : \Omega \to C(I, \mathbb{R}^2)$, the operator given by

$$\mathcal{T}(u, v) = (y_{uv}, y_{uv}^{[1]}). \tag{2.9}$$

Then $\mathcal{T}$ has a fixed point $(\widehat{x}, \widehat{y}) \in S_1 \subset S_0$ with $\widehat{y}(t) = a(t)\Phi(\widehat{x}'(t))$ and $\widehat{x}$ is a solution of (2.5).

Proof. Consider the operator $\mathcal{T}$ given by (2.9). First, let us prove that $\mathcal{T}(\Omega)$ is relatively compact in $C(I, \mathbb{R}^2)$. For any $(u, v) \in \Omega$, the couple $(y_{uv}, y_{uv}^{[1]})$ belongs to $S_1 \subset \Omega$, and so the equiboundedness of $(y_{uv}, y_{uv}^{[1]})$ follows. Since

$$|y_{uv}'(t)| = \frac{|y_{uv}^{[1]}(t)|^{1/\beta}}{\big(H(t, v(t))\big)^{1/\beta}}\ , \tag{2.10}$$

the functions y_{uv}' are equibounded on each compact set $K \subset I$, which yields the equicontinuity of y_{uv}. Further, since (2.7) can be written as

$$(y_{uv}^{[1]}(t))' = -G(t, u(t))\, \Phi_\beta(y_{uv}(t))\,,$$

the equiboundedness of $(y_{uv}^{[1]})'$ follows. Then functions $y_{uv}^{[1]}$ are equicontinuous on each compact set $K \subset I$ and $\mathcal{T}(\Omega)$ is relatively compact in $C(I, \mathbb{R}^2)$.

Now, let us prove that $\mathcal{T}$ is continuous in Ω, that is, if $\{(u_n, v_n)\}$ is a sequence which converges to $(u, v) \in \Omega$, then $\{\mathcal{T}(u_n, v_n)\}$ converges to $\mathcal{T}(u, v)$. Since $\mathcal{T}(\Omega)$ is relatively compact, the sequence $\big\{\mathcal{T}(u_n, v_n)\big\}$ has a converging subsequence. For sake of simplicity, denote this subsequence by $\{(y_n, y_n^{[1]})\}$. Put

$$\lim_n y_n = \widetilde{y}, \quad \lim_n y_n^{[1]} = \widetilde{z}. \tag{2.11}$$

The sequence $\{y_n^{[1]}\}$ is compact, thus it uniformly converges on K. Thus, in view of (2.10), also $\{y_n'\}$ uniformly converges on K. Hence, from (2.11)

$$\lim_n y_n'(t) = \frac{d}{dt} \lim_n y_n(t) = \frac{d}{dt}\widetilde{y}(t) = \widetilde{y}'(t)$$

and the continuity of H, we get

$$\widetilde{z}(t) = \lim_n y_n^{[1]}(t) = H(t, v(t))\, \Phi_\beta(\widetilde{y}'(t)) = \widetilde{y}^{\,[1]}(t)\,. \tag{2.12}$$

We have

$$\lim_n G(t, u_n(t))\, \Phi_\beta(y_n) \, = G(t, u(t))\, \Phi_\beta(\widetilde{y})$$

uniformly on K. From this, (2.7) and (2.12), we obtain for $t \in K$

$$-G(t, u(t))\, \Phi_\beta(\widetilde{y}) = \lim_n \frac{d}{dt}\left(y_n^{[1]}(t)\right) = \frac{d}{dt} \lim_n y_n^{[1]}(t) = \frac{d}{dt}\widetilde{y}^{\,[1]}(t)\,.$$

Hence, $\widetilde{y}$ is a solution of (2.7), which corresponds to $(u, v) \in \Omega$. Moreover, $\widetilde{y}$ is the only possible cluster point of the compact sequence $\{y_n\}$ and the couple $(\widetilde{y}, \widetilde{y}^{[1]})$ belongs to S_1, because S_1 is closed. Then

$$(\widetilde{y}\,, \widetilde{y}^{\,[1]}) = \mathcal{T}(u, v)$$

and $\mathcal{T}$ is continuous in Ω. Since $S_1 \subset \Omega$ we have $\mathcal{T}(\Omega) \subset \Omega$ and so from the Tychonov fixed point theorem there exists $(\widehat{x}, \widehat{y}) \in \Omega$ such that $(\widehat{x}, \widehat{y}) = \mathcal{T}(\widehat{x}, \widehat{y})$.

