J. Math. Anal. Appl. 527 (2023) 127442

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Regular Articles

An Ahmad-Lazer-Paul-type result for indefinite mixed local-nonlocal problems ☆

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A R T I C L E   I N F O

Article history:
Received 5 February 2023
Available online 29 May 2023
Submitted by H. Frankowska

Keywords:
Operators of mixed order
Variational methods
Saddle point theorem
Fountain theorem

A B S T R A C T

We prove the existence and multiplicity of weak solutions for a mixed local-nonlocal problem at resonance. In particular, we consider a not necessarily positive operator which appears in models describing the propagation of flames. A careful adaptation of well known variational methods is required to deal with the possible existence of negative eigenvalues.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set with C^1-smooth boundary $\partial\Omega$ with $n > 2$. We consider the following Dirichlet boundary value problem

$$\begin{cases} \mathcal{L}_\alpha u = \lambda u + f(x,u), & \text{in } \Omega \\ u = 0, & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.1}$$

where

$$\mathcal{L}_\alpha u := -\Delta u + \alpha(-\Delta)^s u\,.$$

☆ G. Giovannardi is supported by INdAM-GNAMPA Project "Analisi geometrica in strutture subriemanniane" (grant no. CUP_E55F22000270001). D. Mugnai is supported by the FFABR "Fondo per il finanziamento delle attività base di ricerca" 2017 and by the INdAM-GNAMPA Project "PDE ellittiche a diffusione mista" (grant no. CUP_E55F22000270001). E. Vecchi is supported by INdAM-GNAMPA Project "PDE ellittiche a diffusione mista" (grant no. CUP_E55F22000270001).

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https://doi.org/10.1016/j.jmaa.2023.127442

Here $\alpha \in \mathbb{R}$ with no a priori restrictions, Δu denotes the classical Laplace operator while $(-\Delta)^s u$, for fixed $s \in (0,1)$ and up to a multiplicative positive constant, is the fractional Laplacian, usually defined as

$$(-\Delta)^s u(x) := C(n,s)\,\text{P.V.} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy\,,$$

where P.V. denotes the Cauchy principal value, that is

$$\text{P.V.} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy = \lim_{\varepsilon\to 0} \int_{\{y\in\mathbb{R}^n\,:\,|y-x|\geq\varepsilon\}} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy,$$

see [29] for more details. Clearly, when $\alpha = 0$, one recovers the classical Laplacian. Finally, $\lambda \in \mathbb{R}$ is a variational Dirichlet eigenvalue of $\mathcal{L}_\alpha$ (hence (1.1) is a problem at resonance), namely there are nontrivial solutions for the problem

$$\begin{cases} \mathcal{L}_\alpha u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

see Section 2 for the precise setting.

We suppose that f satisfies the following assumptions.

Assumptions on f.

(f_{bc}) *$f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a bounded and Carathéodory function, namely:*
1. *$f(x,\cdot)$ is continuous in $\mathbb{R}$ for a.e. $x \in \Omega$*
2. *$f(\cdot,t)$ is measurable in Ω for all $t \in \mathbb{R}$.*

($F_{\pm\infty}$)

$$\lim_{\substack{u\in Ker(\mathcal{L}_\alpha-\lambda),\\ \|u\|\to\infty}} \int_\Omega F(x,u)dx = \pm\infty$$

uniformly in $x \in \Omega$.

Remark 1.1. The assumption ($F_{\pm\infty}$) is the so–called Ahmad-Lazer-Paul condition introduced in [1] and often used in resonant problems, for instance see [21]. A sufficient condition implying it is given by

$$F(x,t) = \int_0^t f(x,\tau)d\tau \to \pm\infty \qquad \text{as } |t| \to +\infty,$$

as can be quite easily checked, see e.g. [20, Lemma 3.4].

