An infinite dimensional version of the Kronecker index and its relation with the Leray--Schauder degree

Z. Anal. Anwend. (Galley proof)
DOI 10.4171/ZAA/1750

An infinite dimensional version of the Kronecker index and its relation with the Leray–Schauder degree

Pierluigi Benevieri, Alessandro Calamai, and Maria Patrizia Pera

Abstract. Let $\mathfrak{f}$ be a compact vector field of class C^1 on a real Hilbert space $\mathbb{H}$. Denote by $\mathbb{B}$ the open unit ball of $\mathbb{H}$ and by $\mathbb{S} = \partial\mathbb{B}$ the unit sphere. Given a point $q \notin \mathfrak{f}(\mathbb{S})$, consider the self-map of $\mathbb{S}$ defined by

$$\mathfrak{f}_q^{\partial}(p) = \frac{\mathfrak{f}(p) - q}{\|\mathfrak{f}(p) - q\|}, \quad p \in \mathbb{S}.$$

If $\mathbb{H}$ is finite dimensional, then $\mathbb{S}$ is an orientable, connected, compact differentiable manifold. Therefore, the Brouwer degree, $\deg_{\mathrm{Br}}(\mathfrak{f}_q^{\partial})$ is well defined, no matter what orientation of $\mathbb{S}$ is chosen, assuming it is the same for $\mathbb{S}$ as domain and codomain of $\mathfrak{f}_q^{\partial}$. This degree may be considered as a modern reformulation of the Kronecker index of the map $\mathfrak{f}_q^{\partial}$. Let $\deg_{\mathrm{Br}}(\mathfrak{f}, \mathbb{B}, q)$ denote the Brouwer degree of $\mathfrak{f}$ on $\mathbb{B}$ with target q. It is known that one has the equality

$$\deg_{\mathrm{Br}}(\mathfrak{f}, \mathbb{B}, q) = \deg_{\mathrm{Br}}(\mathfrak{f}_q^{\partial}).$$

Our purpose is an extension of this formula to the infinite dimensional context. Namely, we will prove that

$$\deg_{\mathrm{LS}}(\mathfrak{f}, \mathbb{B}, q) = \deg_{\mathrm{bf}}(\mathfrak{f}_q^{\partial}),$$

where $\deg_{\mathrm{LS}}(\cdot)$ denotes the Leray–Schauder degree and $\deg_{\mathrm{bf}}(\cdot)$ is the degree earlier introduced by M. Furi and the first author, which extends, to the infinite dimensional case, the Brouwer degree and the Kronecker index. In other words, here, we extend to the Leray–Schauder degree the boundary dependence property which holds for the Brouwer degree in the finite dimensional context.

1. Introduction

The purpose of this paper is to prove that the *boundary dependence property*, which is known to hold for the Brouwer degree in the finite dimensional context, can be extended to the Leray–Schauder degree as well. To illustrate our results, we need to fix some notation that will be used throughout the paper.

Let $\mathbb{H}$ be a real Hilbert space. Denote by $\mathbb{B}$ the open unit ball of $\mathbb{H}$ and by $\mathbb{S}$ the unit sphere $\partial\mathbb{B}$ of $\mathbb{H}$. Let $\mathfrak{f}\colon \mathbb{H} \to \mathbb{H}$ be a compact vector field; namely, a map of the type $I - k$,

Mathematics Subject Classification 2020: 47H11 (primary); 47A53, 55M25, 58C25 (secondary).
Keywords: Kronecker index, Brouwer degree, Leray–Schauder degree, Fredholm maps, Hilbert manifolds.

where I is the identity on $\mathbb{H}$ and k is a compact map, meaning that k is continuous and sends bounded sets into relatively compact sets. Since $\mathfrak{f}$ is a compact perturbation of the identity, which is a proper map on bounded and closed subsets of $\mathbb{H}$, $\mathfrak{f}$ inherits the same property. As a consequence, it maps bounded and closed sets onto bounded and closed sets. By r we mean the *radial retraction* of $\mathbb{H} \setminus \{0\}$ onto $\mathbb{S}$. Namely, r is the smooth map $p \mapsto p/\|p\|$. The *boundary map of* $\mathfrak{f}$ is the restriction $\mathfrak{f}|_{\mathbb{S}}\colon \mathbb{S} \to \mathbb{H}$ of $\mathfrak{f}$ to the boundary $\mathbb{S}$ of $\mathbb{B}$.

If $0 \notin \mathfrak{f}(\mathbb{S})$, then the Leray–Schauder degree of the triple $(\mathfrak{f}, \mathbb{B}, 0)$, respectively, the Brouwer degree when $\mathbb{H}$ is finite dimensional, are well defined. These degrees will be denoted, respectively, by $\deg_{\mathrm{LS}}$ and $\deg_{\mathrm{Br}}$. Moreover, if $0 \notin \mathfrak{f}(\mathbb{S})$, it makes sense to consider the *boundary self-map* $\mathfrak{f}^{\partial}\colon \mathbb{S} \to \mathbb{S}$ of $\mathfrak{f}$ given by the composition $r \circ \mathfrak{f}|_{\mathbb{S}}$.

Observe first that if $0 \notin \mathfrak{f}(\mathbb{S})$ and $\mathbb{H}$ is finite dimensional, then both the Brouwer degrees $\deg_{\mathrm{Br}}(\mathfrak{f}, \mathbb{B}, 0)$ and $\deg_{\mathrm{Br}}(\mathfrak{f}^{\partial})$ are defined; the second one being the degree for maps between oriented, compact, connected, real manifolds of the same finite dimension (see, e.g., [12, 17]). As usual, when $\mathbb{H}$ is finite dimensional, one assumes that the two orientations of $\mathbb{S}$ as domain and codomain of $\mathfrak{f}$ are the same.

As pointed out in [23], there are not so many references devoted to a thorough exposition of the Brouwer degree on manifolds, whereas the *winding number* in two dimensions is a well-known tool in nonlinear analysis. Actually, there is a deep link between these concepts. In fact, we recall that, when $\mathbb{H} = \mathbb{R}^2$, the integer $\deg_{\mathrm{Br}}(\mathfrak{f}^{\partial})$ is called the *winding number (around the origin) of the closed curve* $\mathfrak{f}|_{\mathbb{S}}\colon \mathbb{S} \to \mathbb{R}^2$. For this reason, it is folklore to still call $\deg_{\mathrm{Br}}(\mathfrak{f}^{\partial})$ the *winding number (around the origin) of* $\mathfrak{f}|_{\mathbb{S}}$, whenever $2 < \dim \mathbb{H} < \infty$ and $0 \notin \mathfrak{f}(\mathbb{S})$. The degree $\deg_{\mathrm{Br}}(\mathfrak{f}^{\partial})$ may be thought of as a modern reformulation of the *index of the boundary map* $\mathfrak{f}|_{\mathbb{S}}$ introduced by L. Kronecker in [14]. The Kronecker index, when different from zero, ensures the existence of at least one zero of the map $\mathfrak{f}$ in $\mathbb{B}$ (see, for example, [18]; consider also [10] for details and interesting historical notes).

An important result (see, for example, [8]) implies that if $\mathbb{H}$ is finite dimensional and $0 \notin \mathfrak{f}(\mathbb{S})$, then one gets the following equality between the Brouwer degree of $\mathfrak{f}$ in $\mathbb{B}$ with target zero and the winding number around the origin of $\mathfrak{f}|_{\mathbb{S}}$:

$$\deg_{\mathrm{Br}}(\mathfrak{f}, \mathbb{B}, 0) = \deg_{\mathrm{Br}}(\mathfrak{f}^{\partial}). \tag{1.1}$$

As far as we know, in the literature there is not an infinite dimensional version of this formula, also called the boundary dependence property of the Brouwer degree. In fact, when $\mathbb{H}$ is infinite dimensional and $0 \notin \mathfrak{f}(\mathbb{S})$, only the degree of the triple $(\mathfrak{f}, \mathbb{B}, 0)$ makes sense, and this is the Leray–Schauder degree $\deg_{\mathrm{LS}}(\mathfrak{f}, \mathbb{B}, 0)$. On the other hand, for maps between differentiable manifolds, the Leray–Schauder degree is not defined.

Here, we want to fill this gap by showing that, whatever is the dimension of $\mathbb{H}$, if $\mathfrak{f}$ is a compact vector field of class C^1 such that $0 \notin \mathfrak{f}(\mathbb{S})$, then one gets an extension of equality (1.1). Namely, in our main result, Theorem 4.4 below, we will prove that

$$\deg_{\mathrm{LS}}(\mathfrak{f}, \mathbb{B}, 0) = \deg_{\mathrm{bf}}(\mathfrak{f}^{\partial}), \tag{1.2}$$

where $\deg_{\mathrm{bf}}$ is the degree introduced in [4] for Fredholm maps of index zero between differentiable manifolds, hereafter called *bf-degree* to distinguish it from other classical degrees such as the already mentioned $\deg_{\mathrm{Br}}$ and $\deg_{\mathrm{LS}}$. The definition and properties of the bf-degree can be found in the recent paper [3], and are recalled in Section 2.6 below. In Section 2, we will also show that the definition of the bf-degree is based on a notion of topological orientation for nonlinear Fredholm maps of index zero between finite or infinite dimensional manifolds (see [2, 23] for additional details). As we will see, (1.2) holds if $\mathfrak{f}^{\partial}$ is oriented by a special orientation that we will call *canonical*.

The bf-degree coincides with the Brouwer degree for C^1 maps between finite dimensional oriented manifolds; in the infinite dimensional case and for C^1 compact vector fields, it coincides with the Leray–Schauder degree. Therefore, equality (1.1) can be written, just using the bf-degree, as

$$\deg_{\mathrm{bf}}(\mathfrak{f}, \mathbb{B}, 0) = \deg_{\mathrm{bf}}(\mathfrak{f}^{\partial}), \tag{1.3}$$

which holds with a suitable orientation of $\mathfrak{f}$ on $\mathbb{B}$. In other words, formulas (1.2) and (1.3) are equivalent since $\deg_{\mathrm{bf}}(\mathfrak{f}, \mathbb{B}, 0) = \deg_{\mathrm{LS}}(\mathfrak{f}, \mathbb{B}, 0)$, where $\mathfrak{f}$ has a special orientation that we will call *standard* (see Section 4). As we already pointed out, formulas (1.2) and (1.3) provide an extension to the Leray–Schauder degree and to the bf-degree of the boundary dependence property of the Brouwer degree.

Many authors proposed a definition of an integer-valued degree for Fredholm maps (see, e.g., [1, 19, 23]). Here, we mention the works of Fitzpatrick, Pejsachowicz, and Rabier, who defined in [11] a notion of degree for C^2 Fredholm maps between real Banach manifolds, extended to the C^1 case in [20]. The degree theory in [11, 20] is based on a different notion of orientation. Instead, here we follow the simpler concept of orientation introduced by M. Furi and the first author in [4], which plays a crucial role in our arguments.

As a final remark, we observe that equality (1.2) implies the following easily comprehensible extension of the classical *intermediate value theorem* (cf. [3]): *If the intersection between $\mathfrak{f}(\mathbb{S})$ and a half-line with extreme $q \notin \mathfrak{f}(\mathbb{S})$ is transversal and is the image (under $\mathfrak{f}$) of an odd number of points of $\mathbb{S}$, then any value of the connected component of $\mathbb{H} \setminus \mathfrak{f}(\mathbb{S})$ containing q is assumed by $\mathfrak{f}$ in $\mathbb{B}$. In particular, such a component is bounded, since so is $\mathfrak{f}(\mathbb{B})$.*

2. Preliminaries

In this section, we gather some notation and recall some preliminary notions that we will need later. In particular, we summarize the main concepts related to the degree introduced in [4], hereafter called *bf-degree* as pointed out in the Introduction (see [2, 3, 5, 6] for additional details). A special attention is devoted to the concept of topological orientation for Fredholm maps of index zero between finite or infinite dimensional real differentiable manifolds. Concept that, throughout the paper, we will call *top-orientation*, in order to avoid misunderstanding with the classical *orientation* of finite dimensional differentiable manifolds, as well as with the algebraic orientation for Fredholm linear operators of index zero introduced in [4], here called *alg-orientation*.

2.1. Notation and recalls

Let $\mathbb{H}$ and $\mathbb{K}$ be two real Hilbert spaces. Denote by $\mathcal{L}(\mathbb{H},\mathbb{K})$ the Banach space of the bounded linear operators from $\mathbb{H}$ into $\mathbb{K}$, endowed with the usual operator norm, and, for simplicity, let $\mathcal{L}(\mathbb{H})$ stand for $\mathcal{L}(\mathbb{H},\mathbb{H})$. Let $\mathrm{Iso}(\mathbb{H},\mathbb{K})$ be the open subset of $\mathcal{L}(\mathbb{H},\mathbb{K})$ of the invertible operators; let $\mathrm{GL}(\mathbb{H})$ stand for $\mathrm{Iso}(\mathbb{H},\mathbb{H})$.

The following fact regarding finite dimensional Hilbert spaces belongs to the folklore.

Remark 2.1 (Canonical determinant). Assume that $\mathbb{H}$ and $\mathbb{K}$ have the same finite dimension, and let $L \in \mathcal{L}(\mathbb{H},\mathbb{K})$ be given. If $\mathbb{H}$ and $\mathbb{K}$ are oriented, then the determinant of L, $\det L$, is *canonically well defined*; meaning that it does not depend on the choice of two positively oriented orthonormal bases, one of $\mathbb{H}$ and one of $\mathbb{K}$. Moreover, in the special case when $L \in \mathcal{L}(\mathbb{H})$, then, no matter whether or not $\mathbb{H}$ is oriented, $\det L$ is as well *canonically well defined*; that is, it does not depend on the chosen basis of $\mathbb{H}$, provided it is the same for $\mathbb{H}$ as domain and codomain.