Since $S_1 \subset S_0$, we also obtain $(\widehat{x}, \widehat{y}) \in S_0$. Now, for completing the proof let us show that $\widehat{x}$ is a solution of (2.5) and $\widehat{y}(t) = a(t)\Phi(\widehat{x}'(t))$. Since $\widehat{x}$ is a solution of (2.7) with $u = \widehat{x}$ and $v = \widehat{y}$, we have

$$\frac{d}{dt}\left(H(t, \widehat{y}(t))\Phi_\beta(\widehat{x}\,'(t))\right) + F_1(t, \widehat{x}(t)) = 0,$$

and

$$\widehat{y}(t) = H(t, \widehat{y}(t))\, \Phi_\beta(\widehat{x}\,'(t)), \tag{2.13}$$

i.e.

$$\frac{d}{dt}\widehat{y}(t) + F_1(t, \widehat{x}(t)) = 0. \tag{2.14}$$

In view of (2.2) and (2.8) we have for $|\xi| < \sigma$

$$\Phi^*\left(\frac{|\xi|}{a(t)}\right) = \left(\frac{|\xi|}{a(t)}\right)^{1/\beta} \Psi\left(\frac{|\xi|}{a(t)}\right) = \frac{|\xi|^{1/\beta}}{H^{1/\beta}(t, \xi)}. \tag{2.15}$$

From (2.13) we obtain

$$\widehat{x}'(t) = \frac{|\widehat{y}(t)|^{1/\beta}\ \mathrm{sgn}\ \widehat{y}(t)}{H^{1/\beta}(t, \widehat{y}(t))},$$

and, using (2.15) with $\xi = \widehat{y}(t)$, we have

$$\widehat{x}'(t) = \Phi^*\left(\frac{|\widehat{y}(t)|}{a(t)}\right)\ \mathrm{sgn}\ \widehat{y}(t)$$

or

$$\widehat{y}(t) = a(t)\Phi(\widehat{x}'(t))\,.$$

Then, the assertion follows from (2.14). □

3. Preliminaries on the half-linear equation

Consider the half-linear equation

$$\left(a(t)\,\Phi_\alpha(x')\right)' + b(t)\Phi_\alpha(x) = 0. \tag{3.1}$$

Equation (3.1) is called *half-linear* because the homogeneity property continues to hold for (3.1), while the additive property clearly not. It is known that (3.1) presents a striking similarity with the corresponding linear equation (1.5), especially as it concerns the Sturmian theory and oscillation or nonoscillation criteria, see, e.g., [17] for more details. In particular, the notion of the principal solution, introduced for (1.5) by W. Leighton and M. Morse, see, e.g., [22, Chapter 11, Theorem 6.4], has been extended in [18, 27] to the half-linear equation. It reads as follows. If (3.1) is nonoscillatory, then a nontrivial solution x_0 of (3.1) is said to be *the principal solution* of (3.1) if for every nontrivial solution x of (3.1) such that $x \neq \mu x_0$, $\mu \in \mathbb{R}$, the inequality

$$\frac{x_0'(t)}{x_0(t)} < \frac{x'(t)}{x(t)} \quad \text{for large } t.$$

holds.

The set of principal solutions of (3.1) is nonempty and principal solutions are determined up to a constant factor. We refer to [17, Chapter 4.2] for more details on this topic.

Clearly, the principal solution does not have zeros in a neighborhood of infinity. The positiveness of the principal solution on an *a-priori* closed fixed unbounded interval $[T, \infty), T \geq 1$, is a more subtle question. A sufficient condition for having this property can be obtained by means of a comparison result between two different half-linear equations.

If x is a solution of (3.1), we denote its quasi-derivative $x^{[1]}$ by

$$x^{[1]}(t) = a(t)\,\Phi_\alpha(x'(t)).$$

Consider the half-linear equations

$$\big(a_1(t)\,\Phi_\alpha(z')\big)' + b_1(t)\,\Phi_\alpha(z) = 0, \tag{3.2}$$

and

$$\big(a_2(t)\,\Phi_\alpha(w')\big)' + b_2(t)\,\Phi_\alpha(w) = 0, \tag{3.3}$$

where $a_i, b_i, i = 1, 2$, are positive continuous functions for $t \geq t_0$ such that

$$a_2(t) \leq a_1(t)\ , \quad b_2(t) \geq b_1(t). \tag{3.4}$$

Equation (3.3) is called a *majorant* of (3.2) and, analogously, (3.2) is called a *minorant* of (3.3). If (3.3) is nonoscillatory, then (3.2) is nonoscillatory too. Similarly, if (3.2) is oscillatory, then (3.3) is oscillatory too.

