

STABILITY AND COMPLEX FINSLER–EINSTEIN METRICS

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Dedicated to the memory of Pit-Mann Wong

ABSTRACT. Let $E \rightarrow M$ be a holomorphic vector bundle over a compact Kähler manifold (M, ω) and let F be a strongly pseudoconvex complex Finsler metric on E . We give a self-contained account of the projective geometry of F (the Hessian metric on p^*E , the vertical metric on $T_{\mathbb{P}(E)/M}$, the Chern–Finsler horizontal splitting and the Kobayashi curvature Ψ_F), with complete proofs and explicit normalizations. We then prove a Finsler Chern–Weil inequality: for every saturated subsheaf $\mathcal{S} \subset E$,

$$\mu_\omega(\mathcal{S}) - \mu_\omega(E) \leq \frac{1}{2\pi} \int_M \left(\max_{\mathbb{P}(E_z)} \mathrm{tr}_\omega \Psi_F - \int_{\mathbb{P}(E_z)} \mathrm{tr}_\omega \Psi_F d\nu_{F,z} \right) dV_\omega.$$

The proof uses only Segre forms, a Gauss–Codazzi inequality for the Kobayashi curvature, and a resolution of the singularities of $\mathcal{S}$. It yields at once that Finsler–Einstein bundles (in Kobayashi’s sense) are semistable, and that approximate Finsler–Einstein metrics exist if and only if E is semistable. Optimizing the resulting curvature functional over F recovers $\mu_{\max}(E)$, and the corresponding Finsler Ky Fan functionals recover the vertices of the Harder–Narasimhan polygon. Following Feng–Liu–Wan, we show that Berndtsson’s L^2 direct-image metric on E^* built from a Finsler–Einstein metric is Hermitian–Einstein, so that E admits a Finsler–Einstein metric if and only if it is polystable. The results are organized by a total-space viewpoint, suggested by the late Pit-Mann Wong: Finsler–Einstein geometry on $E \leftrightarrow$ horizontal J -geometry on $\mathbb{P}(E) \rightarrow$ slope stability of E . On $\mathbb{P}(E)$ with the adiabatic Kähler forms $\xi_F + Cp^*\omega$, we study the associated Einstein conditions for $\mathcal{O}(1)$, the direct image E^* , and the pull-back bundle p^*E .

1. INTRODUCTION

Let (M, ω) be a compact Kähler manifold of complex dimension n and $E \rightarrow M$ a holomorphic vector bundle of rank $r \geq 2$. If h is a Hermitian metric on E with Chern curvature Θ_h , the Hermitian–Einstein equation is

$$(1.1) \quad K_h := \sqrt{-1} \Lambda_\omega \Theta_h = \lambda \mathrm{Id}_E.$$

The Kobayashi–Hitchin correspondence asserts that (1.1) is solvable if and only if E is polystable, and that it is solvable up to an arbitrarily small L^∞ error if and only if E is semistable [19, 22, 10, 28, 20, 17].

Kobayashi [21] proposed to replace h by a strongly pseudoconvex complex Finsler metric F . Such an F determines a Hermitian metric on the tautological line bundle $\mathcal{O}_{\mathbb{P}(E)}(-1)$ [18], and the horizontal part of its curvature, the *Kobayashi curvature* Ψ_F , is a real $(1, 1)$ -form in the base directions depending on a point of $\mathbb{P}(E)$. A Finsler metric is *Finsler–Einstein* if $\mathrm{tr}_\omega \Psi_F$ is constant on $\mathbb{P}(E)$; for Hermitian F this is exactly (1.1). Kobayashi asked whether Finsler–Einstein bundles are semistable. Kobayashi’s question has been answered by Feng, Liu and Wan:

- Feng, Liu and Wan [11] proved semistability under an auxiliary hypothesis on exterior powers.

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- In [12] they removed this hypothesis: using a Donaldson-type functional on the space of Finsler metrics, they showed that a Finsler–Einstein bundle carries ω -approximate Hermitian–Einstein structures, hence is semistable.
- In [13] they showed, using Berndtsson’s L^2 metric on the direct image $p_*\mathcal{O}_{\mathbb{P}(E)}(1) = E^*$, that a Finsler–Einstein metric produces a Hermitian–Einstein metric. Hence E admits a Finsler–Einstein metric if and only if it is polystable.

This paper presents, in a unified framework, known and new results connecting Finsler–Einstein geometry on E , horizontal J -geometry on $\mathbb{P}(E)$, and slope stability. Although expository in its foundational sections, the paper contains several new quantitative and adiabatic results. Namely, a Finsler metric on E is the same as a metric on $\mathcal{O}_{\mathbb{P}(E)}(1)$. It is Finsler–Einstein exactly when the curvature of this metric, measured in the directions horizontal to the fibers, solves a J -type equation on $\mathbb{P}(E)$ to leading order as the base is inflated. Testing that equation on the subvarieties $\mathbb{P}(\mathcal{S}) \subset \mathbb{P}(E)$ defined by subsheaves $\mathcal{S} \subset E$ yields precisely the slope inequalities $\mu(\mathcal{S}) \leq \mu(E)$.

The total-space viewpoint was suggested to the author by the late Pit-Mann Wong, in conversations during a research stay at NCTS (Hsinchu, Taiwan) in 2009 [30]: a strongly pseudoconvex Finsler metric F on E is the same thing as a fiberwise positive metric on $\mathcal{O}_{\mathbb{P}(E)}(1)$, and the Einstein conditions for the natural bundles on $\mathbb{P}(E)$ should be read through the adiabatic Kähler forms built from it. In particular, the Hermitian–Einstein equation (14.3) for the Hessian metric on p^*E was proposed by him, and we refer to it as *Wong’s equation*. The ingredients are already in Cao–Wong [6]: the Kähler forms $c_1(\mathcal{O}_{\mathbb{P}(E)}(1)) + \lambda p^*\omega$ on $\mathbb{P}(E)$ [6, §3], and the Hessian metric on the vertical bundle, with its Chern connection and the decomposition of its curvature into horizontal, vertical and mixed components [6, §5]. The first Chern form ξ_F of the metric on $\mathcal{O}_{\mathbb{P}(E)}(1)$ defines the adiabatic Kähler forms

$$\Omega_{F,C} = \xi_F + C p^*\omega \quad (C \rightarrow \infty)$$

on $X = \mathbb{P}(E)$. With respect to the Chern–Finsler splitting, $\Omega_{F,C}$ is block diagonal. Its vertical block is the fiberwise Fubini–Study-type form $\omega_V/2\pi$, which is universal for our purposes. Its *horizontal* block is $A_C = C\omega - \Psi_F/2\pi$, which carries the whole Kobayashi curvature. The argument proceeds in two steps:

- (i) *Finsler–Einstein = horizontal J -geometry*. The Einstein condition for $\mathcal{O}_X(1)$ with respect to $\Omega_{F,C}$ is the degenerate J -equation $\mathrm{tr}_{\Omega_{F,C}}(p^*\omega) = c_C$, whose content is the horizontal trace $\mathrm{tr}_{A_C} \omega$. Its adiabatic expansion has universal leading term n/C . Its first non-universal coefficient is $(k_F - \lambda)/2\pi C^2$, so Kobayashi’s Finsler–Einstein equation is exactly the leading-order horizontal J -equation (Proposition 10.3, Remark 10.4).
- (ii) *From J -geometry to slope stability*. The J -equation comes with a subvariety test. Applied to the projectivized subsheaves $\mathbb{P}(\mathcal{S}) \subset \mathbb{P}(E)$ and expanded with an exact Segre identity (Lemma 10.1), the test reduces at leading order to the slope inequality $\mu(\mathcal{S}) \leq \mu(E)$ (Proposition 11.6). Integrating the underlying pointwise inequality over the fibers instead of bounding it by its maximum gives Theorem A. Optimizing Theorem A over F recovers the maximal Harder–Narasimhan slope (Theorem 12.4).

The program then has three levels on $\mathbb{P}(E)$, all read in the adiabatic limit:

- L1** the line bundle $\mathcal{O}_X(1)$: horizontal J -geometry, semistability, and the Harder–Narasimhan slope (Sections 10–12);
- L2** the direct image $E^* = p_*(K_{X/M} \otimes \mathcal{O}_X(r+1) \otimes p^*\det E)$: Berndtsson’s curvature formula, in which the horizontal block reappears as geodesic curvature, gives a Hermitian–Einstein metric and hence polystability (Section 13);
- L3** the pull-back bundle (p^*E, H_F) : adiabatic slope stability on $\mathbb{P}(E)$ and Wong’s Hermitian–Einstein equation for the Hessian metric. Here the nonlinear fiberwise data enter at leading order, and the level acts as a rigidity test for the Finsler structure (Section 14).

In short: L1 detects semistability and $\mu_{\max}$, L2 detects polystability, and L3 detects pull-back stability and shows that the Hessian constraint is rigid. Hermitian–Einstein metrics on E give approximate solutions at all three levels.

Known results, presented here with complete or streamlined proofs.

- The projective Finsler geometry of $\mathbb{P}(E)$: the Kähler forms $c_1(\mathcal{O}_{\mathbb{P}(E)}(1)) + \lambda p^*\omega$, the horizontal–vertical decomposition of $c_1(\mathcal{O}_{\mathbb{P}(E)}(1))$, the Hessian metric on the vertical bundle and its curvature, and the mixed holomorphic bisectional curvature [18, 6]; the adiabatic Kähler metrics $p^*\omega + \varepsilon \sqrt{-1}\partial\bar{\partial} \log G$ on $\mathbb{P}(E)$ and their scalar and Ricci curvature, for Hermitian metrics [31] and for Finsler metrics [29].
- Finsler–Einstein $\Rightarrow$ semistable [12], and Finsler–Einstein $\Leftrightarrow$ polystable via Berndtsson’s direct-image metric [13] (Theorem B).
- The identification of Kobayashi’s equation with the geodesic–Einstein equation for the metric on the hyperplane bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$, and nonlinear stability for subfibrations [13].
- Finski’s fibered Einstein viewpoint, his remark that the equation resembles the J -equation over curves, and his sup-norm Harder–Narasimhan bounds [14].
- Stability of pull-backs for adiabatic classes [26]; approximate critical Hermitian structures [9, 25, 16].
- Wong’s total-space Einstein equation [30].

New in this paper.

- (1) The exact adiabatic expansion of the degenerate total-space J -equation in arbitrary base dimension, with Kobayashi’s equation as its first non-universal coefficient (Proposition 10.3, Remark 10.4), and the exact Segre expansion behind it (Lemma 10.1).
- (2) The reduction of the J -subvariety test on $\mathbb{P}(\mathcal{S})$ to slope stability, for all saturated subsheaves (Proposition 11.6).
- (3) Theorem A: a fiber-integrated Finsler Chern–Weil inequality for all saturated subsheaves, with an explicit rank-independent constant, proved by Segre forms, a Gauss–Codazzi inequality for the Kobayashi curvature and a resolution of the subsheaf; and the approximate correspondence “semistable $\Leftrightarrow$ approximately Finsler–Einstein” (Corollary 11.3).
- (4) The integrated characterization of $\mu_{\max}(E)$ and of the vertices of the Harder–Narasimhan polygon through a Finsler Ky Fan functional (Theorem 12.4(i)–(iii)); the sup-norm version (iv) follows from [14].
- (5) At Level 3: the full lexicographic Segre expansion of adiabatic slopes, the location of the Finsler data in Wong’s residual (Proposition 14.2), and the fixed-metric adiabatic rigidity theorem (Proposition 14.3).
- (6) Non-Hermitian Finsler–Einstein metrics, both on the trivial bundle (Example 13.4) and, in infinite-dimensional families, on *stable* bundles with irreducible finite holonomy (Proposition 13.5), so that Finsler–Einstein metrics are not unique even on stable bundles; and the fact that a non-split extension $0 \rightarrow L \rightarrow E \rightarrow L \rightarrow 0$ admits approximate but no exact Finsler–Einstein metrics (Corollary 13.2, a consequence of [13]).

The foundational Sections 3–8 are expository, with complete proofs and explicit normalizations; in particular they fix the Einstein constant, the characterization of the vanishing of the vertical Ricci form, the equivalence of strong pseudoconvexity and fiberwise negativity, and the scalar nature of Kobayashi’s condition. Section 15 lists natural questions that remain open.

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Conventions. $X = \mathbb{P}(E)$ is the space of *lines* in E , $p : X \rightarrow M$ the projection, $m = r - 1$, $N = \dim X = n + m$. $E^\circ = E \setminus 0_E$ denotes the punctured total space (we reserve E^* for the dual bundle), and $q : E^\circ \rightarrow X$ the quotient map. For a real $(1, 1)$ -form $\alpha = \sqrt{-1}a_{i\bar{j}}dz^i \wedge d\bar{z}^j$

we write $\alpha(t) = a_{i\bar{j}} t^i \bar{t}^j$ for $t \in T^{1,0}$, and $\text{tr}_\omega \alpha = \Lambda_\omega \alpha = g^{i\bar{j}} a_{i\bar{j}}$ if $\omega = \sqrt{-1} g_{i\bar{j}} dz^i \wedge d\bar{z}^j$, so that

$$(1.2) \quad \alpha \wedge \frac{\omega^{n-1}}{(n-1)!} = (\text{tr}_\omega \alpha) \frac{\omega^n}{n!}, \quad dV_\omega := \frac{\omega^n}{n!}, \quad \text{Vol}(M) := \int_M dV_\omega.$$

Chern forms are $c_1(E, h) = \frac{\sqrt{-1}}{2\pi} \text{tr } \Theta_h$.

2. STABILITY, HERMITIAN–EINSTEIN METRICS, AND KNOWN FINSLER RESULTS

For a torsion-free coherent sheaf $\mathcal{S}$ on M ,

$$\deg_\omega \mathcal{S} = \int_M c_1(\det \mathcal{S}) \wedge \frac{\omega^{n-1}}{(n-1)!}, \quad \mu_\omega(\mathcal{S}) = \frac{\deg_\omega \mathcal{S}}{\text{rk } \mathcal{S}}.$$

E is stable (resp. semistable) if $\mu_\omega(\mathcal{S}) < \mu_\omega(E)$ (resp. $\leq$) for every coherent subsheaf with $0 < \text{rk } \mathcal{S} < \text{rk } E$; it is polystable if it is a direct sum of stable bundles of equal slope. It suffices to test saturated subsheaves, since saturation preserves the rank and does not decrease the degree. A saturated subsheaf of E is reflexive and is a subbundle off an analytic set of codimension ≥ 2 .