We recall that an operator $L \in \mathcal{L}(\mathbb{H},\mathbb{K})$ is said to be *Fredholm* (see, e.g., [22]) if both its kernel, $\mathrm{Ker}\, L$, and its cokernel, $\mathrm{coKer}\, L = \mathbb{K}/L(\mathbb{H})$, are finite dimensional. In this case, its *index* is the integer

$$\mathrm{ind}\, L = \dim(\mathrm{Ker}\, L) - \dim(\mathrm{coKer}\, L).$$

Note that if $L \in \mathcal{L}(\mathbb{H},\mathbb{K})$ is invertible, then it is Fredholm of index zero. Moreover, any operator in $\mathcal{L}(\mathbb{R}^k,\mathbb{R}^s)$ is Fredholm of index $k-s$.

The subset of $\mathcal{L}(\mathbb{H},\mathbb{K})$ of the Fredholm operators will be denoted by $\Phi(\mathbb{H},\mathbb{K})$. In particular, given $n \in \mathbb{Z}$, by $\Phi_n(\mathbb{H},\mathbb{K})$ we denote the set

$$\{L \in \Phi(\mathbb{H},\mathbb{K}) : \mathrm{ind}\, L = n\}.$$

The symbols $\Phi(\mathbb{H})$ and $\Phi_n(\mathbb{H})$, respectively, stand for $\Phi(\mathbb{H},\mathbb{H})$ and $\Phi_n(\mathbb{H},\mathbb{H})$.

Recall that a continuous map between metrizable spaces is said to be *proper* if the preimage of any compact set is a compact set. Note that proper maps are *closed*, meaning that the image of any closed set is a closed set.

Proposition 2.2. *Here are some important properties of the Fredholm operators:*

(1) *if $L \in \Phi(\mathbb{H},\mathbb{K})$, then the image of L is closed in $\mathbb{K}$;*

(2) *the composition of Fredholm operators is Fredholm and its index is the sum of the indices of the composite operators;*

(3) *if $L \in \Phi(\mathbb{H},\mathbb{K})$, then L is proper on any bounded and closed subset of $\mathbb{H}$;*

(4) *for any $n \in \mathbb{Z}$, the set $\Phi_n(\mathbb{H},\mathbb{K})$ is open in $\mathcal{L}(\mathbb{H},\mathbb{K})$;*

(5) *if $L \in \Phi_n(\mathbb{H},\mathbb{K})$ and $K \in \mathcal{L}(\mathbb{H},\mathbb{K})$ is compact, then $L + K \in \Phi_n(\mathbb{H},\mathbb{K})$.*

It is worth noticing the following useful consequence of property (2).

- If $L \in \Phi_n(\mathbb{H}, \mathbb{K})$ and $k \in \mathbb{N}$, then the restriction of L to a k-codimensional closed subspace of $\mathbb{H}$ is Fredholm of index $n - k$. Analogously, if $\check{\mathbb{K}} \subseteq \mathbb{K}$ contains the image L and has codimension k in $\mathbb{K}$, then the operator L, thought as acting from $\mathbb{H}$ to $\check{\mathbb{K}}$, has index $n + k$.

Let $f\colon W \to \mathbb{K}$ be a C^1 map defined on an open subset of $\mathbb{H}$. Recall that f is said to be *Fredholm of index* $n \in \mathbb{Z}$ if, for all $p \in W$, the Fréchet differential df_p of f at p belongs to $\Phi_n(\mathbb{H}, \mathbb{K})$. In the sequel, we will say that f is a Φ_n*-map*.

Throughout the paper, by a *manifold*, for short, we will mean a smooth, boundaryless differentiable manifold embedded in a real Hilbert space. Note that, consequently, any manifold has an induced Riemannian structure.

If $\mathcal{M}$ is a manifold embedded in $\mathbb{H}$ and $p \in \mathcal{M}$, the tangent space of $\mathcal{M}$ at p, denoted by $T_p\mathcal{M}$, will be identified with a closed subspace of $\mathbb{H}$. In fact, one may regard any tangent vector $\dot{p} \in T_p\mathcal{M}$ as the derivative $\gamma'(0) \in \mathbb{H}$ of a C^1 curve $\gamma\colon (-1, 1) \to \mathcal{M}$ such that

$$\gamma(0) = p.$$

The following remark is a direct consequence of Remark 2.1.

Remark 2.3. If $\mathcal{M}$ and $\mathcal{N}$ are two oriented manifolds of the same finite dimension, and $f\colon \mathcal{M} \to \mathcal{N}$ is C^1, then, for any $p \in \mathcal{M}$, the determinant of the differential of $df_p\colon T_p\mathcal{M} \to T_{f(p)}\mathcal{N}$,

$$\det df_p,$$

is well defined. Consequently, if $\mathcal{N} = \mathcal{M}$, the determinant

$$\det df_p,$$

where $df_p\colon T_p\mathcal{M} \to T_{f(p)}\mathcal{M}$, is well defined as well, no matter what orientation of $\mathcal{M}$ is chosen, assuming it is the same for $\mathcal{M}$ as domain and codomain of f.

Similarly to the case of maps between Hilbert spaces, a C^1 map $f\colon \mathcal{M} \to \mathcal{N}$ between two manifolds is *Fredholm of index* n (see [21]) if so is $df_p\colon T_p\mathcal{M} \to T_{f(p)}\mathcal{N}$, for any $p \in \mathcal{M}$. Such a map will be called a Φ_n*-map (between manifolds)*.

As usual, given a map $f\colon \mathcal{M} \to \mathcal{N}$, we will call, respectively, *points* and *values* the elements in the *domain* $\mathcal{M}$ and the *codomain* $\mathcal{N}$ of f.

If $f\colon \mathcal{M} \to \mathcal{N}$ is C^1, an element $p \in \mathcal{M}$ is said to be a *regular point* if the differential $df_p\colon T_p\mathcal{M} \to T_{f(p)}\mathcal{N}$ is surjective; otherwise, p is a *critical point*. An element $q \in \mathcal{N}$ is a *critical value* if its preimage $f^{-1}(q)$ contains at least one critical point; otherwise, q is a *regular value*.

The celebrated Sard's lemma (see, e.g., [17]) implies that if $\mathcal{M}$ and $\mathcal{N}$ have the same finite dimension and f is C^1, then the set of regular values is dense in $\mathcal{N}$. Thus, by a finite dimensional reduction argument, one can show that the same assertion holds true even when f is a proper Φ_0-map.

2.2. Special linear operators and algebraic orientation

With the symbol $\mathcal{F}(\mathbb{H},\mathbb{K})$, or simply by $\mathcal{F}(\mathbb{H})$ when $\mathbb{K}=\mathbb{H}$, we mean the (not necessarily closed) vector subspace of $\mathcal{L}(\mathbb{H},\mathbb{K})$ of the operators with finite dimensional image.

The symbol I stands for the identity operator acting on any vector space.

Let $L \in \mathcal{L}(\mathbb{H})$ be such that $I - L \in \mathcal{F}(\mathbb{H})$. In this case, we will say that L is an *admissible operator (for the determinant)*. By $\mathcal{A}(\mathbb{H})$ we denote the affine subspace $I - \mathcal{F}(\mathbb{H})$ of $\mathcal{L}(\mathbb{H})$ of the admissible operators. Note that if $\mathbb{H}$ is finite dimensional, then $\mathcal{A}(\mathbb{H}) = \mathcal{L}(\mathbb{H})$. In [13], the determinant of an operator $L \in \mathcal{A}(\mathbb{H})$ is defined as

$$\det L = \det L\big|_{\check{\mathbb{H}}},$$

where $L|_{\check{\mathbb{H}}}$ is the restriction of L (as domain and as codomain) to any finite dimensional subspace $\check{\mathbb{H}}$ of $\mathbb{H}$ containing the image of $I - L$, with the convention that $\det L|_{\check{\mathbb{H}}} = 1$ if $\check{\mathbb{H}} = \{0\}$.

Here are three fundamental properties of the determinant (see, for example, [7] for a discussion about other properties).

Remark 2.4. If $L, L_1, L_2 \in \mathcal{A}(\mathbb{H})$ and $R \in \operatorname{Iso}(\mathbb{H},\mathbb{K})$, then

- $\det L \neq 0$ if and only if L is invertible,
- $L_2L_1 \in \mathcal{A}(\mathbb{H})$ and $\det(L_2L_1) = \det L_2 \det L_1$,
- $RLR^{-1} \in \mathcal{A}(\mathbb{K})$ and $\det(RLR^{-1}) = \det L$.

Let $L \in \mathcal{L}(\mathbb{H})$ with $\mathbb{H} = \mathbb{X} \oplus \mathbb{Y}$ and $\dim \mathbb{X} < +\infty$.

Remark 2.5. Suppose that, according to the above splitting of $\mathbb{H}$, L can be represented in a block matrix form as

$$L = \begin{pmatrix} L_{11} & L_{12} \\ 0 & I_{22} \end{pmatrix},$$

where I_{22} is the identity on $\mathbb{Y}$. Then, the image of $I - L$ is a subset of $\mathbb{X}$. Therefore, applying the definition of determinant, one gets $\det L = \det L_{11}$.

Let $L \in \Phi_0(\mathbb{H},\mathbb{K})$ be given. In [4], an operator $A \in \mathcal{F}(\mathbb{H},\mathbb{K})$ is called a *corrector of* L if $L + A \in \operatorname{Iso}(\mathbb{H},\mathbb{K})$. However, based on the opinion of some colleagues, we agree that the word "corrector" is misleading, since an isomorphism need not to be corrected. Therefore, now we use the more appropriate word *companion*. The fact that, given any $L \in \Phi_0(\mathbb{H},\mathbb{K})$, no matter if it is invertible or not, the set $\mathcal{C}(L)$ of its companions is nonempty is of fundamental importance for the construction of the bf-degree. Moreover, it is crucial for us that any $L \in \operatorname{Iso}(\mathbb{H},\mathbb{K})$ has a *natural companion*: the null operator of $\mathcal{L}(\mathbb{H},\mathbb{K})$.

Definition 2.6 (Equivalence relation of companions). Given $L \in \Phi_0(\mathbb{H},\mathbb{K})$, we say that two companions A and B of L are *L-equivalent* if the determinant of the admissible operator $(L + B)^{-1}(L + A)$ is positive. This is an equivalence relation on the set $\mathcal{C}(L)$ of the companions of L with exactly two equivalence classes.

Based on Definition 2.6 one gets the following concept (see also [3]).

Definition 2.7 (Algebraic orientation of a Φ_0-operator). An *algebraic orientation* of an operator $L \in \Phi_0(\mathbb{H}, \mathbb{K})$, for short called *alg-orientation*, is one of the two equivalence classes of $\mathcal{C}(L)$, denoted by $\mathcal{C}_+(L)$ and called the class of *positive companions* of the *alg-oriented operator* L. The equivalence class $\mathcal{C}(L) \setminus \mathcal{C}_+(L)$ is regarded as *opposite* to $\mathcal{C}_+(L)$.

Three special algebraic orientations are in order.

Definition 2.8 (Natural alg-orientation of an isomorphism). Any $L \in \mathrm{Iso}(\mathbb{H}, \mathbb{K})$ admits the *natural alg-orientation*: the one given by considering the null operator of $\mathcal{L}(\mathbb{H}, \mathbb{K})$ as a positive companion of L.

Definitions 2.9 and 2.10 below regard only the finite dimensional case.

Definition 2.9 (Associated alg-orientation of a linear operator). Let $\mathbb{H}$ and $\mathbb{K}$ have the same finite dimension. Assume that they are oriented up to an inversion of both the orientations (or, equivalently, assume that $\mathbb{H} \times \mathbb{K}$ is oriented). Then, any $L \in \mathcal{L}(\mathbb{H}, \mathbb{K})$ admits the alg-orientation which is *associated with the orientations of* $\mathbb{H}$ *and* $\mathbb{K}$: the one obtained by considering as a positive companion of L any $A \in \mathcal{C}(L)$ such that $L + A$ is orientation preserving.

A particular associated orientation is the following.

Definition 2.10 (Canonical alg-orientation of a linear endomorphism). Let $\mathbb{H}$ be finite dimensional. Then, the orientation of $\mathbb{H} \times \mathbb{H}$ does not depend on the chosen orientation of $\mathbb{H}$. Consequently, given $L \in \mathcal{L}(\mathbb{H})$, its associated alg-orientation is well defined, no matter what is the orientation of $\mathbb{H}$. Therefore, we will say that this alg-orientation is *canonical*. Equivalently, a companion A of L defines the *canonical alg-orientation* if and only if the canonical determinant of $L + A$ is positive.

Recall that if $\mathbb{H}$ and $\mathbb{K}$ are oriented spaces with the same finite dimension, then the *(classical) sign* of any $L \in \mathcal{L}(\mathbb{H}, \mathbb{K})$ is defined as follows (see, for example, [17]):

$$\operatorname{sign} L = \begin{cases} 0 & \text{if } L \text{ is not invertible,} \\ +1 & \text{if } L \text{ is orientation preserving,} \\ -1 & \text{if } L \text{ is orientation reversing.} \end{cases}$$

In particular, if $L \in \mathcal{L}(\mathbb{H})$ and $\mathbb{H}$ is finite dimensional, then $\operatorname{sign} L$ is just the sign of $\det L$, which, in this case, is canonically well defined.

In the infinite dimensional case, in [4] the first author and M. Furi introduced the following concept of sign of an alg-oriented operator, called *bf-sign* in [3] in order to distinguish it from the classical notion.

Definition 2.11 (bf-sign of an alg-oriented operator). Let $L \in \Phi_0(\mathbb{H}, \mathbb{K})$ be alg-oriented. Its *bf-sign* is the integer

$$\operatorname{sign}_{\mathrm{bf}} L = \begin{cases} 0 & \text{if } L \text{ is not invertible,} \\ +1 & \text{if } L \text{ is invertible and naturally alg-oriented,} \\ -1 & \text{if } L \text{ is invertible and not naturally alg-oriented.} \end{cases}$$

The proof of the following remark is left to the reader (see also [3]).