The goal of this paper is to extend to a mixed operator an existence and multiplicity result established in [32] for the Laplace operator. We state immediately our first result:

Theorem 1.2. *Let f satisfy (f_{bc}) and ($F_{\pm\infty}$) and suppose that $\lambda \in \mathbb{R}$ is a variational Dirichlet eigenvalue of $\mathcal{L}_\alpha$. Then, the problem (1.1) admits a weak solution $u \in \mathbb{X}(\Omega)$.*

We immediately clarify that Theorem 1.2: it is somehow *easy to get* for $\alpha > 0$ or when

$$-\frac{1}{C_s} < \alpha < 0\,,$$

where $C_s > 0$ is the best constant of the continuous embedding $H_0^1 \subset H^s$ (see e.g. [15]), i.e.

$$[u]_s^2 := \iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dxdy \le C_s \int_{\Omega} |\nabla u|^2\,dx\,.$$

In this perspective, the probably more interesting case is when $\alpha < -\frac{1}{C_s}$ is assumed. Indeed, the situation becomes suddenly more delicate, mainly because the local–nonlocal operator is not more positive definite, *indefinite operators*. As a consequence, the bilinear form naturally associated to it does not induce a scalar product nor a norm, the variational spectrum may exhibit negative eigenvalues and even the maximum principles may fail, see e.g. [4]. As a consequence, some inequalities cannot be adapted to our situations, since dividing by eigenvalues may reverse the sign, nullifying a verbatim adaptation of [32]. For this reason, we need to establish some crucial estimates for eigenspaces, see Lemmas 2.6 and 2.7.

We stress that operators of this latter form (e.g. with $\alpha = -1$) do not have only a purely theoretical mathematical interest, indeed the play a role in applied sciences like combustion theory. We limit ourselves to mention that the stationary part in the original model proposed by Sivashinsky [35] to deal with the instability of the propagation front of flames can be reduced to operator of the form previously described, and this may happen under physically motivated assumptions, see e.g. [26] and the references therein.

We stress that Theorem 1.2 is actually a further generalization of a result by Ahmad, Lazer and Paul [1] where the authors dealt with a local operator at resonance. Despite well known, we want to recall here that the original proof in [1] is by no means of variational flavour, and consists in a delicate construction of the solution using a sort of Galerkin method. As a matter of fact, a striking and not foreseeable consequence of [1] is an existence result with proof relying on the Saddle Point Theorem by Rabinowitz [32], which is one of the cornerstones of variational methods in nonlinear analysis. We will follow this latter approach, which has already been used also in the pure nonlocal case in [20]. We notice that, as in [20], it is possible to work in a slightly more general case considering weighted Dirichlet eigenvalues, where the weight a is a Lipschitz function. However, treating solutions of $\mathcal{L}_\alpha = \lambda a(x)u$ doesn't change the spirit of our result, and for this reason we concentrate on eigenvalues without weight.

In the spirit of [32], under assumption $(F_{-\infty})$, we can also prove a multiplicity result under a few extra assumptions. The precise statement is the following

Theorem 1.3. *Let f satisfy (f_{bc}) and $(F_{-\infty})$ and suppose that $\lambda \in \mathbb{R}$ is a variational Dirichlet eigenvalue of $\mathcal{L}_\alpha$. We further assume that $f(x,0) = 0$ and that $f(x,t)$ is odd in the t variable. Finally, we assume that*

$$\text{there exists } r > 0 \text{ such that } F(x,t) > 0, \quad \text{for } 0 < |t| < r \text{ and } x \in \overline{\Omega}. \tag{F_{pos}}$$

Then, problem (1.1) admits at least $\dim(H_\lambda^0)$ distinct pair of nontrivial weak solutions, where H_λ^0 denotes the eigenspace associated to λ.

Remark 1.4. We remark that, as in [32], we are able to prove the multiplicity result only when $(F_{-\infty})$, since only in such a case we are able to prove a decomposition of the space $\mathbb{X}(\Omega)$ for which the abstract multiplicity Theorem A.1 holds. Hence, the multiplicity result when $(F_{+\infty})$ holds is an open problem.