The following comparison result for principal solutions of (3.2) and (3.3) is an important tool in our later consideration.

Proposition 3.1. *Assume that (3.3) is nonoscillatory and (3.4) is valid. Denote by z_0 and w_0 the principal solutions of (3.2) and (3.3), respectively. If w_0 does not have zeros on $[T, \infty)$, then the following holds.*

(j_1) *The principal solution z_0 does not have zeros on $[T, \infty)$.*

(j_2) *We have for $t \geq T$*

$$\frac{z_0^{[1]}(t)}{\Phi_\alpha(z_0(t))} \leq \frac{w_0^{[1]}(t)}{\Phi_\alpha(w_0(t))}\,,$$

where $z_0^{[1]}$ is the quasi-derivative of z_0 and $w_0^{[1]}$ is the one of w_0.

Proof. First, let us observe that, in the limit case

$$a_2(t) = a_1(t), \quad b_2(t) = b_1(t) \quad \text{on } [1, \infty)$$

the assertion is trivially true. Now, suppose that at least one of the inequality (3.4) is strict on a subinterval of $[T, \infty)$ of positive measure. Applying [17, Theorem 4.2.3] with minor changes, we obtain claim (j_1). Claim (j_2) follows by using a similar argument and applying [17, Theorem 4.2.2], with minor changes. □

Some properties of the principal solution of (3.1) when

$$\int_{t_0}^{\infty} a^{-1/\alpha}(t)\ dt < \infty, \tag{3.5}$$

are given by the following.

Proposition 3.2. *Assume (3.5) and*

$$Y_1 = \int_{t_0}^{\infty} b(t) \left(\int_t^{\infty} a^{-1/\alpha}(s)\ ds \right)^{\alpha}\ dt < \infty. \tag{3.6}$$

Then the following holds.

(j_1) *Equation (3.1) is nonoscillatory and the principal solution u of (3.1), $u(t) > 0$ for large t, satisfies*

$$\lim_{t\to\infty} u(t) = 0, \quad \lim_{t\to\infty} a(t)\Phi_\alpha(u(t)) = \ell_u, \quad -\infty < \ell_u < 0, \tag{3.7}$$

and

$$u(t) = O\left(\int_t^{\infty} a^{-1/\alpha}(s)\ ds \right) \quad \text{as} \ \ t \to \infty. \tag{3.8}$$

where the symbol $O(f) = g$ as $t \to \infty$ means that the limit $\lim_{t\to\infty} f(t)/g(t)$ is finite and different from zero.

(j_2) *Any solution x of (3.1) satisfying*

$$\lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} a(t)\Phi_\alpha(x(t)) = \ell_x, \quad \text{for some } \ell_x \in \mathbb{R}\setminus\{0\},$$

is the principal solution.

(j_3) *In addition, if*

$$J_1 = \int_{t_0}^{\infty} a^{-1/\alpha}(t) \left(\int_{t_0}^{t} b(s)\ ds \right)^{1/\alpha}\ dt < \infty, \tag{3.9}$$

then any nonprincipal solution x of (3.1) does not tend to zero as $t \to \infty$.

Proof. Claim (j_1). From [9, Theorem 2-(i_2)], we get (3.7). Moreover, since $\lim_{t\to\infty} a(t)\Phi_\alpha(u(t)) = \ell_u$, using the l'Hopital rule, the limit

$$\lim_{t\to\infty} \frac{u(t)}{\int_t^{\infty} a^{-1/\alpha}(s)\ ds}$$

is finite and different from zero, i.e. (3.8) is valid.

Claim (j_2) follows again from [9, Theorem 2-(i_2)].

Claim (j_3). By Claims (j_1) and (j_2), the set of eventually positive principal solutions coincides with the set of solutions satisfying (3.7). From here and using [9, Theorem 4 and Theorem 7], the assertion follows. □

We close this section by considering the so-called *reciprocal equation* to (3.1), that is the equation

$$\left(\frac{1}{b^{\sigma}(t)} \Phi_\sigma(y') \right)' + \frac{1}{a^{\sigma}(t)} \Phi_\sigma(y) = 0, \quad t \ge t_0, \tag{3.10}$$

where $\sigma = \alpha^{-1}$, see, e.g., [17, Section 1.2.8]. It is easy to verify that the quasi-derivative $y = x^{[1]}$ of any solution x of (3.1) is a solution of (3.10) and, conversely, the quasi-derivative $y^{[1]}(t) = b^{-\sigma}(t)\Phi_\sigma(y')$ of any solution y of (3.10) is a solution of (3.1). The principal solution of (3.1) and (3.10) are related, as the following result shows.