Remark 2.12. Assume that $\mathbb{H}$ and $\mathbb{K}$ have the same finite dimension and are oriented, if $L \in \mathcal{L}(\mathbb{H}, \mathbb{K})$ has the associated alg-orientation, then $\operatorname{sign}_{\mathrm{bf}} L = \operatorname{sign} L$. In particular, if $L \in \mathcal{L}(\mathbb{H})$, the same equality holds when L is canonically alg-oriented.

From Definitions 2.6, 2.8, and 2.11 one gets the following result (see [2] for details).

Proposition 2.13 (Sign test of an invertible operator). *Let $L \in \operatorname{Iso}(\mathbb{H}, \mathbb{K})$ be alg-oriented. Then, given any $A \in \mathcal{C}_+(L)$, one has*

$$\operatorname{sign}_{\mathrm{bf}} L = \operatorname{sign}(\det(L^{-1}(L + A))) = \operatorname{sign}(\det(I + L^{-1}A)).$$

The following notion regards the composition of alg-oriented operators (see [2] for details).

Definition 2.14 (Composition of alg-oriented operators). Let $L_1 \colon \mathbb{H}_1 \to \mathbb{H}_2$ and $L_2 \colon \mathbb{H}_2 \to \mathbb{H}_3$ be two alg-oriented Φ_0-operators. Their *alg-oriented composition* is the operator $L_2 L_1$ endowed with the alg-orientation obtained by considering as a positive companion any operator of the type $(L_2 + B)(L_1 + A) - L_2 L_1$, where A and B are positive companions of L_1 and L_2, respectively.

One can check that the alg-oriented composition is associative (see [2]).

2.3. Pre-orientation and topological orientation in the flat case

We now sketch the main points regarding the concepts of *pre-orientation* and *topological orientation* (*top-orientation* for short) in the flat case: first, we consider continuous maps from a topological space into $\Phi_0(\mathbb{H}, \mathbb{K})$, and then, we deal with Φ_0-maps $f \colon W \to \mathbb{K}$ defined on $W \subseteq \mathbb{H}$ open.

Let $\check{L} \in \Phi_0(\mathbb{H}, \mathbb{K})$ and $A \in \mathcal{C}(\check{L})$ be given. Since $\operatorname{Iso}(\mathbb{H}, \mathbb{K})$ is open in $\mathcal{L}(\mathbb{H}, \mathbb{K})$, $L + A$ is invertible for all L sufficiently close to $\check{L}$. Thus, because of property (5) of Proposition 2.2, $L \in \Phi_0(\mathbb{H}, \mathbb{K})$ and, consequently, $A \in \mathcal{C}(L)$. This argument leads us to the following definition.

Definition 2.15 (Pre-oriented and top-oriented maps of Φ_0-operators in the flat case). Let $\mathcal{X}$ be a topological space and $\Gamma \colon \mathcal{X} \to \Phi_0(\mathbb{H}, \mathbb{K})$ a continuous map. A *pre-orientation* of Γ is a function ω that to any $x \in \mathcal{X}$ assigns an alg-orientation $\omega(x)$ of $\Gamma(x)$. A pre-orientation ω of Γ is a *topological orientation* (*top-orientation* for short) provided it is *continuous* in the following sense: $\check{x} \in \mathcal{X}$ and $A \in \omega(\check{x})$ imply $A \in \omega(x)$ for all x in a neighborhood of $\check{x}$. The map Γ is called *top-orientable* if it admits a top-orientation, and *top-oriented* if a top-orientation has been chosen. A subset $\mathcal{A}$ of $\Phi_0(\mathbb{H}, \mathbb{K})$ is *top-orientable* or *top-oriented* if so is the inclusion map $\mathcal{A} \hookrightarrow \Phi_0(\mathbb{H}, \mathbb{K})$.

Remark 2.16. In some circumstances, it could be useful to give the definition of top-orientation as follows: *A pre-orientation ω of Γ is a top-orientation if for any $\check{x} \in \mathcal{X}$ there exists $A \in \omega(\check{x})$ such that $A \in \omega(x)$ for all x in a neighborhood of $\check{x}$.*

In [5] it is proved that the definition of top-orientation contained in the above remark is equivalent to that given in Definition 2.15. This is a consequence of Lemma 2.17 below (see [5, Lemma 3.1]).

Lemma 2.17. *Let $A, B \in \mathcal{F}(\mathbb{H}, \mathbb{K})$ be two L-equivalent companions of an operator $L \in \Phi_0(\mathbb{H}, \mathbb{K})$. Then, there exist two neighborhoods U_A and U_B of A and B in $\mathcal{F}(\mathbb{H}, \mathbb{K})$ and a neighborhood V_L of L in $\Phi_0(\mathbb{H}, \mathbb{K})$ such that A' and B' are L'-equivalent for any $A' \in U_A$, $B' \in U_B$, $L' \in V_L$.*

Clearly, if $\mathcal{A} \subseteq \Phi_0(\mathbb{H}, \mathbb{K})$ is top-orientable, then so is any subset of $\mathcal{A}$, as well as any continuous map into $\mathcal{A}$. In [5], by means of the theory of covering spaces, it is proved that any simply connected and locally path connected subset of $\Phi_0(\mathbb{H}, \mathbb{K})$ admits exactly two top-orientations (recall that any simply connected topological space is assumed to be path connected). Actually, it is shown that if $\mathcal{X}$ is a simply connected and locally path connected topological space, then any continuous map $\Gamma\colon \mathcal{X} \to \Phi_0(\mathbb{H}, \mathbb{K})$ admits exactly two top-orientations. Moreover, if $\check{x} \in \mathcal{X}$ and α is any of the two algebraic orientations of $\Gamma(\check{x})$, then there exists a unique top-orientation ω of Γ such that $\omega(\check{x}) = \alpha$. In other words, the two top-orientations of Γ are *opposite* each other (see also Definition 2.7).

Observe that the *Leray–Schauder subset* $\mathrm{LS}(\mathbb{H})$ of $\mathcal{L}(\mathbb{H})$ consisting of the compact linear perturbations of the identity is convex. property (5) of Proposition 2.2 shows that $\mathrm{LS}(\mathbb{H})$ consists of Φ_0-operators. Therefore, since it is simply connected and locally path connected, the following definition makes sense.

Definition 2.18 (Standard top-orientation of the Leray–Schauder subset of $\mathcal{L}(\mathbb{H})$). The unique top-orientation ω of $\mathrm{LS}(\mathbb{H})$ whose alg-orientation $\omega(I)$ of the identity is the natural one will be called *standard*. Given any $L \in \mathrm{LS}(\mathbb{H})$, we will say that $\omega(L)$ is the *standard alg-orientation of L*.

We observe that, with the standard orientation, not all the invertible operators of $\mathrm{LS}(\mathbb{H})$ receive the natural orientation. The simplest case is given by splitting

$$\mathbb{H} = \mathbb{H}_1 \oplus (\mathbb{H}_1)^{\perp},$$

in which $\mathbb{H}_1$ has odd finite dimension, and considering L defined as

$$L(x + y) = -x + y$$

(see also [7, Section 5]).

A surprising result of N. Kuiper (see [15]) asserts that if the Hilbert space $\mathbb{H}$ is infinite dimensional and separable, then the linear group $\mathrm{GL}(\mathbb{H})$ is contractible. In [5] it is shown that, in this case, $\Phi_0(\mathbb{H})$ is not top-orientable.

Let $f: W \to \mathbb{K}$ be a Fredholm map of index zero defined on an open subset of $\mathbb{H}$. Since f is at least of class C^1, the following notions of pre-orientation and top-orientation of f make sense.

Definition 2.19 (Pre-oriented and top-oriented Φ_0-maps in the flat case). A *pre-orientation* and a *top-orientation* of a Φ_0-map $f: W \to \mathbb{K}$ are, respectively, a *pre-orientation* and a *top-orientation* of the differential map $df: W \to \Phi_0(\mathbb{H}, \mathbb{K})$, according to Definition 2.15.

A special and important case of a Φ_0-map defined on an open subset W of $\mathbb{H}$ is a C^1 *compact vector field*. Namely, a C^1 map $\mathfrak{f}: W \to \mathbb{H}$ with the property that $k = I - \mathfrak{f}$ is a compact map; that is, a map sending bounded sets into relatively compact sets. Recalling that the differential of a C^1 compact map at a point of its domain is a compact linear operator (see [9]), one gets that, for any $p \in W$, $d\mathfrak{f}_p$ belongs to the Leray–Schauder subset $\mathrm{LS}(\mathbb{H})$. Therefore, the following definition makes sense.

Definition 2.20 (Standard top-orientation of a C^1 compact vector field). Given an open subset W of $\mathbb{H}$ and a C^1 compact vector field $\mathfrak{f}$ on W, the *standard top-orientation of* $\mathfrak{f}$ is the one inherited by $d\mathfrak{f}: W \to \mathrm{LS}(\mathbb{H})$ from the standard top-orientation of $\mathrm{LS}(\mathbb{H})$, according to Definition 2.18.

Note that any operator $L \in \Phi_0(\mathbb{H}, \mathbb{K})$ can also be regarded as a C^1 map from $W = \mathbb{H}$ into $\mathbb{K}$. Therefore, for L we have two different notions of orientation: the alg-orientation (see Definition 2.7) and the top-orientation (see Definition 2.19). Since $dL: \mathbb{H} \to \mathbb{K}$ is the constant map $dL_p = L$ for all $p \in \mathbb{H}$, we will tacitly assume that the two possible orientations, if given, coincide. More precisely: if $\mathcal{C}_+(L)$ is the class of positive companions for L, then it is as well for dL_p for all $p \in \mathbb{H}$.

2.4. Pre-orientation and topological orientation in the non-flat case

The notions of pre-orientation and top-orientation for Φ_0-maps between open sets of real Hilbert spaces can be extended to the case of maps between manifolds.

Definition 2.21 (Pre-oriented Φ_0-maps between manifolds). A *pre-orientation* of a Fredholm map $f: \mathcal{M} \to \mathcal{N}$ of index zero between two manifolds is given by assigning, to any $p \in \mathcal{M}$, an alg-orientation $\omega(p)$ of $df_p: T_p\mathcal{M} \to T_{f(p)}\mathcal{N}$.

A generalization of the above concept, which will be useful in the sequel, can be given by considering a subset C of $\mathcal{M}$.

Definition 2.22. In notation of Definition 2.21, given a subset C of $\mathcal{M}$, a *pre-orientation* of the pair (f, C) is a map that assigns to any $p \in C$ an alg-orientation of $df_p: T_p\mathcal{M} \to T_{f(p)}\mathcal{N}$.

In the notation of the above definitions, the notions of pre-orientation of f and of $(f, \mathcal{M})$ coincide. In this case, we use the simpler language of Definition 2.21.

Definition 2.23 (Pre-oriented composition of Φ_0-maps between manifolds). The *pre-oriented composition* of two (or more) pre-oriented maps between manifolds is given by assigning, at any point p of the domain of the composite map, the composition of the alg-orientations (according to Definition 2.14) of the differentials in the chain of operators representing the differential at p of the composite map.

A pre-orientation of a map between manifolds will be called a top-orientation if it is continuous in the sense specified in the following definition. Before giving Definition 2.24, we point out that from now on we will make the following tacit assumption.

- Any diffeomorphism between manifolds (such as charts and parametrizations) will be tacitly assumed to be *naturally pre-oriented* by assigning the natural alg-orientation to the differential at any point of its domain (recall Definition 2.8).

Definition 2.24 (Top-oriented Φ_0-maps between manifolds). Let $f\colon \mathcal{M} \to \mathcal{N}$ be a Fredholm map of index zero between two manifolds modeled on $\mathbb{H}$ and $\mathbb{K}$, respectively. A pre-orientation of f is a *top-orientation* if it is continuous in the following sense: given any two charts, $\psi\colon V \to \mathbb{K}$ of $\mathcal{N}$ and $\varphi\colon U \to \mathbb{H}$ of $\mathcal{M}$, such that $f(U) \subseteq V$, the pre-oriented composition

$$\psi \circ f \circ \varphi^{-1}\colon \varphi(U) \to \mathbb{K}$$

(in which φ and ψ are naturally pre-oriented) is a top-oriented map, according to Definition 2.19. The function f is said to be *top-orientable* if it admits a top-orientation, and *top-oriented* if a top-orientation has been chosen.

It is useful (as we will see in Section A) to introduce the term "correlated" for different pairs of "objects".

Definition 2.25. According to Definition 2.24, we say that the following notions are correlated:

(1) the top-orientation of the restriction $f|_U$ of f to U with the top-orientation of $\hat{k} := \psi \circ f \circ \varphi^{-1}$;

(2) a companion A_p of df_p, where $p \in U$, with the companion $\widehat{A}_p$ of $d\hat{k}_{\varphi(p)}$ obtained as

$$\widehat{A}_p = d\psi_{f(p)}\, A_p (d\varphi_p)^{-1};$$

see the diagram

$$\begin{array}{ccc} T_p\mathcal{M} & \xrightarrow{A_p} & T_{f(p)}\mathcal{N} \\ {\scriptstyle d\varphi_p}\big\downarrow & & \big\downarrow{\scriptstyle d\psi_{f(p)}} \\ \mathbb{H} & \xrightarrow[\widehat{A}_p]{} & \mathbb{K} \end{array} \tag{2.1}$$

The following remark is an immediate consequence of Definition 2.24.

Remark 2.26. In the notation of Definitions 2.24 and 2.25, if α and $\widehat{\alpha}$ are two correlated top-orientations of $f|_U$ and $\hat{k}$, respectively, a companion A_p of df_p belongs to $\alpha(p)$ if and only if the correlated companion $\widehat{A}_p$ belongs to $\widehat{\alpha}(p)$.