We close the introduction with a quite short overview on the more recent (elliptic) PDEs oriented literature. Problems driven by operators of mixed type, even with a nonsingular nonlocal operator [16], have raised a certain interest in the last few years, for example in connection with the study of optimal animal foraging strategies (see e.g. [19] and the references therein). From the pure mathematical point of view, the superposition of such operators generates a lack of scale invariance which may lead to unexpected complications.

At the present stage, and without aim of completeness, the investigations have taken into consideration interior regularity and maximum principles (see e.g. [4,12,14,23,24]), boundary Harnack principle [13], boundary regularity and overdetermined problems [10,36], existence of solutions (see e.g. [7–9,17,18,25,31,34,3,22]) and shape optimization problems [5,6].

The paper is organized as follows. In Section 2 we introduce some preliminary definitions and results, such as the Hilbert space $\mathbb{X}(\Omega)$, the notion of weak solution of (1.1) (as critical point of the functional $\mathcal{J}_\lambda$), the variational eigenvalue problem for $\mathcal{L}_\alpha$ and the crucial main lemmas. Section 3 is dedicated to the proofs of the Theorem 1.2 and Theorem 1.3; we first deal with the geometry of the functional $\mathcal{J}_\lambda$ and the Palais-Smale condition, then we verify the hypothesis of the Saddle Point Theorem and [32, Theorem 1.9]. In the Appendix we recall the notion of Krasnoselskii genus and we state [32, Theorem 1.9].

2. Assumptions, notation and preliminary results

Let $\Omega \subseteq \mathbb{R}^n$ be a connected and bounded open set with C^1-smooth boundary $\partial\Omega$. We define the space of solutions of problem (1.1) as

$$\mathbb{X}(\Omega) := \{u \in H^1(\mathbb{R}^n) : u \equiv 0 \text{ a.e. on } \mathbb{R}^n \setminus \Omega\}.$$

Thanks to the regularity assumption on $\partial\Omega$ (see [11, Proposition 9.18]), we can identify the space $\mathbb{X}(\Omega)$ with the space $H_0^1(\Omega)$ in the following sense:

$$u \in H_0^1(\Omega) \iff u \cdot \mathbf{1}_\Omega \in \mathbb{X}(\Omega), \tag{2.1}$$

where $\mathbf{1}_\Omega$ is the indicator function of Ω. From now on, we shall always identify a function $u \in H_0^1(\Omega)$ with $\hat{u} := u \cdot \mathbf{1}_\Omega \in \mathbb{X}(\Omega)$.

By the Poincaré inequality and (2.1), we get that the quantity

$$\|u\|_{\mathbb{X}} := \left(\int_\Omega |\nabla u|^2\,dx\right)^{1/2}, \quad u \in \mathbb{X}(\Omega),$$

endows $\mathbb{X}(\Omega)$ with a structure of (real) Hilbert space, which is isometric to $H_0^1(\Omega)$. To fix the notation, we denote by $\langle \cdot, \cdot \rangle_{\mathbb{X}}$ the scalar product which induces the norm above on $\mathbb{X}(\Omega)$. We briefly recall that the space $\mathbb{X}(\Omega)$ is separable and reflexive, $C_0^\infty(\Omega)$ is dense in $\mathbb{X}(\Omega)$ and eventually that $\mathbb{X}(\Omega)$ compactly embeds in $L^p(\Omega)$ for any $p \in \left[1, \frac{2n}{n-2}\right)$ and in

$$H_0^s(\Omega) := \{H^s(\mathbb{R}^n) : u \equiv 0 \text{ a.e. on } \mathbb{R}^n \setminus \Omega\}$$

by [27, Theorem 16.1].

With the correct functional setting, we are ready to give the suitable notion of weak solution for problem (1.1).

Definition 2.1. A function $u \in \mathbb{X}(\Omega)$ is called a weak solution of (1.1) if

$$\int_{\Omega} \langle \nabla u, \nabla \varphi \rangle \, dx + \alpha \iint_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}} \, dxdy = \lambda \int_{\Omega} u\varphi \, dx + \int_{\Omega} f(x,u)\varphi \, dx$$

for every $\varphi \in \mathbb{X}(\Omega)$.