Integrating the trace of (1.1) and using (1.2) gives the Einstein constant

$$(2.1) \quad \lambda = \frac{2\pi \mu_\omega(E)}{\text{Vol}(M)} = \frac{2\pi n! \mu_\omega(E)}{\int_M \omega^n}.$$

Theorem 2.1 (Kobayashi–Hitchin correspondence). *(i) E admits a Hermitian–Einstein metric if and only if E is polystable [19, 22, 10, 28].*

(ii) For every $\varepsilon > 0$, E admits a Hermitian metric h_ε with $\|K_{h_\varepsilon} - \lambda \text{Id}\|_{L^\infty} < \varepsilon$ if and only if E is semistable [20, 17].

Definition 2.2 (Kobayashi [21]). A strongly pseudoconvex Finsler metric F on E is *Finsler–Einstein* if its Kobayashi curvature (Section 5) satisfies

$$(2.2) \quad k_F := \text{tr}_\omega \Psi_F \equiv \lambda \quad \text{on } \mathbb{P}(E).$$

It is *approximately Finsler–Einstein* if for every $\varepsilon > 0$ there is F_ε with $\|k_{F_\varepsilon} - \lambda\|_{L^\infty(\mathbb{P}(E))} < \varepsilon$.

The constant in (2.2) is necessarily (2.1); see (11.2) below. Aikou [1] introduced a stronger, endomorphism-valued condition (the horizontal mean curvature of the Hessian metric equals λId); contracting it with the radial vector gives (2.2). Since Hermitian–Einstein metrics satisfy Aikou’s condition, every existence statement below holds for both notions, and every obstruction proved for Kobayashi’s notion holds a fortiori for Aikou’s.

Theorem 2.3 (Feng–Liu–Wan). *(i) [12] If E admits a Finsler–Einstein metric, then E admits ω -approximate Hermitian–Einstein structures; in particular E is semistable.*

(ii) [13] E admits a Finsler–Einstein metric if and only if E admits a Hermitian–Einstein metric, i.e. if and only if E is polystable.

The conditional result [11, §3] assumed in addition that the exterior powers $\Lambda^k E$ carry compatible Finsler–Einstein metrics; this hypothesis was removed in [12].

3. COMPLEX FINSLER METRICS

Let $z = (z^1, \dots, z^n)$ be local coordinates on M and $\zeta = (\zeta^1, \dots, \zeta^r)$ fiber coordinates with respect to a local holomorphic frame of E .

Definition 3.1. A complex Finsler metric on E is a continuous $F : E \rightarrow [0, \infty)$ with $F(z, \zeta) = 0$ iff $\zeta = 0$ and $F(z, t\zeta) = |t|F(z, \zeta)$ for $t \in \mathbb{C}$. It is smooth if $G = F^2$ is smooth on E° , and strongly pseudoconvex if the matrix $G_{\alpha\bar{\beta}} = \partial^2 G / \partial \zeta^\alpha \partial \bar{\zeta}^\beta$ is positive definite on E° .

Subscripts on G denote derivatives in the fiber variables; repeated indices are summed.

Lemma 3.2 (Euler identities). *For a smooth Finsler metric,*

$$(3.1) \quad G(z, t\zeta) = |t|^2 G(z, \zeta), \quad G_\alpha \zeta^\alpha = G = G_{\bar{\beta}} \bar{\zeta}^\beta,$$

$$(3.2) \quad G_{\alpha\bar{\beta}} \zeta^\alpha = G_{\bar{\beta}}, \quad G_{\alpha\bar{\beta}} \bar{\zeta}^\beta = G_\alpha, \quad G_{\alpha\bar{\beta}} \zeta^\alpha \bar{\zeta}^\beta = G,$$

$$(3.3) \quad G_{\alpha\bar{\beta}}(z, t\zeta) = G_{\alpha\bar{\beta}}(z, \zeta), \quad G_{\alpha\bar{\beta}\gamma} \zeta^\gamma = 0 = G_{\alpha\bar{\beta}\gamma} \zeta^\alpha.$$

Proof. Treat $t, \bar{t}$ as independent in $G(z, t\zeta) = t\bar{t}G(z, \zeta)$ and differentiate in t at $t = 1$: $G_\alpha \zeta^\alpha = G$; the second identity in (3.1) is the conjugate. Differentiating $G_\alpha \zeta^\alpha = G$ in $\bar{\zeta}^\beta$ gives the first identity in (3.2), and contracting once more gives the third. Differentiating $G(z, t\zeta) = |t|^2 G(z, \zeta)$ twice in ζ shows that $G_{\alpha\bar{\beta}}$ is homogeneous of bidegree $(0, 0)$, whence $G_{\alpha\bar{\beta}\gamma} \zeta^\gamma = 0$ by Euler. Finally $G_{\alpha\bar{\beta}\gamma} \zeta^\alpha = \partial_\gamma(G_{\alpha\bar{\beta}} \zeta^\alpha) - G_{\gamma\bar{\beta}} = G_{\bar{\beta}\gamma} - G_{\gamma\bar{\beta}} = 0$. $\square$

Put $u = \log G$, so that

$$(3.4) \quad u_{\alpha\bar{\beta}} = \frac{G_{\alpha\bar{\beta}}}{G} - \frac{G_\alpha G_{\bar{\beta}}}{G^2}, \quad G_{\alpha\bar{\beta}} = G(u_{\alpha\bar{\beta}} + u_\alpha u_{\bar{\beta}}).$$

Proposition 3.3. *Let F be a smooth Finsler metric. Then F is strongly pseudoconvex if and only if, at every point of E° , the Hermitian form $(u_{\alpha\bar{\beta}})$ is positive semidefinite with kernel exactly $\mathbb{C}\zeta$. In that case $\det(u_{\alpha\bar{\beta}}) = 0$ and $\text{rk}(u_{\alpha\bar{\beta}}) = r - 1$.*

Proof. By (3.1)–(3.2), $u_{\alpha\bar{\beta}} \zeta^\alpha = 0$ always. If $(G_{\alpha\bar{\beta}})$ is positive, Cauchy–Schwarz for it gives, for $v \in \mathbb{C}^r$,

$$u(v, \bar{v}) = \frac{G(v, \bar{v})}{G} - \frac{|G(v, \bar{\zeta})|^2}{G^2} \geq 0,$$

with equality iff $v \in \mathbb{C}\zeta$, since $G(\zeta, \bar{\zeta}) = G$. Conversely, assume $u \geq 0$ with kernel $\mathbb{C}\zeta$. By (3.4), $G_{\alpha\bar{\beta}} v^\alpha \bar{v}^\beta = G(u(v, \bar{v}) + |u_\alpha v^\alpha|^2) \geq 0$, and equality forces $v = c\zeta$ and $0 = |c|^2 |u_\alpha \zeta^\alpha|^2 = |c|^2$, because $u_\alpha \zeta^\alpha = 1$. Hence $(G_{\alpha\bar{\beta}}) > 0$. $\square$

4. PROJECTIVIZATION AND TAUTOLOGICAL BUNDLES

The tautological line bundle is $\mathcal{O}_X(-1) = \{([v], u) \in p^*E : u \in \mathbb{C}v\}$ and $\mathcal{O}_X(1)$ is its dual; the map $\mathcal{O}_X(-1) \rightarrow E$, $([v], u) \mapsto u$, is the blow-up of E along 0_E and restricts to a biholomorphism from the complement of the zero section of $\mathcal{O}_X(-1)$ onto E° .

Proposition 4.1 (Kobayashi correspondence [18]; see also [6, §5, Lemma 5.1]). *Smooth Finsler metrics on E correspond bijectively to smooth Hermitian metrics on $\mathcal{O}_X(-1)$, via $|([v], u)|^2 = G(u)$. Under this correspondence*

$$(4.1) \quad c_1(\mathcal{O}_X(1), G^{-1}) = \xi_F := \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log G,$$

and F is strongly pseudoconvex if and only if ξ_F is positive definite on the tangent space of every fiber $\mathbb{P}(E_z)$.

Proof. Homogeneity makes G a Hermitian norm on the fibers of $\mathcal{O}_X(-1)$, and conversely. On E° the tautological section ζ is a holomorphic frame of $q^*\mathcal{O}_X(-1)$ with $|\zeta|^2 = G$, so $c_1(\mathcal{O}_X(-1)) = -\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log G$, which gives (4.1). The vertical part of $\sqrt{-1} \partial \bar{\partial} \log G$ is $\sqrt{-1} u_{\alpha\bar{\beta}} d\zeta^\alpha \wedge d\bar{\zeta}^\beta$, and the tangent space of $\mathbb{P}(E_z)$ at $[\zeta]$ is $\mathbb{C}^r / \mathbb{C}\zeta$; the last assertion is therefore Proposition 3.3. $\square$

5. THE RELATIVE TANGENT BUNDLE, THE TWO INDUCED METRICS, AND THE HORIZONTAL SPLITTING

5.1. Two Hermitian bundles. The relative Euler sequence is

$$(5.1) \quad 0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(1) \otimes p^*E \rightarrow T_{X/M} \rightarrow 0, \quad T_{X/M} \cong \text{Hom}(\mathcal{O}_X(-1), p^*E/\mathcal{O}_X(-1)),$$

so that $\det T_{X/M} = \mathcal{O}_X(r) \otimes p^* \det E$, i.e.

$$(5.2) \quad K_{X/M} = \mathcal{O}_X(-r) \otimes p^* \det E^*,$$

and for $r = 2$, $T_{X/M} \cong \mathcal{O}_X(2) \otimes p^* \det E$. See [6, (3.10)–(3.11), Thm. 3.5], where $K_{\mathbb{P}(E)} = p^*(K_M \otimes \det E^*) \otimes \mathcal{O}_X(-r)$ is derived in this form.

By (3.3) the matrix $(G_{\alpha\bar{\beta}})$ defines a Hermitian metric H_F on the rank- r bundle p^*E ; on E° this is the Hermitian metric on the vertical bundle $\ker d\pi \cong \pi^*E$ considered in [6, (5.15)–(5.18)], where its Chern connection and the horizontal, vertical and mixed components of its curvature are computed. By Proposition 3.3, $\sqrt{-1}\partial\bar{\partial}\log G$ restricted to fibers defines a Hermitian metric g_F^{vert} on the rank- $(r-1)$ bundle $T_{X/M}$. These must not be confused: $G_{\alpha\bar{\beta}}\zeta^\alpha\bar{\zeta}^\beta = G > 0$, so the Hessian is positive on the radial direction and does not descend to $T_{X/M}$.

Proposition 5.1 ([31], [6, §3], [29]). *If (M, ω) is compact Kähler and F is strongly pseudoconvex, then $\Omega_C = \xi_F + Cp^*\omega$ is a Kähler form on X for all sufficiently large $C > 0$.*

Proof. ξ_F is closed and positive on $T_{X/M}$. Fix a smooth splitting $TX = T_{X/M} \oplus H$. On the compact unit sphere bundle of TX the horizontal and mixed components of ξ_F are bounded, while $p^*\omega$ is uniformly positive on H and vanishes on $T_{X/M}$; completing the square shows that $\Omega_C > 0$ once C is large. (With the splitting of §5.2 there are no mixed terms, and the condition is simply $C\omega > \Psi_F/2\pi$ pointwise.) $\square$

For Hermitian metrics the Kähler forms $p^*\omega + \varepsilon\sqrt{-1}\partial\bar{\partial}\log G$, ε small, go back to Yau [31], who showed that they have positive scalar curvature. Cao and Wong [6, pp. 385–386] showed that $c_1(\mathcal{O}_{\mathbb{P}(E)}(1)) + \lambda p^*\omega$ is Kähler as soon as λ exceeds the supremum of the mixed holomorphic bisectional curvature; the Finsler case also follows from [6, Thm. 5.4]. For Finsler F , Wang and Zhong [29] consider $\Phi = p^*\omega + \varepsilon\sqrt{-1}\partial\bar{\partial}\log G$, which in our notation is $\Phi = 2\pi\varepsilon\Omega_{F,C}$ with $C = 1/(2\pi\varepsilon)$. They show that, for ε small, Φ has positive scalar curvature if and only if the fiber metrics $\omega_V/2\pi$ do, give a sufficient condition for positive Ricci curvature, and show that Φ is Kähler–Einstein when (M, ω) and the fibers are Kähler–Einstein with the same constant and F is locally independent of z [29, Thms. 1.5–1.7]. Their analysis uses the coordinate splitting of $\sqrt{-1}\partial\bar{\partial}\log G$ into horizontal, mixed and vertical parts; the block-diagonal form with respect to the Chern–Finsler splitting (Proposition 5.3) and the curvature questions of this paper are not considered there.

5.2. The Chern–Finsler splitting and the Kobayashi curvature. Write $\partial_i = \partial/\partial z^i$ and let $(G^{\bar{\beta}\alpha})$ be the inverse of $(G_{\alpha\bar{\beta}})$. The Chern–Finsler nonlinear connection is

$$(5.3) \quad N_i^\alpha = G^{\bar{\beta}\alpha} \partial_i G_{\bar{\beta}}, \quad \delta_i = \partial_i - N_i^\alpha \frac{\partial}{\partial \zeta^\alpha}, \quad \delta \zeta^\alpha = d\zeta^\alpha + N_i^\alpha dz^i.$$

Since $N_i^\alpha(z, t\zeta) = tN_i^\alpha(z, \zeta)$, the fields δ_i are $\mathbb{C}^*$ -invariant and span a horizontal subbundle $\mathcal{H}_F \subset T^{1,0}E^\circ$ which descends to X and is complementary to $T_{X/M}$.

Lemma 5.2. $\delta_i G = 0$ and $\delta_i G_{\bar{\beta}} = 0$.

Proof. $\delta_i G_{\bar{\beta}} = \partial_i G_{\bar{\beta}} - N_i^\alpha G_{\alpha\bar{\beta}} = 0$ by (5.3). By (3.2), $G^{\bar{\beta}\alpha} G_\alpha = \bar{\zeta}^\beta$, so $N_i^\alpha G_\alpha = \bar{\zeta}^\beta \partial_i G_{\bar{\beta}} = \partial_i(\bar{\zeta}^\beta G_{\bar{\beta}}) = \partial_i G$. $\square$

Proposition 5.3 (Horizontal–vertical decomposition; cf. [6, (3.7), Thm. 5.4]). *In the coframe $\{dz^i, \delta\zeta^\alpha\}$,*

$$(5.4) \quad 2\pi \xi_F = \sqrt{-1}\partial\bar{\partial}\log G = \omega_V - \Psi_F, \quad \omega_V = \sqrt{-1} u_{\alpha\bar{\beta}} \delta\zeta^\alpha \wedge \delta\bar{\zeta}^\beta, \quad \Psi_F = \sqrt{-1} \Psi_{i\bar{j}} dz^i \wedge d\bar{z}^j,$$

with no mixed terms, and

$$(5.5) \quad \Psi_{i\bar{j}} = \frac{\langle \Theta(\delta_i, \bar{\delta}_j) \zeta, \zeta \rangle_{H_F}}{G},$$

*where Θ is the curvature of the Chern connection of (p^*E, H_F) , pulled back to E° . Ψ_F is the Kobayashi curvature; $\omega_V \geq 0$ with kernel $\mathcal{H}_F \oplus \mathbb{C}\zeta$.*

Proof. Let $f = \log G$. Since $\partial\bar{\partial}f = d\bar{\partial}f$, for $(1,0)$ -fields X and $(0,1)$ -fields $\bar{Y}$ we have $\partial\bar{\partial}f(X, \bar{Y}) = X(\bar{Y}f) - \bar{Y}(\partial f(X)) - \bar{\partial}f([X, \bar{Y}]) = X(\bar{Y}f) - [X, \bar{Y}]^{0,1}f$. Now $[\delta_i, \partial/\partial\bar{\zeta}^\beta] = (\partial_{\bar{\zeta}^\beta} N_i^\alpha) \partial_{\zeta^\alpha}$ has type $(1,0)$, so the mixed component is $\delta_i(G_{\bar{\beta}}/G) = 0$ by Lemma 5.2. The vertical component is $u_{\alpha\bar{\beta}}$, and the horizontal component defines Ψ_F by (5.4).