Proposition 3.3. [17, Theorem 4.2.4] *Let* (3.1) *be nonoscillatory and*

$$\int_{t_0}^{\infty} (a^{-1/\alpha}(t) + b(t))\, dt = \infty.$$

Then a solution u of (3.1) is the principal solution if and only if $v = u^{[1]}$ is the principal solution of (3.10).

4. Global Kneser solutions

In this section we study the existence of Kneser solutions x of (1.1) satisfying (1.2). The following conditions are assumed.

(i_1) We have

$$\inf_{t\geq t_0} a^{1/\alpha}(t) \int_t^{\infty} a^{-1/\alpha}(s) ds = \lambda > 0. \tag{4.1}$$

(i_2) The function F satisfies

$$\lim_{u\to 0+} \frac{F(u)}{u^{\alpha}} = F_0, \quad 0 \leq F_0 < \infty.$$

The following holds.

Theorem 4.1. *Let (i_1) and (i_2) be satisfied and assume $Y_1 < \infty, J_1 < \infty$, where Y_1 and J_1 are defined by (3.6) and (3.9), respectively. Define*

$$M = \sup_{u\in(0,\lambda]} \frac{F(u)}{u^{\alpha}}. \tag{4.2}$$

If the half-linear equation

$$\big(a(t)\Phi_{\alpha}(z')\big)' + M\, b(t)\ \Phi_{\alpha}(z) = 0, \quad t \geq t_0, \tag{4.3}$$

is nonoscillatory and its principal solution z_0 is positive decreasing on $I = [t_0, \infty)$, then for any constant c such that

$$0 < c < \lambda \tag{4.4}$$

equation (1.1) has a solution x satisfying the boundary conditions (1.2).

Proof. For proving the solvability of the BVP (1.1)-(1.2), we will use Theorem 2.1. Fixed c satisfying (4.4), set

$$\Lambda = \frac{\lambda}{(\lambda^{\alpha+1} - c^{\alpha+1})^{1/(\alpha+1)}}. \tag{4.5}$$

Without loss of generality, suppose

$$z_0(t_0) = c^{\Lambda}.$$

By Proposition 3.2, the principal solution z_0 of (4.3) satisfies

$$z_0(t) \leq c^{\Lambda}, \quad z_0'(t) < 0 \ \text{ on } I \text{ and } \lim_{t\to\infty} z_0(t) = 0. \tag{4.6}$$

Let Ω be the set

$$\Omega = \left\{ (u,v) \in C(I,\mathbb{R}^2) : 0 \leq u(t) \leq (z_0(t))^{1/\Lambda}, \quad -\left(\frac{c\Lambda}{\lambda}\right)^{\alpha} a(t) \leq v(t) \leq 0 \right\}.$$

For any $(u,v) \in \Omega$, in view of (4.6), we have $z_0^{1/\Lambda}(t) \le c$ and so $0 \le u(t) \le c$. Denote by S_0 the set

$$S_0 = \left\{(u,v) \in C(I, \mathbb{R}^2) : u(t_0) = c,\ u(t) \ge 0,\ v(t) \le 0,\ \lim_{t\to\infty} u(t) = 0\right\}.$$

and put

$$S_1 = \Omega \cap S_0.$$

Using (4.6), we obtain

$$S_1 = \{(u,v) \in \Omega : u(t_0) = c\}.$$

Fixed $(u,v) \in \Omega$, consider the half-linear equation

$$\big(h_v(t)\,\Phi_\alpha(y')\big)' + b(t)\widetilde{F}_u(t)\,\Phi_\alpha(y) = 0\,, \tag{4.7}$$

where

$$\widetilde{F}_u(t) = \begin{cases} u^{-\alpha}(t)F(u(t)) & \text{if } u(t) > 0, \\ \\ F_0 & \text{if } u(t) = 0, \end{cases}$$

and

$$h_v(t) = a(t)\left(1 + \left(\frac{|v(t)|}{a(t)}\right)^{(\alpha+1)/\alpha}\right)^{\alpha/(\alpha+1)}. \tag{4.8}$$