As usual, weak solutions of (1.1) can be found as critical points of the functional $\mathcal{J}_\lambda : \mathbb{X}(\Omega) \to \mathbb{R}$ defined as

$$\mathcal{J}_\lambda(u) := \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx + \frac{\alpha}{2} \iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} \, dxdy - \frac{\lambda}{2} \int_{\Omega} |u|^2 \, dx - \int_{\Omega} F(x,u) \, dx,$$

where

$$F(x,t) := \int_0^t f(x,\sigma) \, d\sigma, \quad t \in \mathbb{R}.$$

By assumption (f_{bc}) it is standard to prove (see for instance [2]) that the functional $\mathcal{J}_\lambda$ is Fréchet differentiable and that

$$\mathcal{J}_\lambda'(u)(\varphi) = \int_{\Omega} \langle \nabla u, \nabla \varphi \rangle \, dx + \alpha \iint_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}} \, dxdy - \lambda \int_{\Omega} u(x)\varphi(x) \, dx - \int_{\Omega} f(x,u(x))\varphi(x) \, dx \quad \text{for every } \varphi \in \mathbb{X}(\Omega) \, .$$

Now, we consider the bilinear form $\mathcal{B}_\alpha : \mathbb{X}(\Omega) \times \mathbb{X}(\Omega) \to \mathbb{R}$, defined by

$$\mathcal{B}_\alpha(u,v) := \int_{\Omega} \langle \nabla u, \nabla v \rangle \, dx + \alpha \iint_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}} \, dxdy$$

for any $u, v \in \mathbb{X}(\Omega)$. In spite of the fact that α can be such that $\mathcal{B}_\alpha$ is not positive definite, we give the following definition.

Definition 2.2. We say that u and v are $\mathcal{B}_\alpha$-orthogonal if

$$\mathcal{B}_\alpha(u,v) = 0.$$

The terminology adopted above is justified by the fact that, for $\alpha > 0$ (more precisely if $\alpha > -\frac{1}{C_s}$), the bilinear form $\mathcal{B}_\alpha$ defines a true scalar product.

We conclude this section dealing with the eigenvalue problem associated to the operator $\mathcal{L}_\alpha$, that is the following boundary value problem

$$\begin{cases} \mathcal{L}_\alpha u = \lambda u, & \text{in } \Omega \, , \\ u = 0, & \text{in } \mathbb{R}^n \setminus \Omega \, , \end{cases} \tag{2.2}$$

where $\lambda \in \mathbb{R}$. According to Definition 2.1, we give the following definition.

Definition 2.3. A number $\lambda \in \mathbb{R}$ is called a variational Dirichlet eigenvalue of $\mathcal{L}_\alpha$ if there exists a nontrivial weak solution $u \in \mathbb{X}(\Omega)$ of (2.2) or, equivalently, if

$$\int_\Omega \langle \nabla u, \nabla \varphi \rangle \, dx + \alpha \iint_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}} \, dxdy = \lambda \int_\Omega u\varphi \, dx$$

for every $\varphi \in \mathbb{X}(\Omega)$. If such a function u exists, we call it eigenfunction associated to the eigenvalue λ.

Note that the linearity of $\mathcal{L}_\alpha$ guarantees a complete description of its eigenvalues, and relative eigenfunctions, according to the following result, see [28, Proposition 2.4]:

Proposition 2.4. *Let $n > 2$. Then the following statements hold true:*

(a) *$\mathcal{L}_\alpha$ admits a divergent and bounded from below sequence of eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$, i.e., there exists $C > 0$ such that*

$$-C < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_k \to +\infty, \quad \text{as } k \to +\infty.$$