For (5.5): the Chern connection of H_F has connection matrix $\theta_\alpha^\gamma = \partial G_{\alpha\bar{\beta}} G^{\bar{\beta}\gamma}$. By (3.3), $\theta(\delta_i)_\alpha^\gamma \zeta^\alpha = \partial_i G_{\bar{\beta}} G^{\bar{\beta}\gamma} = N_i^\gamma$, while $\delta_i \zeta^\gamma = -N_i^\gamma$. Hence the tautological section satisfies $\nabla_{\delta_i} \zeta = 0$, and $\nabla^{0,1} \zeta = \bar{\partial} \zeta = 0$. By Gauss–Codazzi (Theorem 7.1) for $q^* \mathcal{O}_X(-1) \subset p^* E$, the second fundamental form vanishes on horizontal vectors, so the curvature $-\partial\bar{\partial} \log G$ of $(\mathcal{O}_X(-1), G)$ evaluated on $(\delta_i, \bar{\delta}_j)$ equals $\langle \Theta(\delta_i, \bar{\delta}_j) \zeta, \zeta \rangle / G$. Comparing with (5.4) gives (5.5). The statement on ω_V is Proposition 3.3. $\square$

Cao and Wong [6, §5] obtain (5.4) by a computation in a frame normal at a point; the proof above is frame-free and also identifies the vertical term with the fiber-dependent form ω_V . Their horizontal bundle, the kernel of $X \mapsto \nabla_X P$ for the Chern connection of H_F and the position field P , coincides with $\mathcal{H}_F$, since $\theta(\delta_i) \zeta = N_i$ by the proof above. In their terminology,

$$k(t, \zeta) = \frac{\Psi_F(t, \bar{t})}{|t|_\omega^2}, \quad t \in T_z^{1,0} M,$$

is the *mixed holomorphic bisectional curvature* of F [6, (2.17), §5]. Hence $k_F = \text{tr}_\omega \Psi_F = \sum_i k(e_i, \zeta)$ for an ω -orthonormal basis (e_i) , and Kobayashi’s Finsler–Einstein condition says that this mixed Ricci-type trace is constant on $\mathbb{P}(E)$. Cao and Wong characterize ampleness of E^* (resp. E) by the existence of Finsler metrics with negative (resp. positive) mixed holomorphic bisectional curvature [6, Thms. 5.5, 5.7]; the present paper concerns the trace of the same curvature and slope stability.

If F is the norm of a Hermitian metric h , then $N = \theta_h \zeta$, $\mathcal{H}_F$ is the Chern horizontal distribution, and $\Psi_F = \sqrt{-1} \langle \Theta_h \zeta, \zeta \rangle / |\zeta|^2$; thus $k_F = \langle K_h \zeta, \zeta \rangle / |\zeta|_h^2$, and (2.2) is equivalent to (1.1) (polarize).

The restriction of ξ_F to $\mathbb{P}(E_z)$ is $\omega_V/2\pi$, a Kähler form in $c_1(\mathcal{O}(1))$, so

$$(5.6) \quad d\nu_{F,z} := \left(\frac{\omega_V}{2\pi} \right)^{r-1} \Big|_{\mathbb{P}(E_z)}$$

is a probability measure on $\mathbb{P}(E_z)$. For Hermitian F it is the Fubini–Study measure.

Remark 5.4. For Ω_C of Proposition 5.1, (5.4) shows that $\Omega_C > 0$ iff $2\pi C \omega > \Psi_F$ at every point of X . The positivity condition is open, so “the smallest C ” need not exist; one simply takes $C \gg 0$.

6. THE HESSIAN METRIC ON p^*E

Lemma 6.1 (cf. [2], [6, Lemma 5.2]). *For a strongly pseudoconvex Finsler metric the following are equivalent:*

- (i) *the vertical connection coefficients $G^{\bar{\beta}\gamma} \partial_{\zeta^\mu} G_{\alpha\bar{\beta}}$ vanish;*
- (ii) *$G_{\alpha\bar{\beta}}$ is independent of ζ ;*
- (iii) *F is the norm of a Hermitian metric on E .*

Proof. (i) $\Leftrightarrow$ (ii): (i) says $\partial_{\zeta^\mu} G_{\alpha\bar{\beta}} = 0$; conjugating and using that $G_{\alpha\bar{\beta}}$ is Hermitian gives $\partial_{\bar{\zeta}^\mu} G_{\alpha\bar{\beta}} = 0$. (ii) $\Rightarrow$ (iii): put $H_{\alpha\bar{\beta}}(z) = G_{\alpha\bar{\beta}}$ and $D = G - H_{\alpha\bar{\beta}} \zeta^\alpha \bar{\zeta}^\beta$. Then D is pluriharmonic on $\mathbb{C}^r \setminus \{0\}$, which is simply connected for $r \geq 2$, so $D = \text{Re } f$ with f holomorphic; by Hartogs, f extends to $\mathbb{C}^r$. Expanding $f = \sum_k f_k$ into homogeneous parts and using $D(e^{i\vartheta} \zeta) = D(\zeta)$, averaging in ϑ gives $D = \text{Re } f_0$, a constant c ; then $D(t\zeta) = |t|^2 D(\zeta)$ forces $c = 0$. (iii) $\Rightarrow$ (i) is immediate. $\square$

Proposition 6.2 (Vertical Ricci form). *The vertical part of $c_1(p^*E, H_F)$ is*

$$-\frac{\sqrt{-1}}{2\pi} \partial_{\zeta} \bar{\partial}_{\bar{\zeta}} \log \det(G_{\alpha\bar{\beta}}).$$

It vanishes identically if and only if the function $\det(G_{\alpha\bar{\beta}})(z, \cdot)$ is constant on each fiber $\mathbb{P}(E_z)$; in particular it vanishes for Hermitian F . Its integral over any curve in a fiber is zero.

Proof. $\text{tr } \Theta = \bar{\partial} \partial \log \det G_F$, which gives the formula. By (3.3), $\log \det G_F$ is a function on the compact manifold $\mathbb{P}(E_z)$. A pluriharmonic function on a compact connected complex manifold is constant, which gives the characterization. The last assertion holds because the restriction to a fiber is dd^c -exact. $\square$

7. GAUSS–CODAZZI: HERMITIAN AND FINSLER

Theorem 7.1 (Gauss–Codazzi). *Let (W, h) be a Hermitian holomorphic bundle, $S \subset W$ a holomorphic subbundle, $Q = W/S \cong S^\perp$ (smoothly and isometrically), and $A = \pi_{S^\perp} \nabla|_S \in A^{1,0}(\text{Hom}(S, Q))$. Then*

$$\nabla^W = \begin{pmatrix} \nabla^S & -A^* \\ A & \nabla^Q \end{pmatrix}, \quad \Theta_W = \begin{pmatrix} \Theta_S - A^* \wedge A & -DA^* \\ DA & \Theta_Q - A \wedge A^* \end{pmatrix},$$

$S^\perp$ is holomorphic iff $A = 0$, and $\pi_S K_W \pi_S = K_S + \sum_i A(e_i)^* A(e_i) \geq K_S$ for an ω -orthonormal basis (e_i) .

This is classical [20, I.6]. The Finsler analogue is an *inequality* rather than an identity, because a subbundle of a Finsler bundle has no canonical complement.

Proposition 7.2 (Gauss–Codazzi inequality for the Kobayashi curvature). *Let F be a strongly pseudoconvex Finsler metric on E over a complex manifold B and $S \subset E$ a holomorphic subbundle. Then $F_S := F|_S$ is strongly pseudoconvex, $\mathbb{P}(S) \subset \mathbb{P}(E)$, and for $[v] \in \mathbb{P}(S_z)$ and $t \in T_z^{1,0} B$,*

$$(7.1) \quad \Psi_{F_S}(t) = \Psi_F(t) - \omega_V(w, \bar{w}) \leq \Psi_F(t), \quad w = \tilde{t}^S - \tilde{t}^E,$$

where $\tilde{t}^S, \tilde{t}^E$ are the horizontal lifts of t at v for F_S and F . Equality holds iff $\tilde{t}^S - \tilde{t}^E \in \mathbb{C}\zeta$, i.e. iff $\mathcal{H}_F$ is tangent to $\mathbb{P}(S)$ at $[v]$ in the direction t .

Proof. A positive definite form restricted to a subspace is positive definite, so F_S is strongly pseudoconvex. The restriction of $\mathcal{O}_{\mathbb{P}(E)}(-1)$ to $\mathbb{P}(S)$ is $\mathcal{O}_{\mathbb{P}(S)}(-1)$ with the restricted metric, hence $\xi_{F_S} = \xi_F|_{\mathbb{P}(S)}$. Both lifts project to t , so w is vertical for E . Using (5.4) for F (no mixed terms) and for F_S ,

$$-\Psi_{F_S}(t) = 2\pi\xi_{F_S}(\tilde{t}^S, \overline{\tilde{t}^S}) = 2\pi\xi_F(\tilde{t}^E + w, \overline{\tilde{t}^E + w}) = -\Psi_F(t) + \omega_V(w, \bar{w}).$$

Since $\omega_V \geq 0$ with kernel $\mathbb{C}\zeta$ on vertical vectors, the claims follow. $\square$

This is the inequality behind [13, Lemma 2.4 and Thm. 2.2]; the formulation (7.1) makes the “second fundamental form” w explicit.

8. KÄHLERNESS, PROJECTIVITY, DEGREE AND SLOPE ON $\mathbb{P}(E)$

Proposition 8.1. *Let M be compact. (i) M is Kähler iff $\mathbb{P}(E)$ is Kähler. (ii) M is projective iff $\mathbb{P}(E)$ is projective.*

Proof. (i) If M is Kähler, apply Proposition 5.1 to any Hermitian metric. Conversely, if Ω is Kähler on X , then $\eta = p_*(\Omega^r)$ is a closed real $(1, 1)$ -form, and for $0 \neq t \in T_z M$, $\eta(t) > 0$ because Ω^r restricted to the r -dimensional subspaces $T_x \mathbb{P}(E_z) \oplus \mathbb{C}\tilde{t}$ is a volume form. (ii) If M is projective with ample A , then $\mathcal{O}_X(1) \otimes p^* A^k$ is ample for $k \gg 0$. Conversely, if X is projective, take a Kähler form $\Omega \in c_1(\mathcal{A})$ for an ample $\mathcal{A}$. Then $\eta = p_*(\Omega^r)$ is a Kähler form whose class $p_*(c_1(\mathcal{A})^r)$ is integral, so M is projective by Kodaira’s embedding theorem. $\square$

On (X, Ω) of dimension N , degrees use $\Omega^{N-1}/(N-1)!$ with $N-1 = n+r-2$, and the Einstein constant of a bundle $W \rightarrow X$ is $2\pi\mu_\Omega(W)/\text{Vol}(X, \Omega)$ with $\text{Vol}(X, \Omega) = \int_X \Omega^N/N!$.

9. EINSTEIN METRICS AND STABILITY: THE HERMITIAN CASE

Let $(W, h) \rightarrow (X, \Omega)$ be Hermitian holomorphic over a compact Kähler manifold.

Theorem 9.1 ([19, 22]). *Let $\mathcal{S} \subset W$ be saturated, a subbundle S off Z , $\text{codim } Z \geq 2$, with second fundamental form A . Then*

$$(9.1) \quad \deg_{\Omega} \mathcal{S} = \frac{1}{2\pi} \int_{X \setminus Z} \left(\text{tr}(\pi_S K_h \pi_S) - |A|^2 \right) dV_{\Omega}.$$

Consequently, if $K_h = \lambda \text{Id}$ then $\mu_{\Omega}(\mathcal{S}) \leq \mu_{\Omega}(W)$; equality forces $A = 0$ and a holomorphic orthogonal splitting $W = S \oplus S^{\perp}$ extending over Z , and W is polystable. If $\|K_h - \lambda \text{Id}\|_{L^{\infty}} \leq \varepsilon$, then $\mu_{\Omega}(\mathcal{S}) \leq \mu_{\Omega}(W) + \varepsilon \text{Vol}(X, \Omega)/2\pi$.

The identity (9.1) for subsheaves (rather than subbundles) requires the standard justification via the L_1^2 projection π_S , see [20, Ch. V]. The constant in front of $|A|^2$ is exactly $1/2\pi$ with $dV_{\Omega} = \Omega^N/N!$. For the splitting, the holomorphic projection onto S is bounded on $X \setminus Z$ and extends by Hartogs to an idempotent holomorphic endomorphism of W . Its image is then a subbundle, and both summands are Hermitian–Einstein, so polystability follows by induction on the rank. Note that “polystable” cannot be replaced by “semistable” in Theorem 2.1(i): a non-split extension of equal-slope stable bundles is semistable but has no Hermitian–Einstein metric.

10. THE TOTAL-SPACE VIEWPOINT: THREE EINSTEIN CONDITIONS ON $\mathbb{P}(E)$

Fix a strongly pseudoconvex F and write $\Omega = \Omega_{F,C} = \xi_F + Cp^*\omega$, a Kähler form for $C \gg 0$ (Proposition 5.1), and $\chi = p^*\omega \geq 0$. By (5.4), Ω is block diagonal in the splitting $T^{1,0}X = \mathcal{H}_F \oplus T_{X/M}$:

$$(10.1) \quad \Omega|_{T_{X/M}} = \frac{\omega_V}{2\pi}, \quad \Omega|_{\mathcal{H}_F} = A_C := C\omega - \frac{\Psi_F}{2\pi}.$$

Every fiberwise positive metric on $\mathcal{O}_X(1)$ comes from a Finsler metric (Proposition 4.1). So as F varies, $\Omega_{F,C}$ runs over the Kähler forms in the fixed class $c_1(\mathcal{O}_X(1)) + C[p^*\omega]$ of the form $c_1(\mathcal{O}_X(1), \phi) + Cp^*\omega$.