Equation (4.7) will play the role of (2.7) in Theorem 2.1, with $\beta = \alpha$. Indeed, for the function Ψ given by (2.4), using (4.8) we have

$$a(t)\Psi^{-\alpha}\left(\frac{|v(t)|}{a(t)}\right) = h_v(t).$$

Since $h_v(t) \ge a(t)$, from (4.2) equation (4.7) is a minorant of (4.3). Thus, (4.7) is nonoscillatory. Let η_{uv} be the principal solution of (4.7) such that $\eta_{uv}(t_0) = c$. We want to show that

$$(\eta_{uv}, \eta_{uv}^{[1]}) \in S_1,$$

where $\eta_{uv}^{[1]}$ is the quasiderivative of η_{uv}, that is $\eta_{uv}^{[1]}(t) = h_v(t)\ \Phi_\alpha(\eta_{uv}'(t))$. We have for $t \in I$

$$a(t) \le h_v(t) \le a(t)\left(1 + \left(\frac{c\Lambda}{\lambda}\right)^{\alpha+1}\right)^{\alpha/(\alpha+1)}. \tag{4.9}$$

From (4.5), a standard calculation gives

$$1 + \left(\frac{c\Lambda}{\lambda}\right)^{\alpha+1} = 1 + \frac{c^{\alpha+1}}{\lambda^{\alpha+1} - c^{\alpha+1}} = \frac{\lambda^{\alpha+1}}{\lambda^{\alpha+1} - c^{\alpha+1}} = \Lambda^{\alpha+1}$$

or

$$\left(1 + \left(\frac{c\Lambda}{\lambda}\right)^{\alpha+1}\right)^{\alpha/(\alpha+1)} = \Lambda^\alpha.$$

Thus, from (4.9) we get

$$\frac{1}{\Lambda} \le \left(\frac{a(t)}{h_v(t)}\right)^{1/\alpha} \le 1. \tag{4.10}$$

Applying Proposition 3.1 to (4.3) and its minorant (4.7), we have that η_{uv} is positive decreasing on I and

$$(h_v(t))^{1/\alpha}\frac{\eta'_{uv}(t)}{\eta_{uv}(t)} \le a^{1/\alpha}(t)\frac{z'_0(t)}{z_0(t)}. \tag{4.11}$$

Using (4.10) and taking into account that η'_{uv} and z'_0 are negative on I, from (4.11) we obtain

$$\frac{\eta'_{uv}(t)}{\eta_{uv}(t)} \le \Lambda^{-1}\frac{z'_0(t)}{z_0(t)}.$$

Integrating this inequality on $[t_0, t)$ we get

$$0 < \eta_{uv}(t) \le (z_0(t))^{1/\Lambda}.$$

Let φ be the principal solution of

$$\big(\Lambda^\alpha\, a(t)\Phi_\alpha(x')\big)' = 0$$

such that $\varphi(t_0) = c$, i.e.

$$\varphi(t) = c\left(\int_{t_0}^\infty a^{-1/\alpha}(s)\ ds\right)^{-1}\int_t^\infty a^{-1/\alpha}(s)\ ds.$$

Applying again Proposition 3.1 we have for $t \in I$

$$\frac{\varphi^{[1]}(t)}{\varphi^\alpha(t)} \le \frac{\eta^{[1]}_{uv}(t)}{\eta^\alpha_{uv}(t)}, \tag{4.12}$$

where $\varphi^{[1]}$ is quasiderivative of φ, that is $\varphi^{[1]}(t) = \Lambda^\alpha\ a(t)\Phi_\alpha(\varphi'(t))$. From (4.1) we have

$$\lambda^\alpha \le a(t)\left(\int_t^\infty a^{-1/\alpha}(s)\ ds\right)^\alpha.$$

Using this inequality and

$$\frac{\varphi^{[1]}(t)}{\varphi^\alpha(t)} = -\Lambda^\alpha\left(\int_t^\infty a^{-1/\alpha}(s)\ ds\right)^{-\alpha}, \tag{4.13}$$

since η_{uv} is decreasing, from (4.12) we obtain

$$\eta^{[1]}_{uv}(t) \ge -c^\alpha\frac{\Lambda^\alpha\, a(t)}{a(t)\left(\int_t^\infty a^{-1/\alpha}(s)\ ds\right)^\alpha} \ge -\left(\frac{c\Lambda}{\lambda}\right)^\alpha a(t).$$

Hence, the couple $(\eta_{uv}, \eta^{[1]}_{uv})$ belongs to S_1.