The following orientability result for a Φ_0-map between manifolds includes the flat case in which the first manifold is an open subset of $\mathbb{H}$ and the second one is $\mathbb{K}$ (see [5]). Recall that a simply connected topological space is assumed to be path connected and observe that any manifold is locally path connected.

Proposition 2.27 (Orientability criterion for Φ_0-maps between manifolds). *Let $f: \mathcal{M} \to \mathcal{N}$ be a Φ_0-map between two manifolds, and assume that $\mathcal{M}$ is simply connected. Then, given any $p \in \mathcal{M}$ and one of the two alg-orientations α of df_p, there exists one and only one top-orientation ω of f such that $\omega(p) = \alpha$.*

By the above proposition, we can say that f admits exactly two top-orientations, one *opposite* to the other (see also Definition 2.7).

The special attributes, introduced above, regarding the alg-orientation of some linear operators (see Definitions 2.8 and 2.9) can be adapted to some particular Φ_0-maps between manifolds just by considering their differential. In fact, we introduce the following three notions (see also [3, Definition 3.20]).

Definition 2.28 (Three special top-orientations for Φ_0-maps between manifolds). A special top-orientation of a Φ_0-map between manifolds is the *natural* one, which makes sense whenever $f: \mathcal{M} \to \mathcal{N}$ is a diffeomorphism (or, more generally, a local diffeomorphism): given any $p \in \mathcal{M}$, according to Definition 2.8, one assigns the natural alg-orientation to the differential df_p.

Another special top-orientation, called *associated*, can be given to $f: \mathcal{M} \to \mathcal{N}$ provided it is C^1 and the two manifolds have the same finite dimension and are oriented: according to Definition 2.9, for any $p \in \mathcal{M}$ one considers the associated alg-orientation of df_p.

Finally, the associated top-orientation of a self-map f of a connected, orientable, finite dimensional manifold $\mathcal{M}$ will also be called *canonical*. In fact, it does not depend on the chosen orientation of $\mathcal{M}$, provided it is the same for $\mathcal{M}$ as domain and codomain of f.

2.5. Families and homotopies of Φ_0-maps between manifolds

Definition 2.24 can be extended in order to obtain a notion of top-orientation for homotopies or, more generally, for continuous families of Fredholm maps of index zero between manifolds.

Let $\mathcal{M}$ and $\mathcal{N}$ be two given manifolds modeled on $\mathbb{H}$ and $\mathbb{K}$, respectively, and let Λ denote a simply connected and locally path connected topological space.

Definition 2.29 (Top-oriented families of Φ_0-maps between manifolds). A continuous map $\mathcal{H}: \mathcal{M} \times \Lambda \to \mathcal{N}$ is a Φ_0-*family* if it is continuously differentiable with respect to the first variable and any *partial map* $\mathcal{H}_\lambda = \mathcal{H}(\cdot, \lambda)$, $\lambda \in \Lambda$, is a Φ_0-map. A *top-orientation* of $\mathcal{H}$ is a *continuous* family $\{\omega_\lambda\}$ of top-orientations of all the Φ_0-maps of the family $\{\mathcal{H}_\lambda\}$;

where "continuous" means that, given any open subset Ξ of Λ and any two charts, $\varphi\colon U \to \mathbb{H}$ of $\mathcal{M}$ and $\psi\colon V \to \mathbb{K}$ of $\mathcal{N}$, whenever $\mathcal{H}(U \times \Xi) \subseteq V$, the alg-oriented composition

$$d\psi_{\mathcal{H}(p,\lambda)}\, d(\mathcal{H}_\lambda)_p (d\varphi_p)^{-1} \in \Phi_0(\mathbb{H},\mathbb{K}),$$

in which (we recall) the isomorphisms $(d\varphi_p)^{-1}$ and $d\psi_{\mathcal{H}(p,\lambda)}$ are naturally alg-oriented, depends continuously on $(p,\lambda) \in U \times \Xi$ according to Definition 2.15. The Φ_0-family $\mathcal{H}$ is said to be *top-orientable* if it admits a top-orientation, and *top-oriented* if a top-orientation has been chosen.

One obtains an important particular Φ_0-family when the parameter space is the interval $[0,1]$. In this case, $\mathcal{H}$ is called a Φ_0*-homotopy* and the partial maps $\mathcal{H}_0$ and $\mathcal{H}_1$ are said to be Φ_0*-homotopic*.

The following is a crucial result concerning the transport of the top-orientation along Φ_0-homotopies. The proof can be found in [5] and is based on the theory of covering spaces.

Proposition 2.30 (Transport of the top-orientations along Φ_0-homotopies). *Assume that for a given $t \in [0,1]$ the partial map $\mathcal{H}_t$ of a Φ_0-homotopy $\mathcal{H}$ from $\mathcal{M}$ to $\mathcal{N}$ has a top-orientation α. Then, there exists one and only one top-orientation ω of $\mathcal{H}$ such that the (partial) top-orientation ω_t of $\mathcal{H}_t$ coincides with α. In particular, if two maps from $\mathcal{M}$ to $\mathcal{N}$ are Φ_0-homotopic, then they are both top-orientable or both not top-orientable.*

From Proposition 2.30, we deduce that any C^1 self-map $f\colon \mathcal{M} \to \mathcal{M}$ which is Φ_0-homotopic to the identity is top-orientable. In fact, the identity, being a diffeomorphism, admits the natural top-orientation (even when $\mathcal{M}$ is a non-orientable finite dimensional manifold).

A simple example of a not top-orientable Φ_0-map is given by a constant map f from a non-orientable compact manifold $\mathcal{M}$ into itself (see [5]). Observe that the fact that f is not top-orientable agrees with the non-contractibility of $\mathcal{M}$.

Again, by means of the theory of covering spaces, one obtains the following generalization of Proposition 2.30 to Φ_0-families. The proof is similar to the one given in [5] for Φ_0-homotopies, therefore, it is omitted.

Proposition 2.31 (Spread of the top-orientations for families of Φ_0-maps). *Let $\mathcal{H}\colon \mathcal{M} \times \Lambda \to \mathcal{N}$ be as in Definition 2.29. Assume that, for a given $\lambda \in \Lambda$, the partial map $\mathcal{H}_\lambda$ has a top-orientation α. Then, there exists one and only one top-orientation ω of $\mathcal{H}$ such that the partial top orientation ω_λ coincides with α.*

2.6. Topological degree for strictly admissible triples

The *bf-degree* introduced in [4] is an integer valued function, $\deg_{\mathrm{bf}}(\cdot)$, defined on a wide class of triples, called *bf-admissible*, and satisfying some important properties which, for simplicity, we do not to mention here (see [3,5,6,23]). Instead, we will focus our attention only on a subclass of the bf-admissible triples, called *strictly bf-admissible*, which are necessary, as well as sufficient, for the understanding and proof of some of our results.

We recall first the notion of bf-admissible triple: given a top-oriented Φ_0-map $f: \mathcal{M} \to \mathcal{N}$, an open (possibly empty) subset U of $\mathcal{M}$, and a *target value* $q \in \mathcal{N}$, the triple (f, U, q) is said to be *bf-admissible* if $U \cap f^{-1}(q)$ is compact.

Definition 2.32 (Strictly bf-admissible triple). Let U be an open subset of $\mathcal{M}$, $f: \overline{U} \to \mathcal{N}$ a continuous map which is Φ_0 on U, and $q \in \mathcal{N}$ a target value. The triple (f, U, q) will be called *strictly bf-admissible* if it satisfies the following properties:

(1) f is proper on $\overline{U}$,

(2) $q \notin f(\partial U)$,

(3) f is top-oriented on U.

With the notation and assumptions of Definition 2.32, observe that U is a manifold and the compact set $f^{-1}(q) \cap \overline{U}$ is contained in U. Therefore, the triple (f, U, q) is bf-admissible and, consequently, the integer $\deg_{\mathrm{bf}}(f, U, q)$ is well defined.

The following properties of the restriction of bf-degree to the class of the strictly bf-admissible triples are in order.

- (Additivity) *Let (f, U, q) be strictly bf-admissible. If U_1 and U_2 are two disjoint open subsets of U such that $U \cap f^{-1}(q) \subseteq U_1 \cup U_2$, then*

$$\deg_{\mathrm{bf}}(f, U, q) = \deg_{\mathrm{bf}}(f, U_1, q) + \deg_{\mathrm{bf}}(f, U_2, q).$$

- (Homotopy invariance) *Let U be an open subset of $\mathcal{M}$, and $\mathcal{H}: \overline{U} \times [0, 1] \to \mathcal{N}$ a proper homotopy. Assume that the restriction of $\mathcal{H}$ to $U \times [0, 1]$ is a top-oriented Φ_0-homotopy. If $\gamma: [0, 1] \to \mathcal{N}$ is a continuous path such that $\gamma(t) \notin \mathcal{H}_t(\partial U)$ for all $t \in [0, 1]$, then $\deg_{\mathrm{bf}}(\mathcal{H}_t, U, \gamma(t))$ does not depend on $t \in [0, 1]$.*
- (Excision) *If (f, U, q) is strictly bf-admissible and V is an open subset of U such that $f^{-1}(q) \cap U \subseteq V$, then*

$$\deg_{\mathrm{bf}}(f, U, q) = \deg_{\mathrm{bf}}(f, V, q).$$

- (Computation formula) *If (f, U, q) is strictly bf-admissible and q is a regular value for f in U, then the set $U \cap f^{-1}(q)$ is finite and*

$$\deg_{\mathrm{bf}}(f, U, q) = \sum_{p \in U \cap f^{-1}(q)} \operatorname{sign}_{\mathrm{bf}} df_p,$$

  with the convention that the sum is zero if $U \cap f^{-1}(q)$ is empty.
- (Continuous dependence) *Let $f: \mathcal{M} \to \mathcal{N}$ be a top-oriented Φ_0-map and U an open subset of $\mathcal{M}$. If f is proper on the closure of U, then the function $\deg_{\mathrm{bf}}(f, U, \cdot): \mathcal{N} \setminus f(\partial U) \to \mathbb{Z}$ is locally constant.*

Observe that if $\deg_{\mathrm{bf}}(f, U, q) \neq 0$, then the equation $f(p) = q$ admits at least one solution in U. In fact, if $U \cap f^{-1}(q)$ is empty, then q is a regular value for f in U, and the computation formula applies.

Notice also that the Computation Formula shows that the integer $\deg_{\mathrm{bf}}(f, U, q)$—when q is a regular value—is an algebraic count of the solutions in U of the equation $f(p) = q$.

Let $(f, \mathcal{M}, q)$ be strictly bf-admissible and assume that $\mathcal{N}$ is connected. Then, the Continuous Dependence property implies that $\deg_{\mathrm{bf}}(f, \mathcal{M}, q)$ does not depend on the target $q \in \mathcal{N}$. Thus, we will adopt the following notation.

Notation 2.33. Let $f : \mathcal{M} \to \mathcal{N}$ be a proper top-oriented Φ_0-map. The symbol $\deg_{\mathrm{bf}}(f)$ stands for $\deg_{\mathrm{bf}}(f, \mathcal{M}, q)$, with $q \in \mathcal{N}$ arbitrary, provided that $\mathcal{N}$ is connected and the degree is considered in the whole domain of f.

As a simple example of degree for strictly bf-admissible triples consider a complex polynomial P of degree $n > 0$. Observe that $P : \mathbb{C} \to \mathbb{C}$ is a proper map, since $|P(z)| \to \infty$ as $|z| \to \infty$. Thus, regarding P as a self-map of $\mathbb{R}^2$ with the canonical top-orientation (recall Definition 2.28) and given any $q \in \mathbb{R}^2$, the triple $(P, \mathbb{R}^2, q)$ is strictly bf-admissible (see Definition 2.32). Since $\mathbb{R}^2$ is the whole domain of P and is connected, according to Notation 2.33 we may simply write $\deg_{\mathrm{bf}}(P)$ instead of $\deg_{\mathrm{bf}}(P, \mathbb{R}^2, q)$. We claim that $\deg_{\mathrm{bf}}(P)$ is the same as the algebraic degree of P. To prove this, observe first that, given any $z \in \mathbb{C}$, one has $dP_z(\dot{z}) = P'(z)\dot{z}$, $\dot{z} \in \mathbb{C}$. Consequently, whenever the derivative $P'(z)$ is different from zero, the differential dP_z is orientation preserving and, due to the canonical top-orientation of P, its bf-sign is 1. To compute the bf-degree of P, one may consider a regular value q of P and apply the computation formula. In this case, because of the fundamental theorem of algebra and the factorization of a polynomial, the equation $P(z) = q$ has exactly n solutions. Since the bf-sign of the differential dP at any of these solutions is 1, we get $\deg_{\mathrm{bf}}(P) = n$, as claimed.

Willingly, one may prove the same equality avoiding the use of the Fundamental Theorem of Algebra, and getting one of the many proofs of such a beautiful theorem. In fact, one can check that there exists a proper Φ_0-homotopy joining P with a monomial of the type $Q(z) = az^n$, with $a \neq 0$. Thus, because of the homotopy invariance property of the bf-degree, one gets $\deg_{\mathrm{bf}}(P) = \deg_{\mathrm{bf}}(Q)$, and the fact that $\deg_{\mathrm{bf}}(Q) = n$ can be proved by considering the well-known n solutions of the equation $az^n = a$.

Another important example is given by Leray–Schauder C^1-triples. Namely, triples $(\mathfrak{f}, U, q)$, where $\mathfrak{f}$ is a (continuous) compact vector field on the closure $\overline{U}$ of a bounded open subset U of $\mathbb{H}$, $q \notin \mathfrak{f}(\partial U)$, and the restriction of $\mathfrak{f}$ to U is of class C^1 and standardly top-oriented (see Definition 2.20). With this top-orientation, because of the Computation Formula of the bf-degree, we get the following result that can be found in [1].