Moreover, for every $k \in \mathbb{N}$, λ_k can be characterized as

$$\lambda_k = \min_{\substack{u\in\mathbb{P}_k \\ \|u\|_{L^2(\Omega)}=1}} \left\{ \int_\Omega |\nabla u|^2 \, dx + \alpha \iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} \, dxdy \right\}, \tag{2.3}$$

where

$$\mathbb{P}_1 := \mathbb{X}(\Omega),$$

and, for every $k \ge 2$,

$$\mathbb{P}_k := \{u \in \mathbb{X}(\Omega) : \mathcal{B}_\alpha(u, u_j) = 0 \text{ for every } j = 1, \ldots, k-1\};$$

(b) *for every $k \in \mathbb{N}$ there exists an eigenfunction $u_k \in \mathbb{X}(\Omega)$ corresponding to λ_k, which realizes the minimum in (2.3);*

(c) *the sequence $\{u_k\}_{k\in\mathbb{N}}$ of eigenfunctions constitutes an orthonormal basis of $L^2(\Omega)$; moreover, the eigenfunctions are $\mathcal{B}_\alpha$-orthogonal.*

(d) *for every $k \in \mathbb{N}$, λ_k has finite multiplicity.*

Remark 2.5. Clearly, if $\alpha > -\dfrac{1}{C_s}$ there is an improvement on the lower bound of λ_1, which is thus strictly positive. Moreover, in this case, λ_1 is also simple.

We denote by H_k the linear subspace of $\mathbb{X}(\Omega)$ generated by the first k eigenfunctions of $\mathcal{L}_\alpha$, i.e.

$$H_k = \operatorname{span}_{\mathbb{R}}\{u_1, \ldots, u_k\}.$$

Notice that $\mathbb{P}_{k+1} = (H_k)^{\perp_{\mathcal{B}_\alpha}}$, namely the subspace $\mathcal{B}_\alpha$-orthogonal to H_k. Also we set

$$H_k^0 = \operatorname{span}_{\mathbb{R}}\{u_j \ : \ \lambda_j = \lambda_k\},$$

i.e. the kernel of $\mathcal{L}_\alpha - \lambda_k$, and

$$H_k^- = \text{span}_{\mathbb{R}}\{u_j \ : \ \lambda_j < \lambda_k\}.$$

By Proposition 2.4 (a) we can infer the existence of a positive integer $N_0 \in \mathbb{N}$ such that λ_{N_0} is the first (not necessarily simple) positive eigenvalue. Of course, $\lambda_k > 0$ for every $k > N_0$.
We further notice that, again by Proposition 2.4,

$$\lambda_{k+1} \int_\Omega u^2 \, dx \leq \int_\Omega |\nabla u|^2 \, dx + \alpha \iint_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \, dxdy \tag{2.4}$$

for every $u \in \text{span}(u_1, \ldots, u_k)^\perp = \mathbb{P}_{k+1}$ and

$$\int_\Omega |\nabla u|^2 \, dx + \alpha \iint_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \, dxdy \leq \lambda_k \int_\Omega u^2 \, dx \tag{2.5}$$

for every $u \in H_k$.

We first need the following preliminary result inspired by Rabinowitz [32], see [28, Lemma 4.1] for a proof.

Lemma 2.6. *Let $k \in \mathbb{N}$ be such that*

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{k-1} \leq \lambda_k < \lambda_{k+1} \leq \ldots$$

and decompose the space $\mathbb{X}(\Omega)$ as $\mathbb{X}(\Omega) = H_k \oplus \mathbb{P}_{k+1}$, where $H_k := span(u_1, \ldots, u_k)$. Then, there exists a positive constant β such that for any $u \in \mathbb{P}_{k+1}$

$$\mathcal{B}_\alpha(u, u) - \lambda_k \|u\|^2_{L^2(\Omega)} \geq \beta \|u\|^2_{\mathbb{X}(\Omega)}, \tag{2.6}$$

or, equivalently,

$$\inf_{u \in \mathbb{P}_{k+1} \setminus \{0\}} \left\{ 1 + \frac{\alpha[u]_s^2 - \lambda_k \|u\|^2_{L^2(\Omega)}}{\|u\|^2_{\mathbb{X}(\Omega)}} \right\} > 0.$$

We now prove a *sort of counterpart* of Lemma 2.6 when we restrict our attention to the finite dimensional space H_k. The former and the next lemma will be two of the crucial ingredients to verify that the functional $\mathcal{J}_\lambda$ verifies the saddle point geometry.