Level	Object on X	Einstein condition	Stability content
L1	$(\mathcal{O}_X(1), G^{-1})$	$\text{tr}_{\Omega_{F,C}} \chi = c_C$ (J -type); first correction = Kobayashi	ω -semistability; $\mu_{\max}(E)$ (Thms. A, 12.4)
L2	$p_*(K_{X/M} \otimes L) = E^*$	L^2 metric Hermitian–Einstein	polystability (Thm. B)
L3	(p^*E, H_F)	Hermitian–Einstein w.r.t. $\Omega_{F,C}$ (Wong)	adiabatic slope stability [26]; rigidity (Prop. 14.3)

10.1. Segre identities. Since $X = \mathbb{P}(E)$ parametrizes lines, the Segre classes $s(E) = c(E)^{-1}$ satisfy [15, §3.1]

$$(10.2) \quad p_*(c_1(\mathcal{O}_X(1))^{m+j}) = s_j(E), \quad j \geq 0,$$

and in particular

$$(10.3) \quad p_*(c_1(\mathcal{O}_X(1))^{r-1}) = 1, \quad p_*(c_1(\mathcal{O}_X(1))^r) = s_1(E) = -c_1(E).$$

For (10.3), use the exact sequence $0 \rightarrow \mathcal{O}_X(-1) \rightarrow p^*E \rightarrow Q \rightarrow 0$ with $\text{rk } Q = r - 1$, which gives

$$\sum_{i=0}^r c_i(E) c_1(\mathcal{O}_X(1))^{r-i} = 0.$$

All adiabatic constants in this paper come from the following exact identity.

Lemma 10.1 (Adiabatic Segre expansion). *Fix the conventions above: $\mathbb{P}(V)$ is the space of lines of a holomorphic bundle V of rank s over a compact complex manifold B of dimension n , $h = c_1(\mathcal{O}_{\mathbb{P}(V)}(1))$, $s(V) = c(V)^{-1}$, so $s_1(V) = -c_1(V)$. Let $q = n + s - 1 = \dim \mathbb{P}(V)$, and let ω be any closed real $(1,1)$ -form on B (not necessarily positive). Then for every $C \in \mathbb{R}$,*

$$\begin{aligned} \int_{\mathbb{P}(V)} (h + Cp^*\omega)^q &= \sum_{j=0}^n \binom{q}{n-j} C^{n-j} \int_B s_j(V) \wedge \omega^{n-j}, \\ \int_{\mathbb{P}(V)} p^*\omega \wedge (h + Cp^*\omega)^{q-1} &= \sum_{j=0}^{n-1} \binom{q-1}{n-1-j} C^{n-1-j} \int_B s_j(V) \wedge \omega^{n-j}, \\ p_* \left(\frac{(h + Cp^*\omega)^{q-1}}{(q-1)!} \right) &= \sum_{j=0}^{n-1} \frac{C^{n-1-j}}{(s-1+j)!(n-1-j)!} s_j(V) \wedge \omega^{n-1-j}. \end{aligned}$$

In particular, writing $V_\omega = \int_B \omega^n$ and $d_V = \int_B c_1(V) \wedge \omega^{n-1}$,

$$\begin{aligned} \int_{\mathbb{P}(V)} (h + Cp^*\omega)^q &= \binom{q}{n} C^n V_\omega - \binom{q}{n-1} C^{n-1} d_V + O(C^{n-2}), \\ \int_{\mathbb{P}(V)} p^*\omega \wedge (h + Cp^*\omega)^{q-1} &= \binom{q-1}{n-1} C^{n-1} V_\omega - \binom{q-1}{n-2} C^{n-2} d_V + O(C^{n-3}). \end{aligned}$$

Proof. As usual, $\binom{a}{b} = 0$ for $b < 0$; with this convention all formulas also cover $n = 1$. Expand binomially. By the projection formula and (10.2) (for V), $p_*(h^a \wedge p^*\omega^b) = s_{a-s+1}(V) \wedge \omega^b$, which vanishes for $a < s - 1$. In $(h + Cp^*\omega)^q$, the term $\binom{q}{k} C^k h^{q-k} \omega^k$ contributes only if $q - k \geq s - 1$, i.e. $k \leq n$; put $j = n - k$. In $p^*\omega \wedge (h + Cp^*\omega)^{q-1}$ the term $\binom{q-1}{k} C^k h^{q-1-k} \omega^{k+1}$ contributes with $j = n - 1 - k$. The third identity is the same computation without integrating over B . The particular cases are $j = 0, 1$, using $s_0 = 1$ and $s_1 = -c_1$. $\square$

10.2. Level 1: the Einstein condition for $\mathcal{O}_X(1)$ is a J -type equation.

Lemma 10.2. $\Lambda_\Omega \xi_F = N - C \operatorname{tr}_\Omega \chi$, and $\operatorname{tr}_\Omega \chi = \operatorname{tr}_{A_C} \omega$. Hence $(\mathcal{O}_X(1), G^{-1})$ has constant mean curvature with respect to $\Omega_{F,C}$ if and only if

$$(10.4) \quad \operatorname{tr}_{\Omega_{F,C}}(p^*\omega) = c_C := \frac{N \int_X p^*\omega \wedge \Omega^{N-1}}{\int_X \Omega^N}.$$

Proof. $\xi_F = \Omega - C\chi$ and $\Lambda_\Omega \Omega = N$. Since Ω is block diagonal by (10.1) and χ vanishes on $T_{X/M}$, $\operatorname{tr}_\Omega \chi$ is the trace of ω with respect to the horizontal block A_C . The constant is fixed by integrating against Ω^N . $\square$

Equation (10.4) is Donaldson's J -equation [7, 27, 5] on X in the class $c_1(\mathcal{O}_X(1)) + C[p^*\omega]$, with the degenerate reference form $\chi = p^*\omega$, and with unknowns restricted to $\Omega = \xi_\phi + Cp^*\omega$.

Relation to the literature. Feng–Liu–Wan [13] identify Kobayashi's equation with the geodesic-Einstein equation for the relatively positive metric on $\mathcal{O}_{\mathbb{P}(E)}(1)$, and relate it to nonlinear stability with respect to subfibrations. Finski [14] calls such metrics fibered Einstein. He observes that the fibered Einstein equation resembles the J -equation when the base is one-dimensional, and compares it with the optimal symplectic connection equation of Dervan–Sektnan [8]. The observation used here is more specific: in arbitrary base dimension, Kobayashi's equation is exactly the first non-constant coefficient in the adiabatic expansion of the degenerate total-space J -equation (10.4) (Proposition 10.3). We do not know a previous statement of this expansion.

Proposition 10.3. *Uniformly on X , as $C \rightarrow \infty$,*

$$\operatorname{tr}_{\Omega_{F,C}}(p^*\omega) = \frac{n}{C} + \frac{k_F}{2\pi C^2} + O(C^{-3}), \quad c_C = \frac{n}{C} + \frac{\lambda}{2\pi C^2} + O(C^{-3}),$$

with $\lambda = 2\pi\mu_\omega(E)/\text{Vol}(M)$ as in (2.1). Hence

$$\text{tr}_{\Omega_{F,C}}(p^*\omega) - c_C = \frac{k_F - \lambda}{2\pi C^2} + O(C^{-3}),$$

Consequently $C^2(\text{tr}_{\Omega_{F,C}} p^*\omega - c_C) \rightarrow (k_F - \lambda)/2\pi$ uniformly on X . Hence F is Finsler-Einstein in Kobayashi's sense if and only if the residual in (10.4) is $O(C^{-3})$, that is, if and only if its coefficient of C^{-2} vanishes.

The $O(C^{-3})$ bounds are uniform on X for a fixed F ; the constants depend on bounds for Ψ_F . They are not uniform over families F_C varying with C unless their curvatures are uniformly bounded.

Remark 10.4 (The adiabatic regime, made explicit). (a) *The class.* The Kähler class is

$$[\Omega_{F,C}] = c_1(\mathcal{O}_X(1)) + C[p^*\omega], \quad C \rightarrow \infty.$$

Equivalently, with $\varepsilon = C^{-1}$, $\Omega_{F,C} = C(p^*\omega + \varepsilon \xi_F)$: the base class is scaled up and the fibers keep their Fubini-Study-type size. All expansions are in powers of $C^{-1} = \varepsilon$.

- (b) *The vertical term.* By (10.1), $\Lambda_\Omega \xi_F$ splits as the vertical term $\Lambda_{\omega_V/2\pi}(\omega_V/2\pi) = m$ plus the horizontal term $\text{tr}_{A_C}(-\Psi_F/2\pi)$. The vertical term is exactly m for every F and C .
(c) *First appearance of the Finsler curvature.* By Lemma 10.2 and Proposition 10.3,

$$\Lambda_\Omega \xi_F = m - \frac{k_F}{2\pi C} + O(C^{-2}), \quad \text{equivalently} \quad \text{tr}_\Omega(p^*\omega) = \frac{n}{C} + \frac{k_F}{2\pi C^2} + O(C^{-3}).$$

The Kobayashi curvature first appears at order C^{-1} in the mean curvature of $\mathcal{O}_X(1)$, i.e. at order C^{-2} in the J -residual. The leading terms (m , respectively n/C) are universal.

- (d) *Exact versus limiting equation.* For each finite C , (10.4) is an exact (degenerate) J -equation on X . Kobayashi's equation is not this equation: it is the vanishing of the first non-universal coefficient of its residual, i.e. the leading-order adiabatic limit.

Proof. $A_C = C(\omega - \Psi_F/2\pi C)$, and Ψ_F is bounded on the compact X . Hence $\text{tr}_{A_C} \omega = C^{-1}(n + \text{tr}_\omega \Psi_F/2\pi C + O(C^{-2}))$, by $(I - B)^{-1} = I + B + O(|B|^2)$. For c_C , write $V = \int_M \omega^n$ and $d_E = \int_M c_1(E) \wedge \omega^{n-1}$. By Lemma 10.1 with $(B, V) = (M, E)$ and $q = N$,

$$\begin{aligned} \int_X \Omega^N &= \binom{N}{n} C^n V - \binom{N}{n-1} C^{n-1} d_E + O(C^{n-2}), \\ \int_X \chi \wedge \Omega^{N-1} &= \binom{N-1}{n-1} C^{n-1} V - \binom{N-1}{n-2} C^{n-2} d_E + O(C^{n-3}). \end{aligned}$$

Using $N \binom{N-1}{n-1} = n \binom{N}{n}$, $N \binom{N-1}{n-2} = (n-1) \binom{N}{n-1}$ and $\binom{N}{n-1} / \binom{N}{n} = n/r$, and setting $x = d_E/CV$, we get

$$c_C = \frac{1}{C} \cdot \frac{n - (n-1)\frac{n}{r}x}{1 - \frac{n}{r}x} + O(C^{-3}) = \frac{1}{C} \left(n + \frac{n}{r}x \right) + O(C^{-3}).$$

Finally $\frac{n}{r} \frac{d_E}{V} = \frac{n(n-1)! \deg_\omega E}{r n! \text{Vol}(M)} = \frac{\mu_\omega(E)}{\text{Vol}(M)} = \frac{\lambda}{2\pi}$. $\square$

10.3. Level 2 and Level 3. Level 2 is the direct image of Section 13. The relevant line bundle is $L = \mathcal{O}_X(r+1) \otimes p^* \det E$, whose geodesic curvature (13.2) is the horizontal block of its curvature form in the splitting (10.1). Level 3 is Wong's total-space equation, treated in Section 14. In the rest of the paper we take the levels in order.

11. LEVEL 1, SHARPENED: A FINSLER CHERN-WEIL INEQUALITY AND SEMISTABILITY

11.1. Segre-form representation of c_1 .

Lemma 11.1. *For a strongly pseudoconvex F , the closed $(1,1)$ -form $-p_*(\xi_F^r)$ represents $c_1(E)$ and equals*

$$(11.1) \quad \gamma_F := \frac{r}{2\pi} \int_{\mathbb{P}(E_z)} \Psi_F d\nu_{F,z}, \quad \text{i.e.} \quad \gamma_F(t) = \frac{r}{2\pi} \int_{\mathbb{P}(E_z)} \Psi_F(t) d\nu_{F,z}.$$

In particular, taking tr_ω and integrating,

$$(11.2) \quad \deg_\omega E = \frac{r}{2\pi} \int_M \left(\int_{\mathbb{P}(E_z)} k_F d\nu_{F,z} \right) dV_\omega.$$

Proof. By (5.4), $(2\pi\xi_F)^r = \sum_k \binom{r}{k} \omega_V^{r-k} \wedge (-\Psi_F)^k$. The $\delta\zeta$ -part of ω_V has rank $r-1$, so $\omega_V^r = 0$; for $k \geq 2$ the vertical degree $2(r-k)$ is smaller than the fiber dimension $2(r-1)$, so these terms have zero fiber integral. Evaluating p_* on horizontal lifts, which are annihilated by $\delta\zeta$, gives $p_*(\xi_F^r) = -(2\pi)^{-r} r p_*(\omega_V^{r-1} \wedge \Psi_F) = -\gamma_F$. Fiber integration commutes with d , so γ_F is closed, and (10.3) shows it represents $c_1(E)$. Finally (11.2) follows from (1.2). $\square$

For Hermitian F , $\int \langle a\zeta, \zeta \rangle / |\zeta|^2 d\nu_{FS} = \text{tr } a / r$, and $\gamma_F = c_1(E, h)$. In general γ_F is the first Chern form of [11]. Together with (11.2), (2.2) forces $\lambda = 2\pi\mu_\omega(E) / \text{Vol}(M)$, as in (2.1).