Proposition 2.34 (The bf-degree and the LS-degree). *Let $\mathfrak{f}$ be a compact vector field on the closure $\overline{U}$ of a bounded open subset U of $\mathbb{H}$. Assume that $\mathfrak{f}$ is C^1 on U with the standard top-orientation. Then, given any $q \notin \mathfrak{f}(\partial U)$, one has*

$$\deg_{\mathrm{LS}}(\mathfrak{f}, U, q) = \deg_{\mathrm{bf}}(\mathfrak{f}, U, q).$$

3. Compact and finitely perturbed vector fields

From now on, the Hilbert space $\mathbb{H}$ will be assumed to be infinite dimensional. By $\mathbb{X}$ we will denote a finite dimensional subspace of $\mathbb{H}$ and $\mathbb{Y} = \mathbb{X}^{\perp}$. Then, one has $\mathbb{H} = \mathbb{X} \oplus \mathbb{Y}$. As in the Introduction, here and in the rest of the paper, $\mathbb{B}$ and $\mathbb{S}$ will stand for the unit open ball and the unit sphere of $\mathbb{H}$. In addition, we will always assume that $\dim \mathbb{X} \geq 3$, implying that the unit sphere $\mathbb{S} \cap \mathbb{X}$ of $\mathbb{X}$ is simply connected. For a better comprehension of the statements and the proofs in the sequel, it is useful to distinguish the following special subsets of the space $\mathbb{H}$.

- *Equatorial space*: $\mathbb{X}$.
- *Polar space*: $\mathbb{Y}$.
- *Equatorial sphere*: $\mathbb{E} = \mathbb{S} \cap \mathbb{X}$.

Consider a compact vector field $\mathfrak{f}$ on $\mathbb{H}$. As pointed out in the Introduction, since the identity I is a proper map, $\mathfrak{f}$ is as well proper on any bounded and closed subset of $\mathbb{H}$. Hence, it inherits the same important property. Moreover, as one can check, $\mathfrak{f}$ maps bounded sets into bounded sets. This implies that the image, under $\mathfrak{f}$, of any bounded and closed subset of $\mathbb{H}$ is as well bounded and closed. This is the case of $\mathfrak{f}(\mathbb{S})$.

Denote by r the radial retraction of $\mathbb{H} \setminus \{0\}$ onto the unit sphere $\mathbb{S}$. Observe that $r\colon p \mapsto p/\|p\|$ is a smooth map, since so is the norm function on $\mathbb{H} \setminus \{0\}$.

If $0 \notin \mathfrak{f}(\mathbb{S})$, then an important map can be introduced as follows.

Definition 3.1 (The boundary self-map (of $\mathfrak{f}$)). Given a compact vector field $\mathfrak{f}$ on $\mathbb{H}$, assume that $0 \notin \mathfrak{f}(\mathbb{S})$. Then, the map

$$\mathfrak{f}^{\partial}\colon \mathbb{S} \to \mathbb{S},$$

defined by the composition $r \circ \mathfrak{f}|_{\mathbb{S}}$, will be called *the boundary self-map (of $\mathfrak{f}$)*.

The proof of the following result, which is crucial for us, can be found in [3] (cf. [3, Lemmas 5.1 and 5.4]).

Proposition 3.2. *If the compact vector field $\mathfrak{f}$ is such that $0 \notin \mathfrak{f}(\mathbb{S})$, then the boundary self-map $\mathfrak{f}^{\partial}$ is proper. If, in addition, $\mathfrak{f}$ is C^1, then $\mathfrak{f}^{\partial}$ is Fredholm of index* 0.

From Proposition 3.2, one gets that if $\mathfrak{f}$ is of class C^1, top-oriented and $0 \notin \mathfrak{f}(\mathbb{S})$, then $\deg_{\mathrm{bf}}(\mathfrak{f}^{\partial})$ is well defined, according to Notation 2.33. However, since $\mathfrak{f}^{\partial}$ admits exactly two top-orientations, opposite to each other (Proposition 2.27), one has the following remark.

Remark 3.3. Let $\mathfrak{f}$ be of class C^1 such that $0 \notin \mathfrak{f}(\mathbb{S})$. Then, $\deg_{\mathrm{bf}}(\mathfrak{f}^{\partial})$ is defined up to a sign, depending on the chosen top-orientation of $\mathfrak{f}^{\partial}$.

Let us now consider the subclass of the *finitely perturbed vector fields*.

Definition 3.4. A compact vector field $\mathfrak{g}\colon \mathbb{H} \to \mathbb{H}$ will be called a *finitely perturbed vector field* if the image $h(\mathbb{H})$ of the map $h = I - \mathfrak{g}$ is contained in a finite dimensional subspace of $\mathbb{H}$.

Let us point out that if $\mathfrak{g}$ is a finitely perturbed vector field, then the perturbing map $h = I - \mathfrak{g}$ sends bounded sets into bounded sets.

Hereafter, by $\mathfrak{g} = I - h$ we denote a finitely perturbed vector field such that $h(\mathbb{H}) \subseteq \mathbb{X}$.

Regarding $\mathfrak{g}$ and its relation with the splitting $\mathbb{H} = \mathbb{X} \oplus \mathbb{Y}$ one has further results. The proof of the following remark is left to the reader.

Remark 3.5. The map $\mathfrak{g}$ sends $\mathbb{X}$ into itself. Moreover, the preimage $\mathfrak{g}^{-1}(q)$ of any value $q \in \mathbb{X}$ is contained in $\mathbb{X}$.

The following definition is justified by the fact that, according to Remark 3.5, $\mathfrak{g}$ maps $\mathbb{X}$ into itself.

Definition 3.6. If $0 \notin \mathfrak{g}(\mathbb{E})$, then the composition $r \circ \mathfrak{g}|_{\mathbb{E}}$ will be called the *boundary equatorial self-map (of* $\mathfrak{g}$*)* and denoted by $(\mathfrak{g}|_{\mathbb{X}})^{\partial}\colon \mathbb{E} \to \mathbb{E}$.

Notice that if $\mathfrak{g}$ is C^1 and $0 \notin \mathfrak{g}(\mathbb{S})$, then $(\mathfrak{g}|_{\mathbb{X}})^{\partial}$ is Fredholm of index zero, having domain and codomain of the same finite dimension.

Lemma 3.7. *If* $\mathfrak{g}$ *is* C^1 *and* $0 \notin \mathfrak{g}(\mathbb{S})$*, then a point* p *of* $\mathbb{E}$ *is regular for the boundary self-map* $\mathfrak{g}^{\partial}\colon \mathbb{S} \to \mathbb{S}$ *if and only if so is for the boundary equatorial self-map* $(\mathfrak{g}|_{\mathbb{X}})^{\partial}\colon \mathbb{E} \to \mathbb{E}$.

The proof requires the following three technical remarks. As usual, given $p \in \mathbb{E}$, we denote by $T_p\mathbb{S}$ or $p^{\perp}$ the tangent space of $\mathbb{S}$ at p, and by $T_p\mathbb{E}$ the tangent space of $\mathbb{E}$ at p.

Remark 3.8. Given any $p \in \mathbb{E}$, due to the fact that p is orthogonal to $\mathbb{Y}$, the tangent space of $\mathbb{S}$ at p can be represented as

$$p^{\perp} = (p^{\perp} \cap \mathbb{X}) \oplus \mathbb{Y}. \tag{3.1}$$

Henceforth, any tangent vector $\dot{p}$ to $\mathbb{S}$ at p can be written as $\dot{p} = \dot{x} + \dot{y}$, with $\dot{x} \in \mathbb{X}$ and $\dot{y} \in \mathbb{Y}$. Clearly, $\dot{y} = 0$ if (and only if) $\dot{p} \in T_p\mathbb{E}$.

Remark 3.9. Assume that $\mathfrak{g} = I - h$ is of class C^1. Its differential $d\mathfrak{g}_p$ at any point $p \in \mathbb{H}$ is given by

$$d\mathfrak{g}_p(\dot{p}) = \dot{p} - dh_p(\dot{p}),$$

with $\dot{p} \in \mathbb{H}$. Thus, putting $\dot{p} = \dot{x} + \dot{y}$ with $\dot{x}$ and $\dot{y}$ in $T_p\mathbb{X} = \mathbb{X}$ and $T_p\mathbb{Y} = \mathbb{Y}$, respectively, one gets

$$d\mathfrak{g}_p(\dot{p}) = d\mathfrak{g}_p(\dot{x}) + d\mathfrak{g}_p(\dot{y}) = d(\mathfrak{g}|_{\mathbb{X}})_p(\dot{x}) + \dot{y} - dh_p(\dot{y}),$$

with $d(\mathfrak{g}|_{\mathbb{X}})_p(\dot{x}) - dh_p(\dot{y}) \in \mathbb{X}$ and $\dot{y} \in \mathbb{Y}$.

Remark 3.10. The differential $dr_{\hat{q}}$ at any $\hat{q} \in H \setminus \{0\}$ of the radial retraction r is $\frac{1}{\|\hat{q}\|}\Pi_{q^{\perp}}$, where $\Pi_{q^{\perp}}$ is the orthogonal projection of $\mathbb{H}$ onto the tangent space $q^{\perp}$ of $\mathbb{S}$ at $q = r(\hat{q})$.

Proof of Lemma 3.7. Given $p \in \mathbb{E}$, put $q = \mathfrak{g}^\partial(p) \in \mathbb{E}$. It is sufficient to prove that the differential of $\mathfrak{g}^\partial$ at p,

$$d(\mathfrak{g}^\partial)_p\colon T_p\mathbb{S} \to T_q\mathbb{S},$$

and its restriction

$$d(\mathfrak{g}^\partial)_p|_{T_p\mathbb{E}} = d((\mathfrak{g}|_{\mathbb{X}})^\partial)_p\colon T_p\mathbb{E} \to T_q\mathbb{E}$$

are both injective or both non-injective. In fact, the two linear operators, which are Fredholm of index zero (see Proposition 3.2), are injective if and only if they are surjective (and when this holds, by definition, p is a regular point).

Obviously, if $d(\mathfrak{g}^\partial)_p$ is injective, so is its restriction $d((\mathfrak{g}|_{\mathbb{X}})^\partial)_p$. Let us prove the converse implication.

Assume that $d((\mathfrak{g}|_{\mathbb{X}})^\partial)_p$ is injective and observe that if $\dot{x} \in \mathbb{X}$, then $d(\mathfrak{g}^\partial)_p(\dot{x})$ is the same as $d((\mathfrak{g}|_{\mathbb{X}})^\partial)_p(\dot{x})$. Therefore, denoting $\hat{q} = \mathfrak{g}(p)$, taking into account Remark 3.9, and putting $\dot{p} = \dot{x} + \dot{y}$ as in Remark 3.8, we may write

$$d(\mathfrak{g}^\partial)_p(\dot{p}) = dr_{\hat{q}}(d\,\mathfrak{g}_p(\dot{p})) = d((\mathfrak{g}|_{\mathbb{X}})^\partial)_p(\dot{x}) + \dot{y} - dr_{\hat{q}}(dh_p(\dot{y})),$$

where $h = I - \mathfrak{g}$. It remains to prove that if $d((\mathfrak{g}|_{\mathbb{X}})^\partial)_p(\dot{x}) + \dot{y} - dr_{\hat{q}}(dh_p(\dot{y})) = 0$, then $\dot{x} = 0$ and $\dot{y} = 0$.

In fact, since $d((\mathfrak{g}|_{\mathbb{X}})^\partial)_p(\dot{x}) - dr_{\hat{q}}(dh_p(\dot{y})) \in \mathbb{X}$ (see Remarks 3.9 and 3.10) and $\dot{y} \in \mathbb{Y}$, we obtain

$$d((\mathfrak{g}|_{\mathbb{X}})^\partial)_p(\dot{x}) - dr_{\hat{q}}(dh_p(\dot{y})) = 0 \quad \text{and} \quad \dot{y} = 0.$$

Finally, from $\dot{y} = 0$ one gets $dr_{\hat{q}}(dh_p(\dot{y})) = 0$ and, consequently,

$$d((\mathfrak{g}|_{\mathbb{X}})^\partial)_p(\dot{x}) = 0,$$

which by assumption implies $\dot{x} = 0$. ■

The following result is a consequence of Remark 3.5 and Lemma 3.7.

Corollary 3.11. *If $\mathfrak{g}$ is C^1 and $0 \notin \mathfrak{g}(\mathbb{S})$, then a value q of the equatorial sphere $\mathbb{E}$ is regular for $(\mathfrak{g}|_{\mathbb{X}})^\partial$ if and only if so is for $\mathfrak{g}^\partial$.*

Let $\mathfrak{g}$ be C^1 such that $0 \notin \mathfrak{g}(\mathbb{S})$. Since the equatorial sphere $\mathbb{E}$ is an orientable, finite dimensional connected manifold, the boundary equatorial self-map $(\mathfrak{g}|_{\mathbb{X}})^\partial$ admits the canonical top-orientation (see Definition 2.28). On the other hand, $\mathfrak{g}^\partial$ is Fredholm of index zero (Proposition 3.2) and has a simply connected domain. Therefore it is top-orientable, admitting exactly two top-orientations. Our purpose, in the rest of this section, is to define a particular top-orientation of $\mathfrak{g}^\partial$, induced by the canonical top-orientation of $(\mathfrak{g}|_{\mathbb{X}})^\partial$ and consequently called "canonical".