It is easy to prove that for any $(u, v) \in \Omega$, the solution η_{uv} is the unique solution of (4.7) such that $(\eta_{uv}, \eta^{[1]}_{uv}) \in S_1$. By contradiction , let $\widehat{\eta}$ be another solution of (4.7), $(\widehat{\eta}, \widehat{\eta}^{[1]}) \in S_1$ and $\widehat{\eta} \ne \eta_{uv}$. For the sake of simplicity, the dependence of $\widehat{\eta}$ on the variable (u, v) is omitted. Clearly, $\widehat{\eta}$ is a nonprincipal solution of (4.7). Since $(\widehat{\eta}, \widehat{\eta}^{[1]}) \in S_1$, we have $0 \le \widehat{\eta}(t) \le (z_0(t))^{1/\Lambda}$. Hence

$$\lim_{t\to\infty}\widehat{\eta}(t) = 0.$$

Since $Y_1 < \infty, J_1 < \infty$, from (4.2) and (4.9) we have

$$\int_{t_0}^{\infty} b(t)\widetilde{F}_u(t)\left(\int_t^{\infty} h_v^{-1/\alpha}(s)\ ds\right)^{\alpha}\ dt < \infty,$$

$$\int_{t_0}^{\infty} h_v^{-1/\alpha}(t)\left(\int_{t_0}^{t} b(s)\widetilde{F}_u(s)\ ds\right)^{1/\alpha}\ dt < \infty.$$

Thus, applying Proposition 3.2-(j_3), the solution $\widehat{\eta}$ does not tend to zero as $t \to \infty$, which is a contradiction. Then the solution η_{uv} is the unique solution of (4.7) such that $(\eta_{uv}, \eta_{uv}^{[1]}) \in S_1$.

Applying Theorem 2.1, equation (1.1) has a solution $\widehat{x}$ such that $(\widehat{x}, \widehat{x}^{[1]}) \in S_1$, where $\widehat{x}^{[1]}(t) = a(t)\Phi_R(\widehat{x}'(t))$. Moreover, $\widehat{x}$ satisfies the boundary conditions (1.2), as it is easy to verify. □

A closer examination of the proof of Theorem 4.1 yields also lower and upper bounds for the solution $\widehat{x}$ of the BVP (1.1)-(1.2). Indeed, from Proposition 3.2 any eventually positive principal solution of (4.3) satisfies (3.8). Since $(\widehat{x}, \widehat{x}^{[1]}) \in S_1$ we get

$$(\widehat{x}(t))^{\Lambda} \le O\left(\int_t^{\infty} a^{-1/\alpha}(s)\ ds\right) \quad \text{as} \quad t \to \infty.$$

Concerning the lower bound, from (4.12) and (4.13) we obtain

$$\frac{\eta_{uv}^{[1]}(t)}{\eta_{uv}^{\alpha}(t)} \ge -\Lambda^{\alpha}\left(\int_t^{\infty} a^{-1/\alpha}(s)\ ds\right)^{-\alpha}, \tag{4.14}$$

where η_{uv} is the principal solution of (4.7) such that $\eta_{uv}(t_0) = c$ and $\eta_{uv}^{[1]}(t) = h_v(t)$ $\Phi_\alpha(\eta_{uv}'(t))$. Since $\eta_{uv}^{[1]}$ is negative and $h_v(t) \ge a(t)$, from (4.14) we get

$$\Phi_\alpha\left(\frac{\eta_{uv}'(t)}{\eta_{uv}(t)}\right) \ge -\frac{\Lambda^{\alpha}}{a(t)}\left(\int_t^{\infty} a^{-1/\alpha}(s)\ ds\right)^{-\alpha}$$

or

$$\frac{\eta_{uv}'(t)}{\eta_{uv}(t)} \ge -\Lambda a^{-1/\alpha}(t)\left(\int_t^{\infty} a^{-1/\alpha}(s)\ ds\right)^{-1}.$$

Integrating this inequality on $[t_0, t]$ we have

$$\eta_{uv}(t) \ge c\left(\int_{t_0}^{\infty} a^{-1/\alpha}(s)\ ds\right)^{-\Lambda}\left(\int_t^{\infty} a^{-1/\alpha}(s)\ ds\right)^{\Lambda}. \tag{4.15}$$

Since the solution $\widehat{x}$ of the BVP (1.1)-(1.2) coincides with η_{uv} for some $(u, v) \in \Omega$, the lower bound (4.15) is valid also for $\widehat{x}$. Hence the following holds.