11.2. The inequality.

Theorem A. *Let F be a strongly pseudoconvex Finsler metric on $E \rightarrow (M, \omega)$, M compact Kähler, and let $\mathcal{S} \subset E$ be a saturated subsheaf of rank s . Then*

$$(11.3) \quad \deg_\omega \mathcal{S} \leq \frac{s}{2\pi} \int_M \left(\max_{\mathbb{P}(E_z)} k_F \right) dV_\omega,$$

and therefore

$$(11.4) \quad \mu_\omega(\mathcal{S}) - \mu_\omega(E) \leq \frac{1}{2\pi} \int_M \left(\max_{\mathbb{P}(E_z)} k_F - \int_{\mathbb{P}(E_z)} k_F d\nu_{F,z} \right) dV_\omega \leq \frac{\text{Vol}(M)}{\pi} \inf_{c \in \mathbb{R}} \|k_F - c\|_{L^\infty(\mathbb{P}(E))}.$$

Proof. Step 1: resolution. Let $\mathcal{Q} = E/\mathcal{S}$. Since $\mathcal{S}$ is saturated, $\mathcal{Q}$ is torsion-free. Let $Z \subset M$ be the locus where $\mathcal{S}$ or $\mathcal{Q}$ fails to be locally free; then $\text{codim } Z \geq 2$, and over $M \setminus Z$ the inclusion $\mathcal{S} \hookrightarrow E$ is an inclusion of holomorphic vector bundles. It therefore defines a holomorphic section $z \mapsto \mathcal{S}_z \in \text{Gr}_s(E_z)$ of the Grassmann bundle over $M \setminus Z$, and the closure of its image is an irreducible analytic subvariety of the Grassmann bundle. Here and below $\det \mathcal{S} := (\Lambda^s \mathcal{S})^{**}$, a line bundle on M . Resolving its singularities (Hironaka) gives a compact complex manifold $\tilde{M}$ and a proper bimeromorphic holomorphic map $\pi : \tilde{M} \rightarrow M$, biholomorphic over $M \setminus Z$, together with a holomorphic subbundle $\tilde{\mathcal{S}} \subset \pi^* E$ of rank s that agrees with $\pi^* \mathcal{S}$ over $M \setminus Z$. Since $\det \mathcal{S} \hookrightarrow \Lambda^s E$ and $\det \tilde{\mathcal{S}} \subset \pi^* \Lambda^s E$ is a subbundle, the induced map $\pi^* \det \mathcal{S} \rightarrow \det \tilde{\mathcal{S}}$ is holomorphic and an isomorphism off $\pi^{-1}(Z)$. Hence $\det \tilde{\mathcal{S}} = \pi^* \det \mathcal{S} \otimes \mathcal{O}(D)$ with $D \geq 0$ supported in $\pi^{-1}(Z)$. Every component of D maps to Z , of dimension $\leq n-2$, so $\pi^* \omega^{n-1}$ vanishes on it, and

$$\deg_\omega \mathcal{S} = \int_{\tilde{M}} c_1(\tilde{\mathcal{S}}) \wedge \frac{(\pi^* \omega)^{n-1}}{(n-1)!}.$$

Step 2: pull back and restrict. Put $\tilde{F} = \pi^* F$. It is strongly pseudoconvex, because its fiber Hessian is the pull-back of the fiber Hessian of F . Let $\hat{\pi} : \mathbb{P}(\pi^* E) \rightarrow \mathbb{P}(E)$ be the induced map. In local coordinates w^i on $\tilde{M}$ and $z^a = z^a(w)$ on M , holomorphy of π and (5.3) give

$$\tilde{N}_i^\alpha = N_a^\alpha(\pi(w), \zeta) \frac{\partial z^a}{\partial w^i}, \quad \delta_{\tilde{F}} \zeta^\alpha = d\zeta^\alpha + \tilde{N}_i^\alpha dw^i = \hat{\pi}^*(\delta_F \zeta^\alpha).$$

Pulling back (5.4) therefore gives $2\pi\xi_{\tilde{F}} = \hat{\pi}^* \omega_V - \hat{\pi}^* \Psi_F$, where $\hat{\pi}^* \omega_V$ is built from the forms $\delta_{\tilde{F}} \zeta$ and $\hat{\pi}^* \Psi_F$ only from dw . Since $\{dw^i, \delta_{\tilde{F}} \zeta^\alpha\}$ is a coframe, this is the decomposition (5.4) for $\tilde{F}$, and $\Psi_{\tilde{F}} = \hat{\pi}^* \Psi_F$. The argument does not require $d\pi$ to have full rank. (Alternatively: the identity holds by naturality over the dense set where π is biholomorphic, and both sides are smooth.) Although $\pi^* \omega$ is only semipositive, no Kähler argument on $\tilde{M}$ is required: it enters only through the semipositive $(n-1, n-1)$ -form $(\pi^* \omega)^{n-1}$ and the projection formula. Apply Lemma 11.1 to $(\tilde{\mathcal{S}}, \tilde{F}|_{\tilde{\mathcal{S}}})$ over the compact manifold $\tilde{M}$, and Proposition 7.2 to $\tilde{\mathcal{S}} \subset \pi^* E$. At $x \in \tilde{M}$ this gives the inequality of real $(1, 1)$ -forms

$$\gamma_{\tilde{F}|_{\tilde{\mathcal{S}}}}(x) = \frac{s}{2\pi} \int_{\mathbb{P}(\tilde{\mathcal{S}}_x)} \Psi_{\tilde{F}|_{\tilde{\mathcal{S}}}} d\nu \leq \frac{s}{2\pi} \int_{\mathbb{P}(\tilde{\mathcal{S}}_x)} \hat{\pi}^* \Psi_F d\nu.$$

Step 3: integrate. Wedging with the positive form $(\pi^*\omega)^{n-1}/(n-1)!$ preserves the inequality, and by (1.2) $\hat{\pi}^*\Psi_F \wedge (\pi^*\omega)^{n-1}/(n-1)! = \pi^*(k_F dV_\omega)$ pointwise on $\mathbb{P}(E_{\pi(x)})$. Since ν is a probability measure,

$$\deg_\omega \mathcal{S} \leq \frac{s}{2\pi} \int_{\tilde{M}} \pi^* \left(\max_{\mathbb{P}(E_z)} k_F dV_\omega \right) = \frac{s}{2\pi} \int_M \max_{\mathbb{P}(E_z)} k_F dV_\omega,$$

because π has degree one. This is (11.3). Dividing by s and subtracting (11.2) divided by r gives the first inequality in (11.4). For the second, $\max k - \int k d\nu \leq 2\|k - c\|_\infty$ for every constant c . $\square$

Corollary 11.2 (Kobayashi’s problem; cf. [12, 13]). *If k_F is constant on each fiber $\mathbb{P}(E_z)$ (in particular, if F is Finsler–Einstein), then E is ω -semistable.*

Corollary 11.3 (Approximate correspondence). *E is ω -semistable if and only if E is approximately Finsler–Einstein. More precisely, for every F and every saturated $\mathcal{S}$,*

$$\mu_\omega(\mathcal{S}) - \mu_\omega(E) \leq \frac{\text{Vol}(M)}{\pi} \|k_F - \lambda\|_{L^\infty(\mathbb{P}(E))}.$$

Proof. The inequality is (11.4) with $c = \lambda$, so approximate Finsler–Einstein implies semistable. Conversely, if E is semistable, the norms of the approximate Hermitian–Einstein metrics of Theorem 2.1(ii) satisfy $|k_F - \lambda| = |\langle (K_h - \lambda)\zeta, \zeta \rangle|/|\zeta|^2 \leq \|K_h - \lambda \text{Id}\|$. $\square$

This is an explicit comparison between the Finsler mean curvature and the slopes of subsheaves, with constant $\text{Vol}(M)/\pi$, for the scalar Kobayashi curvature. It requires neither an averaging of the Hessian metric over the fibers nor Finsler metrics on exterior powers.

Proposition 11.4 (Equality case). *Let F be Finsler–Einstein and $\mathcal{S} \subset E$ saturated with $\mu_\omega(\mathcal{S}) = \mu_\omega(E)$. Then over $M \setminus Z$ the horizontal distribution $\mathcal{H}_F$ is tangent to $\mathbb{P}(\mathcal{S})$, $\Psi_{F|_{\mathcal{S}}} = \Psi_{F|_{\mathbb{P}(\mathcal{S})}}$, and $F|_{\mathcal{S}}$ satisfies (2.2) with the same constant.*

Proof. All inequalities in the proof of Theorem A are equalities. Over $M \setminus Z$, where π is biholomorphic and $\omega > 0$, this forces tr_ω of the nonnegative form $\Psi_F - \Psi_{F_S}$ to vanish, hence the form itself vanishes; conclude with Proposition 7.2. $\square$

In the Hermitian case Proposition 11.4 yields a parallel, hence holomorphically split, subbundle. For a general Finsler metric there is no orthogonal complement, and polystability needs a different argument. This is the content of the next section.

11.3. The Level 1 subvariety test. The J -equation has a Nakai–Moishezon-type obstruction: positivity of $\int_Y (c\Omega^p - p\chi \wedge \Omega^{p-1})$ on subvarieties Y of dimension p [27, 5]. On $\mathbb{P}(E)$ the natural test subvarieties are the $\mathbb{P}(\mathcal{S})$, and the test reduces to slope semistability. This gives a second, “total-space” proof of Corollary 11.2.

Lemma 11.5. *Let $\Omega > 0$ and $\chi \geq 0$ be Hermitian forms on a vector space T , and $W \subset T$ a subspace. Then $\text{tr}_{\Omega|_W}(\chi|_W) \leq \text{tr}_\Omega \chi$.*

Proof. Take an Ω -orthonormal basis of T extending one of W . The difference of the two traces is the sum of $\chi(e_i, \bar{e}_i) \geq 0$ over the basis vectors not in W . $\square$

Applied to $T\mathbb{P}(\mathcal{S}) \subset TX$ at points of $\mathbb{P}(\mathcal{S})$, Lemma 11.5 compares $\text{tr}_{(C\omega - \Psi_{F|_{\mathcal{S}}}/2\pi)} \omega$ with $\text{tr}_{(C\omega - \Psi_F/2\pi)} \omega$. It is thus the trace-level counterpart of the Gauss–Codazzi inequality $\Psi_{F|_{\mathcal{S}}} \leq \Psi_F$ of Proposition 7.2.

Proposition 11.6. *Let $\mathcal{S} \subset E$ be saturated of rank s , let Z be as in Step 1 of the proof of Theorem A, let $Y \subset X$ be the closure of $\mathbb{P}(\mathcal{S}|_{M \setminus Z})$, of dimension $p = n + s - 1$, and write $d_{\mathcal{S}} = \int_M c_1(\mathcal{S}) \wedge \omega^{n-1}$. Then:*

$$(i) \int_Y (c_C \Omega^p - p\chi \wedge \Omega^{p-1}) = n \binom{p}{n} C^{m-2} \left(\frac{d_E}{r} - \frac{d_{\mathcal{S}}}{s} \right) + O(C^{m-3}) \text{ and } \int_Y \Omega^p = \binom{p}{n} C^m V - \binom{p}{n-1} C^{m-1} d_{\mathcal{S}} + O(C^{m-2});$$

$$(ii) \int_Y (c_C \Omega^p - p \chi \wedge \Omega^{p-1}) \geq - \left(\max_X \operatorname{tr}_\Omega \chi - c_C \right) \int_Y \Omega^p.$$

Consequently, for every strongly pseudoconvex F ,

$$\mu_\omega(\mathcal{S}) \leq \frac{\operatorname{Vol}(M)}{2\pi} \max_{\mathbb{P}(E)} k_F,$$

and if F is Finsler–Einstein then E is ω -semistable.

Proof. (i) With the resolution $\pi : \tilde{M} \rightarrow M$ and $\tilde{S} \subset \pi^* E$ of Step 1 of the proof of Theorem A, the induced map $\hat{\pi} : \mathbb{P}(\tilde{S}) \rightarrow Y$ is bimeromorphic. Hence $\int_Y \alpha = \int_{\mathbb{P}(\tilde{S})} \hat{\pi}^* \alpha$ for closed α . The class $\hat{\pi}^* \xi_F$ is $c_1(\mathcal{O}_{\mathbb{P}(\tilde{S})}(1))$. Apply Lemma 10.1 to $(B, V) = (\tilde{M}, \tilde{S})$ with the closed semipositive form $\pi^* \omega$, and use Step 1 of the proof of Theorem A to identify $\int_{\tilde{M}} c_1(\tilde{S}) \wedge (\pi^* \omega)^{n-1} = d_S$. This gives

$$\begin{aligned} \int_Y \Omega^p &= \binom{p}{n} C^n V - \binom{p}{n-1} C^{n-1} d_S + O(C^{n-2}), \\ \int_Y \chi \wedge \Omega^{p-1} &= \binom{p-1}{n-1} C^{n-1} V - \binom{p-1}{n-2} C^{n-2} d_S + O(C^{n-3}). \end{aligned}$$

Insert $c_C = \frac{n}{C} + \frac{n}{r} \frac{d_E}{V} \frac{1}{C^2} + O(C^{-3})$. The terms of order C^{n-1} are $n \binom{p}{n} C^{n-1} V - p \binom{p-1}{n-1} C^{n-1} V = 0$, because $p \binom{p-1}{n-1} = n \binom{p}{n}$. The terms of order C^{n-2} are

$$\binom{p}{n} \frac{n}{r} d_E - n \binom{p}{n-1} d_S + p \binom{p-1}{n-2} d_S = \binom{p}{n} \frac{n}{r} d_E - \binom{p}{n-1} d_S = n \binom{p}{n} \left(\frac{d_E}{r} - \frac{d_S}{s} \right),$$

using $p \binom{p-1}{n-2} = (n-1) \binom{p}{n-1}$ and $\binom{p}{n-1} = \frac{n}{s} \binom{p}{n}$.

(ii) On the regular part of Y , $p \chi \wedge \Omega^{p-1}|_Y = \operatorname{tr}_{\Omega|_Y}(\chi|_Y) \Omega^p|_Y \leq (\max_X \operatorname{tr}_\Omega \chi) \Omega^p|_Y$ by Lemma 11.5. Integrate.

For the consequence, combine (i), (ii) and Proposition 10.3, which gives $\max_X \operatorname{tr}_\Omega \chi - c_C = (\max k_F - \lambda)/2\pi C^2 + O(C^{-3})$. Dividing by $\binom{p}{n} C^{n-2}$ and letting $C \rightarrow \infty$,

$$n \left(\frac{d_S}{s} - \frac{d_E}{r} \right) \leq \frac{V}{2\pi} (\max k_F - \lambda),$$

that is, $\mu_\omega(\mathcal{S}) - \mu_\omega(E) \leq \frac{\operatorname{Vol}(M)}{2\pi} (\max k_F - \lambda)$, since $d = (n-1)! \deg$ and $V = n! \operatorname{Vol}(M)$. $\square$

Proposition 11.6 is the sup-norm version of Theorem A. Theorem A refines it by integrating the Gauss–Codazzi inequality over the fibers instead of bounding it by the maximum. The two coincide when k_F is constant. This is the precise sense in which Kobayashi’s equation and the Feng–Liu–Wan “nonlinear semistability” [13, §2] are the adiabatic limit of a J -equation and its subvariety test on $\mathbb{P}(E)$.

12. LEVEL 1, OPTIMIZED: THE MAXIMAL HARDER–NARASIMHAN SLOPE

Theorem A bounds the slope of every subsheaf by a Finsler curvature integral. Here we show that this bound is sharp: optimizing over Finsler metrics recovers the maximal slope of E , and more generally the vertices of its Harder–Narasimhan polygon. The optimal value is already approached by Hermitian metrics, so the content of the result is the lower bound for the much larger class of Finsler metrics, which is Theorem A.