Lemma 3.12. *Let $\mathfrak{g}$ be C^1 and $0 \notin \mathfrak{g}(\mathbb{S})$. Given any $p \in \mathbb{E}$, put $q = \mathfrak{g}^\partial(p) \in \mathbb{E}$. Suppose that $A_p\colon T_p\mathbb{E} \to T_q\mathbb{E}$ is a companion of the differential $d((\mathfrak{g}|_{\mathbb{X}})^\partial)_p$. Then, the linear operator $A_p^s\colon T_p\mathbb{S} \to T_q\mathbb{S}$, defined by $A_p^s(\dot{x} + \dot{y}) = A_p\dot{x}$, where $\dot{x} \in \mathbb{X}$ and $\dot{y} \in \mathbb{Y}$, is a companion of $d(\mathfrak{g}^\partial)_p$.*

Proof. The operator A_p^s has the same finite dimensional image as A_p. Thus, it is sufficient to show that $L_p^s = d(\mathfrak{g}^\partial)_p + A_p^s$ is invertible. To see this, since L_p^s is a Φ_0-operator (see property (5) of Proposition 2.2), it is enough to check that L_p^s is injective. This can be done as in the proof of Lemma 3.7. ■

We will call A_p^s the *normal extension* of A_p. We can generalize this concept in the following definition.

Definition 3.13. Let $T: \mathbb{X} \to \mathbb{H}$ be a linear operator. The operator $T^s: \mathbb{H} \to \mathbb{H}$ defined as $T^s(v + w) = Tv$, where $v \in \mathbb{X}$ and $w \in \mathbb{Y}$, is called the *normal extension* of T.

Remark 3.14. With the notation of Lemma 3.12 and in the sense of Definition 2.6, one has the following assertion: if A and B are $d((\mathfrak{g}|_{\mathbb{X}})^\partial)_p$-equivalent, then the normal extensions A^s and B^s are $d(\mathfrak{g}^\partial)_p$-equivalent. The converse is also true. Therefore, to any pre-orientation α of $(\mathfrak{g}|_{\mathbb{X}})^\partial$ correspond one and only one pre-orientation α^s of the pair $(\mathfrak{g}^\partial, \mathbb{E})$ (recall Definition 2.22). The pre-orientation α^s will be called the *normal extension* of α.

In the case when α is a top-orientation of $(\mathfrak{g}|_{\mathbb{X}})^\partial$, the following crucial result says that α not only induces a pre-orientation on $(\mathfrak{g}^\partial, \mathbb{E})$, but also induces a top-orientation on $\mathfrak{g}^\partial$. The proof of Theorem 3.15 below requires some technical steps and it is postponed to the appendix (see Section A).

Theorem 3.15. *Let $\mathfrak{g}$ be C^1 and such that $0 \notin \mathfrak{g}(\mathbb{S})$. Assume that $(\mathfrak{g}|_{\mathbb{X}})^\partial$ is canonically top-oriented and denote by ω its canonical top-orientation. Denote by ω^s the pre-orientation of $(\mathfrak{g}^\partial, \mathbb{E})$, which is the normal extension of ω, according to Remark 3.14. Then, there exists one and only one top-orientation of $\mathfrak{g}^\partial$ that coincides with ω^s at any point of $\mathbb{E}$.*

Based on the above theorem, we can now present the definition of canonical top-orientation of $\mathfrak{g}^\partial$.

Definition 3.16 (Canonical top-orientation of $\mathfrak{g}^\partial$). Let $\mathfrak{g}$ be C^1 and such that $0 \notin \mathfrak{g}(\mathbb{S})$. Assume that $(\mathfrak{g}|_{\mathbb{X}})^\partial$ is canonically top-oriented and call ω its canonical top-orientation. Denote by ω^s the pre-orientation of $(\mathfrak{g}^\partial, \mathbb{E})$, which is the normal extension of ω, according to Remark 3.14. The *canonical top-orientation of* $\mathfrak{g}^\partial$ is the one that coincides with ω^s at any point of $\mathbb{E}$.

To obtain a result on the transport of the canonical top-orientation in the infinite dimensional setting (see Proposition 3.18 below), we need the following lemma about top-oriented Φ_0-homotopies of self-maps in the finite dimensional case.

Lemma 3.17. *Let $\mathcal{M}$ be an orientable, connected, finite dimensional manifold and $\mathcal{K}: \mathcal{M} \times [0, 1] \to \mathcal{M}$ a top-oriented Φ_0-homotopy. Then, if one partial top-orientation of $\mathcal{K}$ is canonical, so are all the others.*

Proof. Call $\mathbb{X}'$ a finite dimensional Hilbert space which $\mathcal{M}$ is modeled on. For any $t \in [0,1]$, denote by $\mathcal{K}_t$ the partial map $\mathcal{K}_t(\cdot) = \mathcal{K}(\cdot, t)$. Let $\bar{t} \in [0,1]$ be given and assume that $\mathcal{K}_{\bar{t}}$ is canonically top-oriented. Fix $\bar{p} \in \mathcal{M}$ and take two charts of $\mathcal{M}$, (U, φ) at $\bar{p}$ and (V, ψ) at $\bar{q} := \mathcal{K}(\bar{p}, \bar{t})$. For a suitable neighborhood J of $\bar{t}$, assume, without loss of generality, that $\mathcal{K}(U \times J) \subseteq V$.

Let $A_{(\bar{p},\bar{t})}: T_{\bar{p}}\mathcal{M} \to T_{\bar{q}}\mathcal{M}$ be a positive companion of $d(\mathcal{K}_{\bar{t}})_{\bar{p}}$ and call $\widehat{A}_{(\bar{p},\bar{t})}: \mathbb{X}' \to \mathbb{X}'$ the correlated operator according to Definition 2.25. See also the following commutative diagram:

$$\begin{array}{ccc} T_{\bar{p}}\mathcal{M} & \xrightarrow{A_{(\bar{p},\bar{t})}} & T_{\bar{q}}\mathcal{M} \\ {\scriptstyle d(\varphi_{\bar{t}})_{\bar{p}}}\downarrow & & \downarrow{\scriptstyle d\psi_{\bar{q}}} \\ \mathbb{X}' & \xrightarrow[\widehat{A}_{(\bar{p},\bar{t})}]{} & \mathbb{X}' \end{array}$$

Consider the map $\hat{k}: \varphi(U) \times J \to \mathbb{X}'$ given by

$$\hat{k}(x,t) = \hat{k}_t(x) := \psi(\mathcal{K}_t(\varphi^{-1}(x))).$$

Taking U and J smaller if necessary, assume that $\widehat{A}_{(\bar{p},\bar{t})}$ is a companion of $d(\hat{k}_t)_x$ for every $(x,t) \in \varphi(U) \times J$. Consequently, the determinant of $d(\hat{k}_t)_x + \widehat{A}_{(\bar{p},\bar{t})}$ does not change its sign in $\varphi(U) \times J$ since it depends continuously on (x,t).

Consider, for every $(p,t) \in U \times J$, the companion $A_{(p,t)}$ of $d(\mathcal{K}_t)_p$ which is correlated with the constant companion $\widehat{A}_{(\bar{p},\bar{t})}$. Recalling that $A_{(\bar{p},\bar{t})}$ is a positive companion of $d(\mathcal{K}_{\bar{t}})_{\bar{p}}$ for its canonical alg-orientation (Definition 2.10), it follows that

$$\det(d(\mathcal{K}_{\bar{t}})_{\bar{p}} + A_{(\bar{p},\bar{t})}) > 0.$$

It is important to point out that the above determinant is well defined thanks to Remark 2.3. Consequently, we have

$$\det(d(\mathcal{K}_t)_p + A_{(p,t)}) > 0 \quad \forall (p,t) \in U \times J.$$

Therefore, $A_{(p,t)}$ is a positive companion of $d(\mathcal{K}_t)_p$ for its canonical algebraic orientation for every $(p,t) \in U \times J$.

Thus, we obtain that the set $\mathcal{C}$ of $(p,t) \in \mathcal{M} \times [0,1]$ such that $d(\mathcal{K}_t)_p$ is canonically alg-orientated is open. Recalling, by the assumption of the lemma, that $\mathcal{C} \neq \emptyset$ and $\mathcal{M} \times [0,1]$ is connected, it follows that $\mathcal{C} = \mathcal{M} \times [0,1]$, and the lemma is proved. ■

Proposition 3.18 (Homotopic transport of the canonical top-orientation). *Let $\mathfrak{g}_0$ and $\mathfrak{g}_1$ be two C^1 finitely perturbed vector fields on $\mathbb{H}$. Assume that*

$$0 \notin ((1-t)\mathfrak{g}_0 + t\mathfrak{g}_1)(\mathbb{S}) \quad \forall t \in [0,1],$$

and consider the Φ_0-homotopy $\mathcal{H}: \mathbb{S} \times [0,1] \to \mathbb{S}$, given by

$$\mathcal{H}_t = ((1-t)\mathfrak{g}_0 + t\mathfrak{g}_1)^{\partial}, \quad t \in [0,1].$$

Then, given a top-orientation of $\mathcal{H}$, if one partial top-orientation is canonical, so are all the others.

Proof. It can be obtained proceeding as in the proof of Lemma 3.17 for the particular case of a boundary equatorial self-map, and applying the definition of canonical top-orientation of a boundary self-map. ∎

Lemma 3.19. *Suppose that the finitely perturbed vector field $\mathfrak{g}$ is C^1 such that $0 \notin \mathfrak{g}(\mathbb{S})$. If $\mathfrak{g}^\partial$ is canonically top-oriented, then*

$$\operatorname{sign}_{\mathrm{bf}} d(\mathfrak{g}^\partial)_p = \operatorname{sign} d((\mathfrak{g}|_{\mathbb{X}})^\partial)_p \quad \forall p \in \mathbb{E}.$$

Proof. Let $p \in \mathbb{E}$ be given. From Lemma 3.7 one gets that if one of the two operators, $d(\mathfrak{g}^\partial)_p$ or $d((\mathfrak{g}|_{\mathbb{X}})^\partial)_p$, is not invertible, so is the other; and in this case, their sign is zero. We may therefore assume that both the differentials are invertible, so that each of their signs is either 1 or -1. Hence, it is sufficient to show that if one of them has sign 1, so has the other. Meaning that if one of the two differentials, $d(\mathfrak{g}^\partial)_p$ or $d((\mathfrak{g}|_{\mathbb{X}})^\partial)_p$, admits the null operator as a positive companion, the same holds true for the other one; and this is a consequence of the relation between $\omega(p)$ and its normal extension $\omega^s(p)$, introduced in Remark 3.14. ∎

Proposition 3.20. *Suppose that $\mathfrak{g}$ is C^1 and $0 \notin \mathfrak{g}(\mathbb{S})$. If $\mathfrak{g}^\partial$ has the canonical top-orientation, then*

$$\deg_{\mathrm{bf}}(\mathfrak{g}^\partial) = \deg_{\mathrm{Br}}(\mathfrak{g}|_{\mathbb{X}}, \mathbb{B} \cap \mathbb{X}, 0).$$

Proof. According to the Computation Formula of the bf-degree one has

$$\deg_{\mathrm{bf}}(\mathfrak{g}^\partial) = \sum_{p \in (\mathfrak{g}^\partial)^{-1}(q)} \operatorname{sign}_{\mathrm{bf}} d(\mathfrak{g}^\partial)_p,$$

where $q \in \mathbb{S}$ is any regular value for $\mathfrak{g}^\partial$. Corollary 3.11 implies that as q we may choose a regular value for the boundary equatorial self-map $(\mathfrak{g}|_{\mathbb{X}})^\partial$, whose existence is ensured by Sard's lemma. Because of Remark 3.5, one has

$$(\mathfrak{g}^\partial)^{-1}(q) = ((\mathfrak{g}|_{\mathbb{X}})^\partial)^{-1}(q).$$

Thus, Lemma 3.19 implies

$$\deg_{\mathrm{bf}}(\mathfrak{g}^\partial) = \sum_{p \in (\mathfrak{g}^\partial)^{-1}(q)} \operatorname{sign}_{\mathrm{bf}} d(\mathfrak{g}^\partial)_p = \sum_{p \in ((\mathfrak{g}|_{\mathbb{X}})^\partial)^{-1}(q)} \operatorname{sign} d((\mathfrak{g}|_{\mathbb{X}})^\partial)_p = \deg_{\mathrm{Br}}((\mathfrak{g}|_{\mathbb{X}})^\partial).$$

Finally, from the classical boundary dependence property of the Brouwer degree

$$\deg_{\mathrm{Br}}((\mathfrak{g}|_{\mathbb{X}})^\partial) = \deg_{\mathrm{Br}}(\mathfrak{g}|_{\mathbb{X}}, \mathbb{B} \cap \mathbb{X}, 0),$$

recalled in the Introduction (see, e.g., [8]), we get the result. ∎

4. The Leray–Schauder case

Let $\mathfrak{g}$ be a finitely perturbed vector field of class C^1 such that $0 \notin \mathfrak{g}(\mathbb{S})$. In this case, the Leray–Schauder degree on $\mathbb{B}$ (with target 0) of $\mathfrak{g}$ is defined by the so called "finite dimensional reduction" (see [16]). Namely,

$$\deg_{\mathrm{LS}}(\mathfrak{g}, \mathbb{B}, 0) := \deg_{\mathrm{Br}}(\mathfrak{g}|_{\mathbb{X}}, \mathbb{B} \cap \mathbb{X}, 0). \tag{4.1}$$

It is known that this is a well-posed definition since $\deg_{\mathrm{Br}}(\mathfrak{g}|_{\mathbb{X}}, \mathbb{B} \cap \mathbb{X}, 0)$ does not depend on the choice of $\mathbb{X}$.

If the boundary self-map $\mathfrak{g}^{\partial}$ is canonically top-oriented, from Proposition 3.20 and equality (4.1) one gets

$$\deg_{\mathrm{LS}}(\mathfrak{g}, \mathbb{B}, 0) = \deg_{\mathrm{bf}}(\mathfrak{g}^{\partial}). \tag{4.2}$$

Assume now that $\mathfrak{f}$ is a compact vector field of class C^1 such that $0 \notin \mathfrak{f}(\mathbb{S})$. We will prove that the same equality still holds for $\mathfrak{f}$, provided that the corresponding boundary self-map $\mathfrak{f}^{\partial}$ has "the orthodox topological orientation" that we will define below. In fact, according to Remark 3.3, $\deg_{\mathrm{bf}}(\mathfrak{f}^{\partial})$ is defined up to a sign, depending on the choice of one of the two possible top-orientations of $\mathfrak{f}^{\partial}$. We will show that one of them is, in some sense, more natural than the other. This top-orientation will be the one for which the equality (4.2) holds, with $\mathfrak{f}$ instead of $\mathfrak{g}$.