Corollary 4.2. *Under the assumptions of Theorem 4.1, equation (1.1) has a solution $\widehat{x}$ satisfying (1.2) and*

$$(\widehat{x}(t))^{\Lambda} \le O\left(\int_t^{\infty} a^{-1/\alpha}(s)\, ds\right) \quad \text{as} \quad t \to \infty,$$

$$(\widehat{x}(t))^{1/\Lambda} \ge c^{1/\Lambda}\left(\int_{t_0}^{\infty} a^{-1/\alpha}(s)\, ds\right)^{-1}\int_t^{\infty} a^{-1/\alpha}(s)\, ds \quad \text{for } t \ge t_0.$$

Theorem 4.1 requires that there exists a suitable half-linear equation which is nonoscillatory and its principal solution is positive decreasing on the whole interval $I = [t_0, \infty)$. This assumption may be verified by using Proposition 3.1 and a half-linear equation whose principal solution is known and has the desired properties. An example in this direction can be obtained using the half-linear Euler differential equation

$$\big(\Phi_\beta(x')\big)' + \left(\frac{\beta}{\beta+1}\right)^{\beta+1} t^{-\beta-1}\Phi_\beta(x) = 0, \qquad t \ge t_0 > 0. \tag{4.16}$$

It is known that (4.16) is nonoscillatory. Moreover, the function

$$x_0(t) = t^{\beta/(\beta+1)}$$

is the principal solution of (4.16), see [17, Section 1.4.2.]. The change of variable

$$y = \Phi_\beta(x')$$

transforms (4.16) into the equation

$$\big(t^{(\beta+1)/\beta}\Phi_{1/\beta}(y')\big)' + \left(\frac{\beta}{\beta+1}\right)^{(\beta+1)/\beta} \Phi_{1/\beta}(y) = 0, \qquad t \ge t_0 > 0, \tag{4.17}$$

which, as claimed, is the reciprocal equation to (4.16). Setting $\alpha = \beta^{-1}$, equation (4.17) becomes

$$\big(t^{1+\alpha}\Phi_\alpha(y')\big)' + \left(\frac{1}{1+\alpha}\right)^{1+\alpha} \Phi_\alpha(y) = 0, \qquad t \ge t_0 > 0. \tag{4.18}$$

Let v_0 be the function

$$v_0(t) = \Phi_\beta(x_0'(t)) = \left(\frac{1}{1+\alpha}\right)^{1/\alpha} \left(\frac{1}{t}\right)^{1/(1+\alpha)}.$$

Hence, from Proposition 3.3 the function v_0 is the principal solution of (4.18).

Now, if the functions a, b satisfy for $t \ge t_0 > 0$

$$a(t) \ge t^{1+\alpha} \quad \text{and} \quad M\ b(t) \le \left(\frac{1}{1+\alpha}\right)^{1+\alpha}, \tag{4.19}$$

where M is given by (4.2), equation (4.18) is a majorant of (4.3). Moreover, (4.18) is nonoscillatory and its principal solution is positive decreasing for $t \ge t_0$. Hence, from Proposition 3.1, the principal solution of (4.3) is positive decreasing on $[t_0, \infty)$. Thus, from Theorem 4.1 we get the following.

Corollary 4.3. *Let (i_1) and (i_2) be satisfied and assume $Y_1 < \infty, J_1 < \infty$. If (4.19) is satisfied for $t \ge t_0 > 0$, where M given by (4.2), then for any constant c, such that $0 < c < \lambda$, equation (1.1) has a solution x satisfying the boundary conditions (1.2).*

Clearly, other criteria can be obtained by using as majorant of (4.3) any half-linear equation whose principal solution is positive decreasing on I.

5. Concluding Remarks

(1) The following results illustrate some properties of the nonoscillatory solutions of (1.1). They show other similarities between the generalized relativistic operator Φ_R and the α-Laplacian operator Φ_α.

Proposition 5.1. *For any solution x of (1.1) such that $x(t) \neq 0$ on (t_1, t_2), $t_0 \leq t_1 < t_2$, the derivative x' has at most one zero on (t_1, t_2). Consequently, any nonoscillatory solution x of (1.1) satisfies either $x(t)x'(t) > 0$ or $x(t)x'(t) < 0$ for large t.*

Proof. Set $G(t) = a(t)x(t)\Phi_R(x'(t))$. Then

$$G'(t) = -b(t)F(x(t))x(t) + a(t)x'(t)\Phi_R(x'(t)).$$

Clearly, the solution x is not constant on (t_1, t_2). Suppose that there exist two consecutive zeros of x' on (t_1, t_2), say s_1, s_2, with $t_1 < s_1 < s_2 < t_2$. Then $G'(s_1) < 0$, $G'(s_2) < 0$ and $x(t) \neq 0, x'(t) \neq 0$ on (t_1, t_2). Since $G(s_1) = G(s_2) = 0$, we get $G(t) \neq 0$ for $t \in (t_1, t_2)$, which is a contradiction. Hence, the derivative x' has at most one zero on (t_1, t_2). Consequently, if $x(t) \neq 0$ for large t, then there exists $t_x \geq t_0$ such that $x(t)x'(t) \neq 0$ for $t \in [t_x, \infty)$. □

It is easy to obtain a necessary condition for the existence of solutions x to (1.1) satisfying

$$x(t)x'(t) < 0 \text{ for large } t. \tag{5.1}$$

The following holds.