12.1. A Finsler Ky Fan functional. For $z \in M$ and $1 \leq s \leq r$ put

$$(12.1) \quad \Sigma_s^F(z) = \max_{V \in \operatorname{Gr}_s(E_z)} s \int_{\mathbb{P}(V)} k_F d\nu_{F|_V},$$

where $\nu_{F|_V}$ is the probability measure (5.6) of the restricted metric $F|_V$ on $\mathbb{P}(V)$. The maximum exists because the integrand depends continuously on V in the compact Grassmannian. Two cases are immediate. $\Sigma_1^F(z) = \max_{\mathbb{P}(E_z)} k_F$, since $\mathbb{P}(V)$ is a point and ν a Dirac mass. $\Sigma_r^F(z) = r \int_{\mathbb{P}(E_z)} k_F d\nu_{F,z}$, so that $\int_M \Sigma_r^F dV_\omega = 2\pi \deg_\omega E$ by (11.2). If F is the norm

of a Hermitian metric h , then $\nu_{F|V}$ is the Fubini–Study measure, $s \int_{\mathbb{P}(V)} \langle K_h \zeta, \zeta \rangle / |\zeta|^2 d\nu = \text{tr}(\pi_V K_h \pi_V)$, and by Ky Fan’s maximum principle

$$(12.2) \quad \Sigma_s^{F_h}(z) = \lambda_1(z) + \cdots + \lambda_s(z), \quad \lambda_1 \geq \cdots \geq \lambda_r \text{ the eigenvalues of } K_h(z).$$

Proposition 12.1 (Refined Theorem A). *For every strongly pseudoconvex F and every saturated $\mathcal{S} \subset E$ of rank s ,*

$$\deg_\omega \mathcal{S} \leq \frac{1}{2\pi} \int_M \Sigma_s^F dV_\omega \leq \frac{s}{2\pi} \int_M \max_{\mathbb{P}(E_z)} k_F dV_\omega.$$

Proof. In Step 2 of the proof of Theorem A, the measure on $\mathbb{P}(\tilde{S}_x)$ is $\nu_{F|V}$ with $V = \tilde{S}_x \subset E_{\pi(x)}$. Stopping the estimate in Step 3 before bounding k_F by its maximum gives

$$\gamma_{\tilde{F}|_{\tilde{\mathcal{S}}}} \wedge \frac{(\pi^* \omega)^{n-1}}{(n-1)!} \leq \frac{s}{2\pi} \left(\int_{\mathbb{P}(V)} k_F d\nu_{F|V} \right) \pi^* dV_\omega \leq \frac{1}{2\pi} \pi^* (\Sigma_s^F dV_\omega),$$

and integration over $\tilde{M}$ proves the first inequality. The second holds because $\nu_{F|V}$ is a probability measure. $\square$

12.2. The Harder–Narasimhan filtration. Let $0 = E_0 \subset E_1 \subset \cdots \subset E_\ell = E$ be the Harder–Narasimhan filtration of E with respect to ω . The quotients E_k/E_{k-1} are torsion-free and semistable, of ranks r_k and slopes $\mu_1 > \cdots > \mu_\ell$, and $\mu_{\max}(E) = \mu_1$ is the maximal slope of a subsheaf. The Harder–Narasimhan polygon $P_E : [0, r] \rightarrow \mathbb{R}$ is the concave piecewise linear function with $P_E(0) = 0$ and slope μ_k on $[\sigma_{k-1}, \sigma_k]$, where $\sigma_k = r_1 + \cdots + r_k$; its vertices are $P_E(\sigma_k) = \deg_\omega E_k$. With $\lambda_k = 2\pi\mu_k/\text{Vol}(M)$, a smooth Hermitian metric h defines, off the singular set of the filtration, the endomorphism

$$\Psi_h^{\text{HN}} = \sum_{k=1}^{\ell} \lambda_k (\pi_k^h - \pi_{k-1}^h),$$

where π_k^h is the h -orthogonal projection onto E_k . Following [9], h is an L^p δ -approximate critical Hermitian structure if $\|K_h - \Psi_h^{\text{HN}}\|_{L^p(\omega)} < \delta$.

Theorem 12.2 (Jacob [16, Thm. 1] for $p = 2$; Sibley [25, Thm. 5.12] for $1 \leq p < \infty$; see [9] for surfaces). *For every $\delta > 0$ and $1 \leq p < \infty$, E admits an L^p δ -approximate critical Hermitian structure.*

When the filtration consists of subbundles the statement holds with $p = \infty$ [25, Thm. 5.2]. We include an elementary argument.

Lemma 12.3. *Assume that every E_k is a subbundle of E . Then for every $\delta > 0$ there is a smooth Hermitian metric h with $\|K_h - \Psi_h^{\text{HN}}\|_{L^\infty} < \delta$.*

Proof. Write $Q_k = E_k/E_{k-1}$, which is a semistable vector bundle. Fix a C^∞ splitting $E \cong \bigoplus_k Q_k$ compatible with the filtration. Then $\bar{\partial}_E = \bar{\partial}_0 + \beta$, where $\bar{\partial}_0 = \bigoplus \bar{\partial}_{Q_k}$ and β is a smooth $(0, 1)$ -form with components in $\text{Hom}(Q_j, Q_i)$ for $i < j$ only. By Theorem 2.1(ii), choose metrics h_k on Q_k with $\|K_{h_k} - \lambda_k \text{Id}\|_{L^\infty} < \delta/2$, and put $h_0 = \bigoplus h_k$.

For $0 < t < 1$ let $g_t = \bigoplus_k t^{-k} \text{Id}_{Q_k}$ and $h_t = g_t^* h_0$, that is, $|v|_{h_t} = |g_t v|_{h_0}$. Then g_t is an isometry from $(E, \bar{\partial}_E, h_t)$ onto $(E, \bar{\partial}_0 + \beta_t, h_0)$, where $\beta_t = g_t \beta g_t^{-1}$ has $\text{Hom}(Q_j, Q_i)$ -component $t^{j-i} \beta_{ij} = O(t)$. Since g_t is constant on each block, $D_0 \beta_t = g_t (D_0 \beta) g_t^{-1} = O(t)$ as well. The Chern connection of $(\bar{\partial}_0 + \beta_t, h_0)$ is $D_0 + \beta_t - \beta_t^*$, so its curvature is $\bigoplus_k \Theta_{h_k} + O(t)$ uniformly on M . Hence $\|K - \bigoplus_k \lambda_k \text{Id}_{Q_k}\|_{L^\infty} < \delta/2 + O(t)$. The map g_t preserves each E_k and the splitting, so it carries $\Psi_{h_t}^{\text{HN}}$ to $\bigoplus_k \lambda_k \text{Id}_{Q_k}$. Take t small. $\square$

12.3. The characterization.

Theorem 12.4. *Let (M, ω) be compact Kähler and $E \rightarrow M$ holomorphic. Then:*

- (i) $\mu_{\max}(E) = \frac{1}{2\pi} \inf_F \int_M \max_{\mathbb{P}(E_z)} k_F dV_\omega$;
- (ii) $\deg_\omega E_k = P_E(\sigma_k) = \frac{1}{2\pi} \inf_F \int_M \Sigma_{\sigma_k}^F dV_\omega$ for every vertex σ_k of the Harder–Narasimhan polygon, and $\frac{1}{2\pi} \inf_F \int_M \Sigma_s^F dV_\omega \leq P_E(s)$ for every $s \in \{1, \dots, r\}$;
- (iii) $\mu_{\max}(E) - \mu_\omega(E) = \frac{1}{2\pi} \inf_F \int_M \left(\max_{\mathbb{P}(E_z)} k_F - \int_{\mathbb{P}(E_z)} k_F d\nu_{F,z} \right) dV_\omega$. In particular, E is semistable if and only if this infimum is 0;
- (iv) if the Harder–Narasimhan filtration consists of subbundles, then

$$\mu_{\max}(E) = \frac{\text{Vol}(M)}{2\pi} \inf_F \max_{\mathbb{P}(E)} k_F.$$

In each formula the infimum is over strongly pseudoconvex Finsler metrics on E , and the same value is obtained by restricting to Hermitian metrics.

Proof. Lower bounds. These hold for every Finsler metric. Applying Proposition 12.1 to the saturated subsheaf E_k gives $\deg_\omega E_k \leq \frac{1}{2\pi} \int \Sigma_{\sigma_k}^F dV_\omega$, which is the lower bound in (ii). For $k = 1$, using $\Sigma_{\sigma_1}^F \leq \sigma_1 \max k_F$ and $\deg_\omega E_1 = \sigma_1 \mu_1$, it gives (i). For (iv), note that $\int_M \max_{\mathbb{P}(E_z)} k_F dV_\omega \leq \text{Vol}(M) \max_{\mathbb{P}(E)} k_F$.

Upper bounds. These are attained by Hermitian metrics. For Hermitian matrices, the Ky Fan sum $\Sigma_s(A) = \max\{\text{tr}(\pi A \pi) : \pi \text{ an orthogonal projection of rank } s\}$ satisfies $|\Sigma_s(A) - \Sigma_s(B)| \leq s \|A - B\|$. Off the singular set, the eigenvalues of Ψ_h^{HN} are λ_k with multiplicities r_k , so $\int_M \Sigma_s(\Psi_h^{\text{HN}}) dV_\omega = 2\pi P_E(s)$. By (12.2), for $F = F_h$,

$$\int_M \Sigma_s^{F_h} dV_\omega = \int_M \Sigma_s(K_h) dV_\omega \leq 2\pi P_E(s) + s \|K_h - \Psi_h^{\text{HN}}\|_{L^1}.$$

Theorem 12.2 (with any $p \geq 1$, using $\|\cdot\|_{L^1} \leq \text{Vol}(M)^{1-1/p} \|\cdot\|_{L^p}$) makes the last term arbitrarily small. This proves the upper bound in (ii) for every s . The case $s = 1$, where $\Sigma_1 = \lambda_{\max}$ and $P_E(1) = \mu_1$, is the upper bound in (i). For (iv), Lemma 12.3 gives $\max_{\mathbb{P}(E)} k_{F_h} = \max_M \lambda_{\max}(K_h) \leq \lambda_1 + \delta$.

Part (iii). By (11.2), $\int_M \int_{\mathbb{P}(E_z)} k_F d\nu_{F,z} dV_\omega = 2\pi \mu_\omega(E)$ for every F , so (iii) is (i) minus this constant. Finally, $\mu_{\max}(E) = \mu_\omega(E)$ exactly when E is semistable. $\square$

Remark 12.5 (Relation with Finski's lower bounds). Finski [14] studies relatively positive metrics h^L on a relatively ample line bundle L over a fibration. For $X = \mathbb{P}(E)$ and $L = \mathcal{O}_X(1)$, these are exactly the Finsler metrics (Proposition 4.1). Finski uses the Chern form $\omega(h^L) = \frac{\sqrt{-1}}{2\pi} R^L$ and the contraction $\wedge_\omega \alpha = \alpha \wedge \omega^{n-1} / \omega^n = \frac{1}{n} \Lambda_\omega \alpha$. Since the horizontal component of $c_1(\mathcal{O}_X(1), G^{-1})$ is $-\Psi_F / 2\pi$ by (5.4), his horizontal mean curvature is

$$\wedge_\omega \omega_H = -\frac{k_F}{2\pi n}.$$

His degree $\deg^{\text{Fin}} \mathcal{S} = \int_M c_1(\det \mathcal{S}) \wedge \omega^{n-1}$ carries neither the factorial nor a volume denominator, so his slope is $\mu^{\text{Fin}} = (n-1)! \mu_\omega$. On the scale of the pointwise mean curvature, the corresponding volume-normalized slope is

$$\hat{\mu}^{\text{Fin}} := \frac{\mu^{\text{Fin}}}{\int_M \omega^n} = \frac{\mu_\omega}{n \text{Vol}(M)}.$$

To apply Finski's estimates, rescale the base form to $\hat{\omega} = (\int_M \omega^n)^{-1/n} \omega$, so that $\int_M \hat{\omega}^n = 1$, and then translate the resulting inequalities back to ω . The relevant direct images are $p_* \mathcal{O}_X(k) = \text{Sym}^k E^*$, and $\mu_{\min}(\text{Sym}^k E^*) = k \mu_{\min}(E^*) = -k \mu_{\max}(E)$ in characteristic zero. Theorems 1.7 and 1.10 of [14] therefore yield the following:

- (a) for every F , $\max_{\mathbb{P}(E)} k_F \geq \frac{2\pi \mu_{\max}(E)}{\text{Vol}(M)}$ and $\min_{\mathbb{P}(E)} k_F \leq \frac{2\pi \mu_{\min}(E)}{\text{Vol}(M)}$;
- (b) $\inf_F \max_{\mathbb{P}(E)} k_F = \frac{2\pi \mu_{\max}(E)}{\text{Vol}(M)}$.

Thus Theorem 12.4(iv) holds *without* the hypothesis that the Harder–Narasimhan filtration consists of subbundles; Lemma 12.3 gives an elementary proof in that special case. Part (i) of Theorem 12.4 has a stronger lower bound, since $\int_M \max_{\mathbb{P}(E_z)} k_F dV_\omega \leq \text{Vol}(M) \max_{\mathbb{P}(E)} k_F$, while (a) supplies a bound for $\mu_{\min}$ that our methods do not give. Finski also proves lower bounds for L^p fibered Yang–Mills functionals in terms of the limiting Harder–Narasimhan distribution of the direct images.

Remark 12.6. (a) Part (i) is the Finsler counterpart of the Hermitian identity $\mu_{\max}(E) = \frac{1}{2\pi} \inf_h \int_M \lambda_{\max}(K_h) dV_\omega$, which follows from Theorem 12.2 together with the Hermitian case of Theorem A. The theorem says that enlarging the class of metrics from Hermitian to Finsler does not lower the infimum. Finsler metrics therefore carry no destabilizing information beyond that of Hermitian ones, but every Finsler metric gives a computable *upper* bound for $\mu_{\max}(E)$.

- (b) Part (iii) sharpens the oscillation inequality (11.4): its right-hand side, minimized over F , equals the Harder–Narasimhan defect $\mu_{\max} - \mu$ exactly.
- (c) For non-vertex s , Proposition 12.1 still gives $\max\{\deg_\omega \mathcal{S} : \text{rk } \mathcal{S} = s\} \leq \frac{1}{2\pi} \inf_F \int \Sigma_s^F$. Equality with $P_E(s)$ for all s would follow if $s \mapsto \int_M \Sigma_s^F dV_\omega$ were concave for every F . This holds for Hermitian F , by (12.2), since partial sums of decreasing eigenvalues are concave in s . For Finsler F it is open (Problem 15.5).
- (d) By Remark 12.5(a), $\min_{\mathbb{P}(E)} k_F \leq 2\pi \mu_{\min}(E) / \text{Vol}(M)$ for every F . Whether equality holds in the form

$$\sup_F \min_{\mathbb{P}(E)} k_F = 2\pi \mu_{\min}(E) / \text{Vol}(M)$$

is open in general; it is part of Finski’s optimality conjecture [14]. An integrated version, analogous to (i), would require Finsler metrics on quotient bundles. These are not induced by a strongly pseudoconvex F unless F is also convex.