We will need the following Remark (see, for example, [3, Lemma 5.11]).

Remark 4.1. Given $\varepsilon > 0$, the C^1 compact vector field $\mathfrak{f}$ can be uniformly ε-approximated on the unit disk $\overline{\mathbb{B}}$ by a C^1 finitely perturbed vector field.

Recall that $\mathfrak{f}(\mathbb{S})$ is closed. Therefore, if $0 \notin \mathfrak{f}(\mathbb{S})$, the same holds for any other sufficiently close compact vector field.

Now, we extend the notion of canonical top-orientation, given in Definition 3.16 for boundary self-maps of finitely perturbed vector fields, to the general setting of boundary self-maps of compact vector fields.

Definition 4.2 (Canonical top-orientation of $\mathfrak{f}^{\partial}$). Given a C^1 compact vector field $\mathfrak{f}\colon \mathbb{H} \to \mathbb{H}$ such that $0 \notin \mathfrak{f}(\mathbb{S})$, the *canonical top-orientation* of its corresponding boundary self-map $\mathfrak{f}^{\partial}$ is the one transported (according to Proposition 2.30) by the Φ_0-homotopy

$$t \mapsto ((1-t)\mathfrak{g} + t\mathfrak{f})^{\partial}, \quad t \in [0,1],$$

where $\mathfrak{g}$ is any C^1 finitely perturbed vector field sufficiently close to $\mathfrak{f}$ whose $\mathfrak{g}^{\partial}$ is canonically top-oriented.

The following result ensures that Definition 4.2 is well posed.

Proposition 4.3. *Let $\mathfrak{f}\colon \mathbb{H} \to \mathbb{H}$ be a compact vector field of class C^1 such that $0 \notin \mathfrak{f}(\mathbb{S})$. If $\mathfrak{g}_0$ and $\mathfrak{g}_1$ are two C^1 finitely perturbed vector fields sufficiently close to $\mathfrak{f}$, and $\mathfrak{g}_0^{\partial}$ and $\mathfrak{g}_1^{\partial}$ are canonically top-oriented, then the Φ_0-homotopies $\mathcal{H}^0$ and $\mathcal{H}^1$, defined by*

$$\mathcal{H}^0_t = ((1-t)\mathfrak{g}_0 + t\mathfrak{f})^{\partial}, \quad \mathcal{H}^1_t = ((1-t)\mathfrak{g}_1 + t\mathfrak{f})^{\partial}, \quad t \in [0,1],$$

transport the same orientation to $\mathfrak{f}^{\partial}$.

Proof. Given any $\lambda = (t,s) \in [0,1] \times [0,1]$, consider the boundary self-map

$$\mathcal{K}_{(t,s)} = \big((1-s)\mathcal{H}_t^0 + s\mathcal{H}_t^1\big)^\partial,$$

which is the partial map $\mathcal{K}_\lambda$ of the following family of boundary self-maps:

$$\mathcal{K}\colon \mathbb{S} \times [0,1] \times [0,1] \to \mathbb{S}, \quad (p,t,s) \mapsto ((1-s)\mathcal{H}_t^0 + s\mathcal{H}_t^1)^\partial(p).$$

Since the parameter space $\Lambda = [0,1] \times [0,1]$ is simply connected and locally path connected, Proposition 2.31 applies yielding a unique top-orientation ω of $\mathcal{K}$ whose partial top-orientation $\omega(\cdot,0,0)$ coincides with the canonical top-orientation of $\mathfrak{g}_0^\partial = \mathcal{K}_{(0,0)}$. By considering the Φ_0-homotopy $s \mapsto \mathcal{K}_{(0,s)}$ joining $\mathfrak{g}_0^\partial$ with $\mathfrak{g}_1^\partial$, and applying Proposition 3.18, one gets that the partial top-orientation $\omega(\cdot,0,1)$ coincides as well with the canonical top-orientation of $\mathfrak{g}_1^\partial = \mathcal{K}_{(0,1)}$. Consequently, recalling the uniqueness of ω with the assigned top-orientation $\omega(\cdot,0,0)$, the top-orientations transported to $\mathfrak{f}^\partial$ by $\mathcal{H}^0$ and $\mathcal{H}^1$ are, respectively, $\omega(\cdot,1,0)$ and $\omega(\cdot,1,1)$. Finally, these two partial top-orientations coincide, since they are both transported by the family $\mathcal{K}$ to the same map $\mathfrak{f}^\partial$. ∎

In [16], the degree of a compact vector field $\mathfrak{f}$ such that $0 \notin \mathfrak{f}(\mathbb{S})$ is defined as

$$\deg_{\mathrm{LS}}(\mathfrak{f},\mathbb{B},0) := \deg_{\mathrm{LS}}(\mathfrak{g},\mathbb{B},0), \tag{4.3}$$

where $\mathfrak{g}$ is any finitely perturbed vector field, sufficiently close to $\mathfrak{f}$. As shown in [16], this definition is well posed, since if $\mathfrak{g}_0$ and $\mathfrak{g}_1$ are two finitely perturbed vector fields sufficiently close to $\mathfrak{f}$, then $\deg_{\mathrm{LS}}(\mathfrak{g}_0,\mathbb{B},0)$ and $\deg_{\mathrm{LS}}(\mathfrak{g}_1,\mathbb{B},0)$ are well defined and coincide.

From the equalities (4.2) and (4.3) we obtain that if $\mathfrak{g}$ is a C^1 finitely perturbed vector field sufficiently close to $\mathfrak{f}$, with $\mathfrak{g}^\partial$ canonically top-oriented, then

$$\deg_{\mathrm{LS}}(\mathfrak{f},\mathbb{B},0) = \deg_{\mathrm{bf}}(\mathfrak{g}^\partial). \tag{4.4}$$

We finally get our main result.

Theorem 4.4. *Let $\mathfrak{f}\colon \mathbb{H} \to \mathbb{H}$ be a compact vector field of class C^1 such that $0 \notin \mathfrak{f}(\mathbb{S})$. Assume that the corresponding boundary self-map $\mathfrak{f}^\partial\colon \mathbb{S} \to \mathbb{S}$ is canonically top-oriented. Then*

$$\deg_{\mathrm{LS}}(\mathfrak{f},\mathbb{B},0) = \deg_{\mathrm{bf}}(\mathfrak{f}^\partial).$$

Proof. Let $\mathfrak{g}$ be a finitely perturbed vector field of class C^1, and assume that it is so close to $\mathfrak{f}$ such that $0 \notin ((1-t)\mathfrak{g} + t\mathfrak{f})(\mathbb{S})$ for any $t \in [0,1]$. Let $\mathfrak{g}^\partial$ be canonically top-oriented. According to Definition 4.2, $\mathfrak{g}^\partial$ induces, through the Φ_0-homotopy

$$((1-t)\mathfrak{g} + t\mathfrak{f})^\partial\colon \mathbb{S} \to \mathbb{S}, \quad t \in [0,1],$$

the canonical top-orientation of $\mathfrak{f}^\partial$. Moreover, because of the homotopy invariance property of the bf-degree, one gets $\deg_{\mathrm{bf}}(\mathfrak{f}^\partial) = \deg_{\mathrm{bf}}(\mathfrak{g}^\partial)$. Thus, the assertion follows from formula (4.4). ∎

From Theorem 4.4 and Proposition 2.34 we get the following infinite dimensional analogue of (1.1).

Corollary 4.5. *Let* $\mathfrak{f}\colon \mathbb{H} \to \mathbb{H}$ *be a compact vector field of class* C^1 *such that* $0 \notin \mathfrak{f}(\mathbb{S})$. *Assume that* $\mathfrak{f}$ *is standardly top-oriented and the corresponding boundary self-map* $\mathfrak{f}^\partial\colon \mathbb{S} \to \mathbb{S}$ *is canonically top-oriented. Then*

$$\deg_{\mathrm{bf}}(\mathfrak{f}, \mathbb{B}, 0) = \deg_{\mathrm{bf}}(\mathfrak{f}^\partial).$$

As an application of our main results, we prove the following version of the classical intermediate value theorem (compare with [3, Theorem 6.9]).

Theorem 4.6 (Intermediate value theorem via a half-line). *Let* $\mathfrak{f}$ *be a compact vector field of class* C^1 *on* $\mathbb{H}$. *Given* $q \notin \mathfrak{f}(\mathbb{S})$, *let* Λ_q *be a half-line with extreme* q. *If the intersection of* $\mathfrak{f}(\mathbb{S})$ *with* Λ_q *is transverse and its preimage under* $\mathfrak{f}|_{\mathbb{S}}$ *is made up of an odd number of points, then the connected component of* $\mathbb{H} \setminus \mathfrak{f}(\mathbb{S})$ *containing* q *is a bounded open subset of* $\mathfrak{f}(\mathbb{B})$.

Proof. As shown in [3, Lemma 6.8], the transversality assumption implies that $|\deg_{\mathrm{bf}}((\mathfrak{f} - q)^\partial)|$, which is well defined according to Remark 3.3, is different from zero.

Now, assume that $(\mathfrak{f} - q)^\partial$ is canonically top-oriented and denote by $\mathcal{C}$ the connected component of $\mathbb{H} \setminus \mathfrak{f}(\mathbb{S})$ containing q. Since $\mathbb{H} \setminus \mathfrak{f}(\mathbb{S})$ is open, so is the component $\mathcal{C}$. Thus, recalling that a compact vector field maps bounded sets into bounded sets, it remains to prove that $\mathcal{C}$ is contained in $\mathfrak{f}(\mathbb{B})$; which means that $\mathfrak{f} - u$ vanishes somewhere in $\mathbb{B}$, whatever is $u \in \mathcal{C}$. To see this, let $u \in \mathcal{C}$ be given, and observe that $\deg_{\mathrm{LS}}(\mathfrak{f} - q, \mathbb{B}, 0) = \deg_{\mathrm{LS}}(\mathfrak{f} - u, \mathbb{B}, 0)$ because of the homotopy invariance property of the LS-degree. Thus, Theorem 4.4, applied to the compact vector field $\mathfrak{f} - q$, yields

$$0 \neq \deg_{\mathrm{bf}}((\mathfrak{f} - q)^\partial) = \deg_{\mathrm{LS}}(\mathfrak{f} - q, \mathbb{B}, 0) = \deg_{\mathrm{LS}}(\mathfrak{f} - u, \mathbb{B}, 0),$$

and the assertion follows from the existence property of the LS-degree. ∎

As shown in [3], given any real Hilbert space $\mathbb{H}$ and any $n \in \mathbb{Z}$, there exists a top-oriented compact vector field $\mathfrak{f}_n$ on $\mathbb{H}$ such that $\deg_{\mathrm{bf}}(\mathfrak{f}_n^\partial) = n$ (compare with [3, Example 6.10]). Note that Theorem 4.6 does not apply if the integer n is even.

A. Proof of Theorem 3.15

Proof. As the uniqueness is a straightforward consequence of Proposition 2.27, we need to prove the existence of a top-orientation of $\mathfrak{g}^\partial$ that coincides with ω^s at any point of $\mathbb{E}$, where, we recall, ω^s is the normal extension to $(g^\partial, \mathbb{E})$ of the canonical top-orientation ω of $(\mathfrak{g}|_{\mathbb{X}})^\partial$, according to Remark 3.14.

Denote by γ and δ the two top-orientations of $\mathfrak{g}^\partial$ and consider

$$\Gamma := \{p \in \mathbb{E} : \gamma(p) = \omega^s(p)\} \quad \text{and} \quad \Delta := \{p \in \mathbb{E} : \delta(p) = \omega^s(p)\}.$$

One has $\Gamma \cup \Delta = \mathbb{E}$ and $\Gamma \cap \Delta = \emptyset$. Recalling that $\mathbb{E}$ is connected, if we prove that Γ and Δ are open, then one of them coincides with $\mathbb{E}$ and thus Theorem 3.15 is proved. Indeed, if, without loss of generality, $\Gamma = \mathbb{E}$, the top-orientation γ of $\mathfrak{g}^\partial$ satisfies $\gamma(p) = \omega^s(p)$ for every $p \in \mathbb{E}$.

Summarizing the above argument, our purpose is to prove that Γ is open, assuming that it is not empty. From now on, $\bar{p} \in \mathbb{E}$ is a given point and we suppose that

$$\gamma(\bar{p}) = \omega^s(\bar{p}). \tag{A.1}$$

We want to prove that $\gamma(p) = \omega^s(p)$ for all p in a suitable neighborhood of $\bar{p}$ in $\mathbb{E}$. We proceed in two steps.

Step 1. This is a technical step: we start from a positive companion $A_{\bar{p}}$ of $d((\mathfrak{g}|_{\mathbb{X}})^\partial)_{\bar{p}}$, that is, $A_{\bar{p}} \in \omega(\bar{p})$, and we determine, depending on $A_{\bar{p}}$, a selection of positive companions of $d((\mathfrak{g}|_{\mathbb{X}})^\partial)_p$ for p in a suitable neighborhood of $\bar{p}$ in $\mathbb{E}$.