Proposition 5.2. *If*

$$\int_{t_0}^{\infty} \Phi_R^*\left(\frac{k}{a(s)}\right) ds = \infty \quad \textit{for any positive constant } k, \tag{5.2}$$

then (1.1) does not have solutions x satisfying (5.1).

Proof. By contradiction, let x be a solution of (1.1) satisfying (5.1) and, without loss of generality, suppose $x(t) > 0, x'(t) < 0$ on $[T, \infty), T \geq t_0$. Thus, $a(t)\Phi_R(x'(t))$ is nonincreasing on $[T, \infty)$, that is we have for $t \geq T$

$$a(t)\Phi_R(x'(t)) \leq a(T)\Phi_R(x'(T)) < 0. \tag{5.3}$$

From this, we get

$$x'(t) \leq \Phi_R^*\left(\frac{a(T)\Phi_R(x'(T))}{a(t)}\right), \tag{5.4}$$

where Φ_R^* is the inverse of Φ_R, see (2.3). Integrating (5.4) on (T, t) we obtain

$$x(t) \leq x(T) + \int_T^t \Phi_R^*\left(\frac{a(T)\Phi_R(x'(T))}{a(s)}\right) ds.$$

From this, in virtue of (5.3), we get a contradiction with the positiveness of x as t tends to infinity. □

(2) When (5.2) holds, another interesting problem concerns the existence of unbounded solutions of (1.1), which are positive on the whole interval I.

A partial answer to this problem has been given in [15] for equation (1.3) and the equation with the Euclidean mean curvature operator Φ_E

$$(a(t)\Phi_E(x'))' + b(t)F(x) = 0. \tag{5.5}$$

More precisely, in [15] an asymptotic proximity between the unbounded solutions of (1.3) [(5.5)] and the ones of the linear equation (1.5) has been investigated using the fact that the set of solutions of (1.5) is a two-dimensional space.

It should be interesting to extend these results to equation (1.1) for obtaining a qualitative similarity between (1.1) and the half-linear equation (3.1). Clearly, this problem requires a different approach than the one in [15], because the additive property does not holds for solutions of (3.1) when $\alpha \neq 1$.

(3) In the linear case, that is when $\alpha = 1$, the integrals Y_1 and J_1 coincide. Thus, from Proposition 3.2 we can state that if (1.4) is valid and

$$\int_{t_0}^{\infty} a^{-1}(t) \int_{t_0}^{t} b(s)\ ds\ dt < \infty,$$

then nonprincipal solutions of (1.5) have a non-zero limit at infinity. This property has been used in [14], in which the asymptotic behavior of global Kneser solutions for (1.3) is studied. On the other hand, when $\alpha \neq 1$, the integrals

$$\int_{t_0}^{t} b(r)\left(\int_{r}^{t} a^{-1/\alpha}(s)\ ds\right)^{\alpha} dr \quad \text{and} \quad \int_{t_0}^{t} a^{-1/\alpha}(r)\left(\int_{t_0}^{r} b(s)\ ds\right)^{1/\alpha} dr$$

can have a different asymptotic behavior, that is there are examples in which $Y_1 < \infty, J_1 = \infty$ or $Y_1 = \infty, J_1 < \infty$, see [16]. For this fact, in Theorem 4.1 it is assumed that both integrals Y_1 and J_1 are convergent.

It is an open problem if (1.1)-(1.2) is solvable also in the cases in which

$$Y_1 = \infty,\ J_1 < \infty \tag{5.6}$$

or

$$Y_1 < \infty,\ J_1 = \infty \tag{5.7}$$

are valid. Observe that (5.6) and (5.7) are the most interesting cases for the half-linear equation (3.1), because they do not have correspondence in the linear case.

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(Z. Došlá) Masaryk University, 611 37 Brno, Czech Republic
E-mail address: dosla@math.muni.cz

(M. Marini) University of Florence, 50139 Florence, Italy
E-mail address: mauro.marini@unifi.it

(S. Matucci) University of Florence, 50139 Florence, Italy
E-mail address: serena.matucci@unifi.it