13. LEVEL 2: THE DIRECT IMAGE AND POLYSTABILITY

At Level 2 we pass from the line bundle on X to its direct image on M . The Einstein condition on X is transferred to a Hermitian–Einstein metric on E^* through Berndtsson’s curvature formula, whose leading term is the horizontal block of the curvature of the twisting line bundle.

Since $H^0(\mathbb{P}(E_z), \mathcal{O}(1)) = E_z^*$ and, by (5.2),

$$\mathcal{O}_X(1) = K_{X/M} \otimes L, \quad L := \mathcal{O}_X(r+1) \otimes p^* \det E,$$

we have $E^* = p_*(K_{X/M} \otimes L)$. Given F and a Hermitian metric h_d on $\det E$, equip L with the metric $G^{-(r+1)} \otimes h_d$, whose Chern form is

$$(13.1) \quad \theta = (r+1)\xi_F + p^* \rho, \quad \rho = c_1(\det E, h_d).$$

θ is positive on the fibers, its θ -orthogonal complement of $T_{X/M}$ is $\mathcal{H}_F$ (no mixed terms by (5.4)), and its horizontal part (the geodesic curvature) is

$$(13.2) \quad c(\theta) = -\frac{r+1}{2\pi} \Psi_F + p^* \rho.$$

Berndtsson’s L^2 metric on E^* is, explicitly,

$$(13.3) \quad \|u\|_{F, h_d}^2 = C_r \int_{\mathbb{P}(E_z)} \frac{|u(v)|^2}{G(v)^{r+1}} \|e_1 \wedge \cdots \wedge e_r\|_{h_d}^2 (\sqrt{-1})^{(r-1)^2} \Omega_0 \wedge \bar{\Omega}_0,$$

$$\Omega_0 = \sum_{\alpha} (-1)^{\alpha-1} v^\alpha dv^1 \wedge \cdots \widehat{dv^\alpha} \cdots \wedge dv^r,$$

where the factor $(\sqrt{-1})^{(r-1)^2}$ makes $(\sqrt{-1})^{(r-1)^2}\Omega_0 \wedge \bar{\Omega}_0$ a positive form. The integrand is homogeneous of degree 0 and frame-independent. The constant $C_r > 0$ is immaterial for curvature; with our convention for Ω_0 , the choice

$$C_r = \frac{r!}{(2\pi)^{r-1}}$$

normalizes the metric so that, for $G = h(v, v)$ and $h_d = \det h$, it is exactly the dual metric h^* . Intrinsically, (13.3) is Berndtsson's $\|u\|^2 = \int_{X_z} |u|^2 e^{-\phi}$ [13]. It differs from the L^2 metric defined with the volume form ω_{FS}^{r-1} [11, §4] by the fiberwise factor $\det(G_{\alpha\bar{\beta}}) G^{-r}$, up to constants (cf. [29, Prop. 2.3], where $(\omega_V/2\pi)^{r-1}$ is computed in an affine chart); the version (13.3) is the one to which Berndtsson's formula applies directly.

Theorem 13.1 (Berndtsson [3, 4]; see [13, Thm. 3.1]). *For $u \in E_z^*$ and $t \in T_z^{1,0}M$,*

$$\frac{\sqrt{-1}}{2\pi} \langle \Theta^{E^*}(t, \bar{t})u, u \rangle = \int_{\mathbb{P}(E_z)} c(\theta)(t) |u|^2 + R_t(u), \quad R_t(u) \geq 0,$$

where the integral and $R_t(u)$ use the fiber measure implicit in the L^2 metric (13.3).

Theorem B (Feng–Liu–Wan [13]). *If F is Finsler–Einstein and h_d is Hermitian–Einstein on $\det E$, then $\|\cdot\|_{F, h_d}$ is a Hermitian–Einstein metric on E^* , with constant $-\lambda$. Consequently E admits a Finsler–Einstein metric if and only if E is polystable.*

Proof. $\det E$ has Einstein constant $2\pi \deg_\omega E / \text{Vol}(M) = r\lambda$, so $\text{tr}_\omega \rho = r\lambda/2\pi$, and by (13.2) $\text{tr}_\omega c(\theta) = -\frac{r+1}{2\pi} k_F + \frac{r\lambda}{2\pi} = -\frac{\lambda}{2\pi}$. Summing Theorem 13.1 over an ω -orthonormal basis gives $T := K^{E^*} + \lambda \text{Id} \geq 0$ as a Hermitian endomorphism. On the other hand $\int_M \text{tr} T dV_\omega = 2\pi \deg_\omega E^* + r\lambda \text{Vol}(M) = 0$ by (2.1). Hence $T \equiv 0$. By Theorem 9.1, E^* and therefore E are polystable. The converse is Theorem 2.1(i), since a Hermitian–Einstein metric is Finsler–Einstein. $\square$

Corollary 13.2. *Let E be semistable but not polystable, e.g. a non-split extension $0 \rightarrow L \rightarrow E \rightarrow L \rightarrow 0$ over a curve. Then E admits approximate Finsler–Einstein metrics but no Finsler–Einstein metric, in either Kobayashi's or Aikou's sense. Thus:*

$$E \text{ polystable} \iff E \text{ admits a Finsler–Einstein metric,}$$

$$E \text{ semistable} \iff E \text{ admits approximate Finsler–Einstein metrics.}$$

Remark 13.3 (Approximate version of Theorem B). If $\|k_F - \lambda\|_\infty \leq \varepsilon$, the same computation gives $\text{tr}_\omega c(\theta) \geq -(\lambda + (r+1)\varepsilon)/2\pi$, hence $K^{E^*} \geq -(\lambda + (r+1)\varepsilon)\text{Id}$, i.e. the dual metric on E has $K \leq (\lambda + (r+1)\varepsilon)\text{Id}$, which by Theorem 9.1 gives a second proof of Corollary 11.3 with constant $(r+1)\text{Vol}(M)/2\pi$. Theorem A gives the sharper, rank-independent constant.

Example 13.4 (Non-Hermitian Finsler–Einstein metrics: the trivial bundle). Let $E = M \times \mathbb{C}^r$ and $G(z, \zeta) = G_0(\zeta)$ with

$$G_0(\zeta) = |\zeta|^2 + \epsilon \left(\sum_\alpha |\zeta^\alpha|^4 \right)^{1/2}, \quad \epsilon > 0.$$

G_0 is $|t|^2$ -homogeneous and smooth on $\mathbb{C}^r \setminus \{0\}$, and its Levi form is $\geq \text{Id}$, because $\zeta \mapsto \|(\zeta_\alpha^2)\|$ is plurisubharmonic. Since G does not depend on z , $N = 0$ and $\Psi_F = 0$, so F is Finsler–Einstein with $\lambda = 0$, and it is not Hermitian by Lemma 6.1. Metrics of this kind, locally independent of z , are the *locally complex Minkowski metrics* of [29]; for them the Kähler metrics $\Omega_{F,C}$ have Ricci form $\text{Ric}(\omega) + \text{Ric}(\omega_V)$ [29, Thm. 1.6]. The trivial bundle is polystable but not stable for $r \geq 2$, and its holonomy is trivial. The next proposition shows that the phenomenon persists on stable bundles with irreducible holonomy.

Proposition 13.5 (Non-Hermitian Finsler–Einstein metrics on stable bundles). *Let (M, ω) be a compact Kähler manifold and let $\rho : \pi_1(M) \rightarrow \Gamma \subset U(r)$, $r \geq 2$, be an irreducible unitary representation with finite image. Then the associated flat bundle $E_\rho = \widetilde{M} \times_\rho \mathbb{C}^r$ is ω -stable*

of degree zero. Moreover, let $C_0^\infty(\mathbb{P}^{r-1})^\Gamma$ denote the space of smooth Γ -invariant functions on $\mathbb{P}^{r-1}$ with zero average. Then there is an open neighborhood $\mathcal{U}$ of 0 in $C_0^\infty(\mathbb{P}^{r-1})^\Gamma$, for the C^2 topology, such that

$$G_\psi(v) = |v|^2 e^{\psi([v])}, \quad \psi \in \mathcal{U},$$

defines strongly pseudoconvex Finsler–Einstein metrics F_ψ on E_ρ with $\Psi_{F_\psi} = 0$. Distinct ψ give distinct metrics, and F_ψ is induced by a Hermitian metric only for $\psi = 0$. In particular, E_ρ carries an infinite-dimensional family of non-Hermitian Finsler–Einstein metrics.

Proof. Stability. The flat unitary metric has zero mean curvature. Hence the Chern–Weil formula (9.1) gives $\mu_\omega(\mathcal{S}) \leq 0 = \mu_\omega(E_\rho)$ for every saturated subsheaf $\mathcal{S} \subset E_\rho$. If equality holds, its second fundamental form vanishes, so $\mathcal{S}$ is a parallel subbundle (Theorem 9.1). Irreducibility of ρ excludes proper parallel subbundles. Therefore E_ρ is stable of degree zero. For curves this is part of the Narasimhan–Seshadri theorem [23].

Invariant functions. Choose a small open ball $U \subset \mathbb{P}^{r-1}$ such that the finite union $\bigcup_{\gamma \in \Gamma} \gamma U$ is not all of $\mathbb{P}^{r-1}$. Averaging over Γ a nonzero bump function supported in U produces a nonconstant smooth Γ -invariant function. Using infinitely many such balls whose Γ -orbits are pairwise disjoint, one obtains infinitely many linearly independent invariant functions with disjoint supports, so $C_0^\infty(\mathbb{P}^{r-1})^\Gamma$ is infinite-dimensional.

Strong pseudoconvexity and curvature. $G_\psi(tv) = |t|^2 G_\psi(v)$ and G_ψ is Γ -invariant. Moreover, $\sqrt{-1}\partial\bar{\partial}\log G_\psi$ restricts on $\mathbb{P}^{r-1}$ to $\omega_{FS} + \sqrt{-1}\partial\bar{\partial}\psi$, which is positive when $\|\psi\|_{C^2}$ is small; let $\mathcal{U}$ be the corresponding open C^2 -neighborhood of 0. By Proposition 3.3, G_ψ is strongly pseudoconvex. The transition functions of E_ρ in flat holomorphic frames are constant and lie in Γ , so these locally base-independent functions glue to a Finsler metric F_ψ on E_ρ . In flat holomorphic frames G_ψ does not depend on z ; hence $N_i^\alpha = 0$, $\Psi_{F_\psi} = 0$ and $k_{F_\psi} = 0 = \lambda$.

Injectivity and non-Hermitian character. The metric determines $\psi = \log(G_\psi/|v|^2)$, and the zero-average normalization removes the rescalings $G \mapsto cG$. If $G_\psi(v) = h(v, v)$ for a Hermitian form h , the Γ -invariance of G_ψ makes h Γ -invariant; by irreducibility and Schur’s lemma $h = c\text{Id}$, so ψ is constant, hence $\psi = 0$. $\square$

For example, when M is a curve of genus $g \geq 2$, compose a surjection $\pi_1(M) \twoheadrightarrow S_3$ with the irreducible two-dimensional unitary representation of S_3 . The degree-zero metrics F_ψ on E_ρ are locally complex Minkowski metrics in the sense of [29], as in Example 13.4. The construction also gives nonzero slope: if $L \rightarrow M$ carries a Hermitian–Einstein metric h_L , then $G(l \otimes v) = |l|_{h_L}^2 G_\psi(v)$ is a non-Hermitian Finsler–Einstein metric on the stable bundle $L \otimes E_\rho$, with Kobayashi curvature $\sqrt{-1}\Theta_{h_L}$. If h_L is not flat, these twisted metrics are generally not locally complex Minkowski.

More generally, the construction applies to any irreducible unitary representation whose holonomy closure $H \subset U(r)$ does not act transitively on $\mathbb{P}^{r-1}$. If H acts transitively, for instance if $H \supset SU(r)$, then every H -invariant function on $\mathbb{P}^{r-1}$ is constant, so every parallel strongly pseudoconvex Finsler norm obtained in this way is Hermitian up to scale. It does not follow that every Finsler–Einstein metric is then Hermitian, but this source of non-Hermitian solutions disappears.

14. LEVEL 3: ADIABATIC STABILITY ON THE PROJECTIVE BUNDLE

At Level 3 the object is the rank- r bundle p^*E on $(X, \Omega_{F,C})$. The stability side is entirely topological: adiabatic slopes of pull-backs, which reduce to slope stability of E at leading order (§14.1–§14.2). The metric side is Wong’s equation [30], the Hermitian–Einstein equation for the Hessian metric H_F ; the projective Finsler geometry behind it goes back to Kobayashi [18] and Cao–Wong [6]. Precisely, it is the Hermitian–Einstein equation, with respect to the Kähler forms $c_1(\mathcal{O}_{\mathbb{P}(E)}(1)) + C p^*\omega$ of [6, §3], for the Hessian metric on the vertical bundle whose Chern connection and curvature components are computed in [6, §5]. Section 14.3 isolates exactly where the Finsler representative enters, and §14.4 proves a rigidity theorem.

14.1. The polarization and the adiabatic slope. As in Remark 10.4, the polarization is $[\Omega_{F,C}] = c_1(\mathcal{O}_X(1)) + C[p^*\omega]$, with $C \rightarrow \infty$ (equivalently $\varepsilon = C^{-1} \rightarrow 0$). Put

$$\beta_C = p_* \left(\frac{\Omega_{F,C}^{N-1}}{(N-1)!} \right),$$

a closed strictly positive $(n-1, n-1)$ -form on M . By the projection formula, $\deg_\Omega(p^*\mathcal{S}) = \int_M c_1(\det \mathcal{S}) \wedge \beta_C$ for every coherent $\mathcal{S}$ on M . We call

$$\mu_{\beta_C}(\mathcal{S}) := \frac{1}{\text{rk } \mathcal{S}} \int_M c_1(\det \mathcal{S}) \wedge \beta_C = \mu_\Omega(p^*\mathcal{S})$$

the *adiabatic slope*. By the third identity of Lemma 10.1 (with $V = E$, $s = r$),

$$(14.1) \quad [\beta_C] = \sum_{j=0}^{n-1} \frac{C^{n-1-j}}{(m+j)!(n-1-j)!} s_j(E) \wedge [\omega]^{n-1-j},$$

and therefore, for every saturated $\mathcal{S}$ of rank s ,

$$(14.2) \quad \mu_{\beta_C}(\mathcal{S}) - \mu_{\beta_C}(E) = \sum_{j=0}^{n-1} \frac{C^{n-1-j}}{(m+j)!(n-1-j)!} \int_M \left(\frac{c_1(\mathcal{S})}{s} - \frac{c_1(E)}{r} \right) \wedge s_j(E) \wedge \omega^{n-1-j}.$$

Similarly $\text{Vol}(X, \Omega_{F,C}) = C^n \text{Vol}(M)/m! + O(C^{n-1})$, which gives $\lambda_C = \lambda/C + O(C^{-2})$ for the Einstein constant of p^*E .