First of all, denote by $\mathbb{H}'$ and $\mathbb{X}'$ two Hilbert spaces locally diffeomorphic to $\mathbb{S}$ and $\mathbb{E}$, respectively. Without loss of generality, we can assume that $\mathbb{X}'$ is a subspace of $\mathbb{H}'$. Consider two charts $\varphi\colon U \to \mathbb{H}'$ and $\psi\colon V \to \mathbb{H}'$ of $\mathbb{S}$ such that $\bar{p} \in U$, $\mathfrak{g}^\partial(U) \subseteq V$ and put $\bar{q} := \mathfrak{g}^\partial(\bar{p}) \in V$. Denote by $\hat{k}\colon \varphi(U) \to \psi(V)$ the composition

$$\hat{k} := \psi \circ \mathfrak{g}^\partial \circ \varphi^{-1},$$

and by $\widehat{A}_{\bar{p}}\colon \mathbb{X}' \to \mathbb{X}'$ the linear operator given by

$$\widehat{A}_{\bar{p}} = d(\psi|_{V\cap\mathbb{E}})_{\bar{q}}\, A_{\bar{p}}(d(\varphi|_{U\cap\mathbb{E}})_{\bar{p}})^{-1}.$$

According to Definition 2.25, $A_{\bar{p}}$ and $\widehat{A}_{\bar{p}}$ are correlated companions of $d((\mathfrak{g}|_{\mathbb{X}})^\partial)_{\bar{p}}$ and $d(\hat{k}|_{\varphi(U\cap\mathbb{E})})_{\varphi(\bar{p})}$, respectively (see also diagram (2.1)).

Without loss of generality, let $U \cap \mathbb{E}$ be connected and sufficiently small in such a way that $\widehat{A}_{\bar{p}}$ is a companion of $d(\hat{k}|_{\varphi(U\cap\mathbb{E})})_{\varphi(p)}$ for any $p \in U \cap \mathbb{E}$. Thus, using the definition of top-orientation based on Remark 2.16, $\widehat{A}_{\bar{p}}$ defines a top-orientation, say $\hat{\alpha}$, on the restriction $\hat{k}|_{\varphi(U\cap\mathbb{E})}$, where $\widehat{A}_{\bar{p}} \in \hat{\alpha}(p)$ for each $p \in U \cap \mathbb{E}$.

For every $p \in U \cap \mathbb{E}$, consider the companion A_p of $d((\mathfrak{g}|_{\mathbb{X}})^\partial)_p$ defined in the diagram

$$\begin{array}{ccc} T_p\mathbb{E} & \xrightarrow{\;A_p\;} & T_{\mathfrak{g}^\partial(p)}\mathbb{E} \\ {\scriptstyle d(\varphi|_{U\cap\mathbb{E}})_p}\big\downarrow & & \big\downarrow{\scriptstyle d(\psi|_{V\cap\mathbb{E}})_{\mathfrak{g}^\partial(p)}} \\ \mathbb{X}' & \xrightarrow[\;\widehat{A}_{\bar{p}}\;]{} & \mathbb{X}' \end{array} \tag{A.2}$$

Every A_p is correlated with $\widehat{A}_{\bar{p}}$ and hence, by Remark 2.26,

$$A_p \in \alpha(p) \quad \forall p \in U \cap \mathbb{E}, \tag{A.3}$$

where α is the top-orientation of the restriction to $U \cap \mathbb{E}$ of $(\mathfrak{g}|_{\mathbb{X}})^\partial$, which is correlated with $\hat{\alpha}$. Recalling that $A_{\bar{p}} \in \omega(\bar{p})$, we have $\alpha(\bar{p}) = \omega(\bar{p})$ and consequently, since $U \cap \mathbb{E}$

is connected, $\alpha(p) = \omega(p)$ for every $p \in U \cap \mathbb{E}$ (see Proposition 2.27). Therefore, (A.3) implies

$$A_p \in \omega(p) \quad \forall p \in U \cap \mathbb{E}. \tag{A.4}$$

Step 2. In this step we develop the rest of the proof. Let us introduce two selections of companions of the operators $d(\mathfrak{g}^\partial)_p$, where $p \in U$ for the first selection and $p \in U \cap \mathbb{E}$ for the second one.

(1) *The first selection of companions*. Consider the splitting

$$\mathbb{H}' = \mathbb{X}' \oplus (\mathbb{X}')^\perp.$$

Recalling Definition 3.13, consider the normal extension $\widehat{A}^s_{\bar{p}}\colon \mathbb{H}' \to \mathbb{H}'$ of $\widehat{A}_{\bar{p}}$, defined, we recall, as

$$\widehat{A}^s_{\bar{p}}(v + w) = \widehat{A}_{\bar{p}}(v),$$

where $v \in \mathbb{X}'$ and $w \in (\mathbb{X}')^\perp$. The operator $\widehat{A}^s_{\bar{p}}$ is a companion of $d\hat{k}_{\varphi(\bar{p})}$ (Lemma 3.12). Hence, taking U smaller if necessary, $\widehat{A}^s_{\bar{p}}$ is a companion of $d\hat{k}_{\varphi(p)}$ for each $p \in U$. Consider now the linear operators

$$B_p\colon T_p\mathbb{S} \to T_q\mathbb{S}, \quad B_p := d\psi_q^{-1}\widehat{A}^s_{\bar{p}}\, d\varphi_p, \quad p \in U,$$

where $q := \mathfrak{g}^\partial(p)$. Every B_p is correlated with $\widehat{A}^s_{\bar{p}}$ and hence, it is a companion of $d(\mathfrak{g}^\partial)_p$. Thus, we obtain the first selection $\mathcal{B}\colon p \mapsto B_p$, $p \in U$, of companions of the differentials $d(\mathfrak{g}^\partial)_p$.

(2) *The second selection of companions*. First of all, take, for any $p \in U \cap \mathbb{E}$, the companion A_p of $d((\mathfrak{g}|_{\mathbb{X}})^\partial)_p$, defined by the diagram (A.2). Consider, for any $p \in U \cap \mathbb{E}$, the normal extension $A^s_p\colon T_p\mathbb{S} \to T_q\mathbb{S}$ of A_p. Any A^s_p is a companion of $d(\mathfrak{g}^\partial)_p$ (Lemma 3.12), and hence, we obtain another selection of companions of the operators $d(\mathfrak{g}^\partial)_p$, that is, $\mathcal{A}\colon p \mapsto A^s_p$, $p \in U \cap \mathbb{E}$.

We point out that B_p is defined for each $p \in U$, while A^s_p only for $p \in U \cap \mathbb{E}$. A crucial and not difficult property is

$$B_p = A^s_p \quad \forall p \in U \cap E \tag{A.5}$$

(we omit the proof). In step 1, we showed that $A_p \in \omega(p)$ for every $p \in U \cap \mathbb{E}$. Hence, by the definition of the normal extension ω^s of ω, it follows that

$$A^s_p \in \omega^s(p) \quad \forall p \in U \cap \mathbb{E}. \tag{A.6}$$

Now, we want to show that $B_p \in \gamma(p)$, for every $p \in U$. We know that $\widehat{A}^s_{\bar{p}}$ defines a top-orientation of $\hat{k}$, call it $\hat{\beta}$, in which $\widehat{A}^s_{\bar{p}} \in \hat{\beta}(p)$ for every $p \in U$. Then, by Remark 2.26, the correlated top-orientation β on $\mathfrak{g}^\partial|_U$ is such that

$$B_p \in \beta(p) \quad \forall p \in U.$$

Assumption (A.1) and formulas (A.5) and (A.6) imply $B_{\bar{p}} \in \gamma(\bar{p})$ and hence, $\beta(\bar{p}) = \gamma(\bar{p})$. Consequently, taking U connected,

$$\beta(p) = \gamma(p) \quad \forall p \in U,$$

because β and γ are top-orientations. Thus, we have

$$B_p \in \gamma(p) \quad \forall p \in U \tag{A.7}$$

as claimed. Therefore, we can conclude that

$$\gamma(p) = \omega^s(p), \quad p \in U \cap E,$$

that is, $\bar{p} \in \Gamma$ admits a neighborhood $\mathbb{E}$ which is contained in Γ, i.e., Γ is open in $\mathbb{E}$. The proof is complete. ∎

Acknowledgments. The authors would like to warmly thank Massimo Furi for the stimulating discussions and his helpful suggestions during the preparation of the paper. Also they would like to extend thanks to the anonymous referees for the careful reading of the manuscript and the useful comments.

Funding. The second and third author are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). P. Benevieri is partially supported by FAPESP, project no. 2022/14913-7 (Brazil). A. Calamai is partially supported by GNAMPA - INdAM (Italy).

References

[1] P. Benevieri, A. Calamai, and M. Furi, On the degree for oriented quasi-Fredholm maps: its uniqueness and its effective extension of the Leray–Schauder degree. *Topol. Methods Nonlinear Anal.* **46** (2015), no. 1, 401–430 Zbl 1460.47027 MR 3443693

[2] P. Benevieri, A. Calamai, M. Furi, and M. P. Pera, A degree associated to linear eigenvalue problems in Hilbert spaces and applications to nonlinear spectral theory. *J. Dynam. Differential Equations* **34** (2022), no. 1, 555–581 Zbl 07491619 MR 4383298

[3] P. Benevieri, A. Calamai, M. Furi, and M. P. Pera, An infinite dimensional version of the intermediate value theorem. *J. Fixed Point Theory Appl.* **25** (2023), no. 3, article no. 70 Zbl 07725412 MR 4614675

[4] P. Benevieri and M. Furi, A simple notion of orientability for Fredholm maps of index zero between Banach manifolds and degree theory. *Ann. Sci. Math. Québec* **22** (1998), no. 2, 131–148 Zbl 1058.58502 MR 1677235

[5] P. Benevieri and M. Furi, On the concept of orientability for Fredholm maps between real Banach manifolds. *Topol. Methods Nonlinear Anal.* **16** (2000), no. 2, 279–306 Zbl 1007.47026 MR 1820510

[6] P. Benevieri and M. Furi, On the uniqueness of the degree for nonlinear Fredholm maps of index zero between Banach manifolds. *Commun. Appl. Anal.* **15** (2011), no. 2-4, 203–216 Zbl 1239.47048 MR 2867346

[7] P. Benevieri, M. Furi, M. P. Pera, and M. Spadini, About the sign of oriented Fredholm operators between Banach spaces. *Z. Anal. Anwendungen* **22** (2003), no. 3, 619–645 Zbl 1060.47016 MR 2015667
[8] H. Brezis and L. Nirenberg, Degree theory and BMO. I. Compact manifolds without boundaries. *Selecta Math. (N.S.)* **1** (1995), no. 2, 197–263 Zbl 0852.58010 MR 1354598
[9] K. Deimling, *Nonlinear functional analysis*. Springer, Berlin, 1985 Zbl 0559.47040 MR 0787404
[10] G. Dinca and J. Mawhin, *Brouwer degree—the core of nonlinear analysis*. Progr. Nonlinear Differential Equations Appl. 95, Birkhäuser/Springer, Cham, 2021 Zbl 07301627 MR 4390178
[11] P. M. Fitzpatrick, J. Pejsachowicz, and P. Rabier, Orientability of Fredholm families and topological degree for orientable nonlinear Fredholm mappings. *J. Funct. Anal.* **124** (1994), no. 1, 1–39 Zbl 0802.47056 MR 1284601
[12] M. W. Hirsch, *Differential topology*. Grad. Texts in Math. 33, Springer, New York, 1976 Zbl 0356.57001
[13] T. Kato, *Perturbation theory for linear operators*. Classics Math., Springer, Berlin, 1995 Zbl 0836.47009 MR 1335452
[14] L. Kronecker, Über Systeme von Funktionen mehrerer Variabeln. *Monatsber. Königl. Preussischen Akademie Wiss. Berlin* (1869), 150–193 and 688–698 Zbl 02.0203.02
[15] N. H. Kuiper, The homotopy type of the unitary group of Hilbert space. *Topology* **3** (1965), 19–30 Zbl 0129.38901 MR 0179792
[16] J. Leray and J. Schauder, Topologie et équations fonctionnelles. *Ann. Sci. École Norm. Sup. (3)* **51** (1934), 45–78 Zbl 60.0322.02 MR 1509338
[17] J. W. Milnor, *Topology from the differentiable viewpoint*. University Press of Virginia, Charlottesville, VA, 1965 Zbl 0136.20402 MR 0226651
[18] E. Outerelo and J. M. Ruiz, *Mapping degree theory*. Grad. Stud. Math. 108, American Mathematical Society, Providence, RI; Real Sociedad Matemática Española, Madrid, 2009 Zbl 1183.47056 MR 2566906
[19] J. Pejsachowicz, Orientation of Fredholm maps. *J. Fixed Point Theory Appl.* **2** (2007), no. 1, 97–116 Zbl 1138.47039 MR 2336502
[20] J. Pejsachowicz and P. J. Rabier, Degree theory for C^1 Fredholm mappings of index 0. *J. Anal. Math.* **76** (1998), 289–319 Zbl 0932.47046 MR 1676979
[21] S. Smale, An infinite dimensional version of Sard's theorem. *Amer. J. Math.* **87** (1965), 861–866 Zbl 0143.35301 MR 0185604
[22] A. E. Taylor and D. C. Lay, *Introduction to functional analysis*. 2nd edn., John Wiley & Sons, New York-Chichester-Brisbane, 1980 Zbl 0501.46003 MR 0564653
[23] M. Väth, *Topological analysis*. De Gruyter Ser. Nonlinear Anal. Appl. 16, Walter de Gruyter, Berlin, 2012 Zbl 1248.47001 MR 2961860

Received 24 July 2023; revised 21 January 2024.

Pierluigi Benevieri
Instituto de Matemática e Estatística, Universidade de São Paulo, Rua do Matão 1010, SP 05508-090 São Paulo, Brasil; pluigi@ime.usp.br

Alessandro Calamai
Dipartimento di Ingegneria Civile, Edile e Architettura, Università Politecnica delle Marche, Via Brecce Bianche, 60131 Ancona, Italy; a.calamai@univpm.it

Maria Patrizia Pera
Dipartimento di Matematica e Informatica "Ulisse Dini", Università degli Studi di Firenze, Via S. Marta 3, 50139 Florence, Italy; mpatrizia.pera@unifi.it