Reduction to slope stability. The leading coefficient of (14.2) is $\frac{C^{n-1}}{m!}(\mu_\omega(\mathcal{S}) - \mu_\omega(E))$. For $n \geq 2$, β_C -(semi)stability for all $C \gg 0$ is therefore a lexicographic sequence of slope conditions: first with respect to ω^{n-1} , then, among subsheaves of equal ω -slope, with respect to $s_1(E)\omega^{n-2} = -c_1(E)\omega^{n-2}$, then $s_2(E)\omega^{n-3}$, and so on. This refinement is not intrinsic to E . Since $\mathbb{P}(E) = \mathbb{P}(E \otimes N)$ while $\mathcal{O}(1)$ shifts by $p^*c_1(N)$, the correction terms change under twisting; the natural setting allows $\Omega = \xi + p^*(C\omega + \beta)$ with an arbitrary class β . For $n = 1$, β_C is constant and there is no refinement.

14.2. Hermitian–Einstein metrics on p^*E and pull-back stability.

Proposition 14.1. *If p^*E admits any Hermitian–Einstein metric with respect to $\Omega_{F,C}$, then E is β_C -polystable. If it is approximately Hermitian–Einstein, E is β_C -semistable.*

Proof. For saturated $\mathcal{S} \subset E$, Theorem 9.1 applied to $p^*\mathcal{S} \subset p^*E$ gives $\mu_{\beta_C}(\mathcal{S}) \leq \mu_{\beta_C}(E)$. In case of equality, $p^*\mathcal{S}$ is a holomorphic direct summand. Since p is faithfully flat, local freeness of $p^*\mathcal{S}$ implies local freeness of $\mathcal{S}$. Moreover, using $p_*\mathcal{O}_X = \mathcal{O}_M$, the projection lies in $H^0(X, \text{End } p^*E) = H^0(M, \text{End } E)$, so $E = \mathcal{S} \oplus \mathcal{S}'$. Both summands inherit Hermitian–Einstein metrics on their pull-backs (Theorem 7.1); conclude by induction on the rank. $\square$

The proof never uses that the metric is a Finsler Hessian. Hermitian–Einstein metrics on p^*E for adiabatic classes exist exactly when p^*E is $\Omega_{F,C}$ -polystable, by Donaldson–Uhlenbeck–Yau on X . Sektnan and Tipler [26] studied this problem for holomorphic submersions with connected fibers and adiabatic classes. If the base bundle is stable, its pull-back is stable for large adiabatic parameter. For strictly semistable bundles, under hypotheses on a Jordan–Hölder filtration, they give a criterion in terms of intersection numbers on the base. For $\mathbb{P}(E) \rightarrow M$ with the classes above, these intersection numbers are the coefficients in (14.2). Letting $C \rightarrow \infty$ along approximate solutions with $C\delta \rightarrow 0$, where δ is the L^∞ error, yields only ω -semistability, which Theorem A gives for every Finsler metric.

14.3. Where the Finsler representative enters. The adiabatic slopes (14.2) are cohomological, so they do not depend on F at any order. The Finsler metric enters only through the metric equation. Wong’s equation is

$$(14.3) \quad \sqrt{-1}\Lambda_{\Omega_{F,C}}\Theta_{H_F} = \lambda_C \text{Id}_{p^*E}.$$

By (10.1), $\Omega_{F,C}$ is block diagonal, so the left side splits exactly as

$$(14.4) \quad \sqrt{-1}\Lambda_{\Omega_{F,C}}\Theta_{H_F} = K_F^{\text{fib}} + \text{tr}_{A_C}(\sqrt{-1}\Theta_{H_F}^{\mathcal{H}\mathcal{H}}), \quad A_C = C\omega - \frac{\Psi_F}{2\pi}.$$

Here K_F^{fib} is the mean curvature of $(p^*E, H_F)|_{\mathbb{P}(E_z)}$ with respect to $\omega_V/2\pi$; it is independent of C . The mixed curvature components do not contribute.

Proposition 14.2 (Adiabatic expansion of Wong's residual). *Let F be fixed and $R_{F,C} = \sqrt{-1}\Lambda_{\Omega_{F,C}}\Theta_{H_F} - \lambda_C \text{Id}$. Then, uniformly on X ,*

$$R_{F,C} = K_F^{\text{fib}} + \frac{1}{C} \left(\text{tr}_\omega(\sqrt{-1}\Theta_{H_F}^{\mathcal{H}\mathcal{H}}) - \lambda \text{Id} \right) + O(C^{-2}).$$

Moreover:

- (i) the order- C^0 term K_F^{fib} depends only on the restrictions $F|_{E_z}$, and $K_F^{\text{fib}} \equiv 0$ if and only if F is induced by a Hermitian metric on E ;
- (ii) the radial component of the order- C^{-1} term is Kobayashi's residual: $\langle (\text{tr}_\omega \sqrt{-1}\Theta_{H_F}^{\mathcal{H}\mathcal{H}} - \lambda)\zeta, \zeta \rangle_{H_F} / G = k_F - \lambda$;
- (iii) if $F = F_h$ is Hermitian, then $K_F^{\text{fib}} = 0$, the order- C^{-1} term is $p^*(K_h - \lambda \text{Id})$, which is independent of the fiber point, and the fiber point first enters at order C^{-2} , through $\frac{1}{2\pi} \langle \sqrt{-1}\Theta_h, \Psi_h \rangle_\omega$.

Proof. The expansion follows from (14.4), $A_C^{-1} = C^{-1}\omega^{-1} + C^{-2}\omega^{-1}\frac{\Psi_F}{2\pi}\omega^{-1} + O(C^{-3})$, and $\lambda_C = \lambda/C + O(C^{-2})$.

(i) The restriction of (p^*E, H_F) to $\mathbb{P}(E_z)$ is the trivial bundle with the metric $G_{\alpha\bar{\beta}}(z, \cdot)$, which depends only on $F|_{E_z}$. If F is induced by a Hermitian metric, this metric is constant and $K_F^{\text{fib}} = 0$. Conversely, if $K_F^{\text{fib}} = 0$, then $H_F|_{\mathbb{P}(E_z)}$ is Hermitian–Einstein with constant 0 for $\omega_V/2\pi$. For a holomorphic section e , $\sqrt{-1}\Lambda\partial\bar{\partial}|e|^2 = |\nabla' e|^2 \geq 0$, so $|e|^2$ is subharmonic, hence constant, and $\nabla e = 0$. The constant frame is therefore parallel, $G_{\alpha\bar{\beta}}$ is constant along $\mathbb{P}(E_z)$, and F is induced by a Hermitian metric by Lemma 6.1.

(ii) This is (5.5) traced with ω .

(iii) For $F = F_h$, $H_F = p^*h$ and $\Theta_{H_F} = p^*\Theta_h$ is horizontal, so the first-order term is $p^*(\sqrt{-1}\Lambda_\omega\Theta_h - \lambda)$. The next term of A_C^{-1} involves $\Psi_h = \sqrt{-1}\langle \Theta_h\zeta, \zeta \rangle / |\zeta|^2$, which depends on $[\zeta]$. $\square$

This is the precise sense in which Levels 1 and 3 differ. At Level 1 the leading terms of the residual (m , respectively n/C) are universal, and the Finsler data first appear at the next order, through k_F (Remark 10.4). At Level 3 the nonlinear fiberwise data appear already at order C^0 , through K_F^{fib} . The base Hermitian–Einstein condition, whose radial part is Kobayashi's equation, appears at order C^{-1} . The stability obstruction (14.2) sees neither: it is topological at every order.

14.4. Adiabatic rigidity.

Proposition 14.3 (Fixed-metric adiabatic rigidity). *Let F be a fixed strongly pseudoconvex Finsler metric on E . Suppose that, for an unbounded sequence $C_\nu \rightarrow \infty$, H_F satisfies (14.3) with respect to Ω_{F,C_ν} . Then F is the norm of a Hermitian metric h on E , and h is Hermitian–Einstein with respect to ω .*

Proof. Since F is fixed, Ψ_F and Θ_{H_F} are bounded on the compact X . Hence

$$\text{tr}_{A_C}(\sqrt{-1}\Theta_{H_F}^{\mathcal{H}\mathcal{H}}) = C^{-1} \text{tr}_\omega(\sqrt{-1}\Theta_{H_F}^{\mathcal{H}\mathcal{H}}) + O(C^{-2})$$

uniformly, and $\lambda_C = O(C^{-1})$. Letting $\nu \rightarrow \infty$ in (14.4) gives $K_F^{\text{fib}} = 0$.

By Proposition 14.2(i), $F = F_h$ for a Hermitian metric h .

Now $H_F = p^*h$, $K_F^{\text{fib}} = 0$, and $\Theta_{H_F} = p^*\Theta_h$. Multiplying (14.4) by C_ν and letting $\nu \rightarrow \infty$ gives $\sqrt{-1}\Lambda_\omega\Theta_h = \lambda \text{Id}_E$. $\square$

So a single genuinely non-Hermitian Finsler metric cannot solve Wong's equation along an unbounded adiabatic sequence. The conclusion is sharp in the following sense. If h satisfies $\sqrt{-1}\Theta_h = \frac{\lambda}{n}\omega \otimes \text{Id}_E$ (projectively flat, with curvature proportional to ω), then $\Psi_h = \frac{\lambda}{n}\omega$ and $A_C = (C - \frac{\lambda}{2\pi n})\omega$. Hence $\text{tr}_{A_C}(\sqrt{-1}\Theta_h) = \lambda(C - \frac{\lambda}{2\pi n})^{-1}\text{Id}$ is constant, and F_h solves (14.3) *exactly for every* large C . For a general Hermitian–Einstein h , the coefficient of C^{-2} in (14.4) is $\frac{1}{2\pi}\langle \sqrt{-1}\Theta_h, \Psi_h \rangle_\omega$, which depends on the fiber point. The equation for all large C therefore imposes conditions beyond the Hermitian–Einstein condition; we do not pursue them here.

The varying-metric analogue, where $F = F_C$ depends on C , requires uniform curvature bounds, normalization and a compactness argument; see Problem 15.7. The finite- C question is Problem 15.6.

15. OPEN QUESTIONS

There are natural and interesting questions in this framework that remain open and deserve attention. We explicitly state a few of them.

Problem 15.1 (Uniqueness and rigidity). By Proposition 13.5, Finsler–Einstein metrics are not unique, and need not be Hermitian, even on stable bundles. Is uniqueness up to scaling, or at least the Hermitian character of every Finsler–Einstein metric, restored under a large-holonomy condition, for instance when E is stable and the closure of the projective holonomy of its Hermitian–Einstein connection acts transitively on $\mathbb{P}^{r-1}$? Conversely, on a flat unitary bundle, is every Finsler–Einstein metric with $\lambda = 0$ locally independent of z in flat frames, as in Proposition 13.5? When $n = 1$, Finsler–Einstein means $\Psi_F = \lambda\omega$, and the kernel of the closed semipositive form $2\pi\xi_F + \lambda p^*\omega$ is the integrable distribution $\mathcal{H}_F$. Does the flat projective connection determined by $\mathcal{H}_F$ coincide, up to complex gauge or conjugacy in $\text{PGL}(r, \mathbb{C})$, with the projectivization of the Narasimhan–Seshadri connection?

Problem 15.2 (Moduli of Finsler–Einstein metrics). Describe the space of Finsler–Einstein metrics on a polystable bundle. Is it the set of minima of the Donaldson-type functional of [12], and is that functional convex along the $C^{1,1}$ geodesics of [12]?

Problem 15.3 (Equality case without Berndtsson). Give a direct, Finsler-intrinsic proof of polystability from Proposition 11.4: if $\mathcal{H}_F$ is tangent to $\mathbb{P}(\mathcal{S})$ for an equal-slope subsheaf, construct a holomorphic complement.

Problem 15.4 (The Level 1 J -equation at finite C). For fixed large C , when does (10.4) admit a solution $\Omega = \xi_\phi + Cp^*\omega$, with ϕ a fiberwise positive metric on $\mathcal{O}_X(1)$, that is, a Finsler metric? Is solvability for all $C \gg 0$ equivalent to a stability condition on E , for instance an adiabatic refinement of slope stability analogous to (14.2)? The existence theory [27, 5] assumes χ Kähler; here $\chi = p^*\omega$ degenerates along the fibers.

Problem 15.5 (Concavity of the Finsler Ky Fan functional). Is $s \mapsto \int_M \Sigma_s^F dV_\omega$ concave on $\{0, \dots, r\}$ for every strongly pseudoconvex F ? Is it concave pointwise in z ? A positive answer would give $P_E(s) = \frac{1}{2\pi} \inf_F \int_M \Sigma_s^F dV_\omega$ for all s . Is the infimum in Theorem 12.4(i) ever attained by a non-Hermitian F when E is not semistable?

Problem 15.6 (Hessian Hermitian–Einstein metrics on p^*E). For which E and C is the Hermitian–Einstein metric of an Ω_C -polystable p^*E the fiber Hessian of a Finsler metric on E ? Does this force a special structure on E , for example projective flatness?

Problem 15.7 (Varying-metric rigidity). Let F_ν solve (14.3) for $C_\nu \rightarrow \infty$, with F_ν depending on ν . Under which uniform bounds (for instance on Ψ_{F_ν} , on $\Theta_{H_{F_\nu}}$ and on the normalization of G_ν) does a subsequence converge, after constant holomorphic gauge transformations, to a Hermitian–Einstein metric? Is there an approximate version, in which a residual $o(1)$ in (14.3) forces asymptotically Hermitian fibers and a residual $o(C^{-1})$ forces the limiting base metric to be Hermitian–Einstein?

Problem 15.8 (Chern number inequalities). For Finsler–Einstein F , does $\int_M (2rc_2(E) - (r-1)c_1(E)^2) \wedge \omega^{n-2} \geq 0$ follow intrinsically from Finsler Chern forms (cf. the Kobayashi–Lübke type inequality in [11])? By Theorem B it holds, but a Finsler proof would explain which Finsler Chern forms are pointwise controlled.

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