Lemma 2.7. *Let $k \in \mathbb{N}$ be such that $\lambda = \lambda_k < \lambda_{k+1}$. Then there exists a positive constant $\gamma > 0$, such that*

$$\mathcal{B}_\alpha(u, u) - \lambda_k \|u\|^2_{L^2(\Omega)} \leq -\gamma \|u^-\|^2_{\mathbb{X}(\Omega)} \tag{2.7}$$

for each $u \in H_k$, where $u = u^0 + u^-$, $u^0 \in H_k^0$ and $u^- \in H_k^-$.

Proof. If $u \equiv 0$, then the assertion is trivial. Hence we assume $u \in H_k \setminus \{0\}$. Thanks to Proposition 2.4 (c), a simple computation yields that

$$\mathcal{B}_\alpha(u, u) - \lambda_k \|u\|^2_{L^2(\Omega)} = \mathcal{B}_\alpha(u^-, u^-) - \lambda_k \|u^-\|^2_{L^2(\Omega)},$$

which is nonpositive by (2.5). Then it suffices to prove that there exists a positive constant $\gamma > 0$ such that

$$\sup_{u^-\in H_k^-\setminus\{0\}}\left\{1+\frac{\alpha[u^-]^2-\lambda_k\|u^-\|^2_{L^2(\Omega)}}{\|u^-\|^2_{\mathbb{X}(\Omega)}}\right\}=-\gamma. \tag{2.8}$$

To this aim, we argue as in [28, Lemma 4.1] assuming by contradiction that there exists a sequence $\{u_n^-\}_{n\in\mathbb{N}}\in H_k^-\setminus\{0\}$ such that

$$1+\frac{\alpha[u_n^-]^2-\lambda_k\|u_n^-\|^2_{L^2(\Omega)}}{\|u_n^-\|^2_{\mathbb{X}(\Omega)}}\to 0, \qquad \text{as } n\to+\infty. \tag{2.9}$$

We then consider the normalized (in $\mathbb{X}(\Omega)$) sequence

$$v_n^- := \frac{u_n^-}{\|u_n^-\|_{\mathbb{X}(\Omega)}}\in H_k^-\setminus\{0\},$$

and, since $H_k^-\setminus\{0\}$ is finite dimensional, we can infer the existence of a function $v^-\in H_k^-$ with $\|v^-\|_{\mathbb{X}(\Omega)}=1$ and such that $v_n^-\to v^-$. Therefore, by the compact embedding of $\mathbb{X}(\Omega)$ in $H_0^s(\Omega)$ and in $L^2(\Omega)$, we find

$$1+\alpha[v_n^-]^2-\lambda_k\|v_n^-\|^2_{L^2(\Omega)}\to 1+\alpha[v^-]^2-\lambda_k\|v^-\|^2_{L^2(\Omega)}, \quad \text{as } n\to+\infty,$$

but at the same time,

$$1+\alpha[v_n^-]^2-\lambda_k\|v_n^-\|^2_{L^2(\Omega)}\to 0, \quad \text{as } n\to+\infty.$$

Since $v^-\in H_k^-\setminus\{0\}$, we also have that

$$0>\mathcal{B}_\alpha(v^-,v^-)-\lambda_k\|v^-\|^2_{L^2(\Omega)}=\|v^-\|^2_{\mathbb{X}(\Omega)}-1=0,$$

and a contradiction arises. □

The next Lemma is taken verbatim from [20].

Lemma 2.8. *Let f satisfy (f_{bc}). Then there exists a positive constant $\tilde M>0$, depending on Ω, such that*

$$\left|\int_\Omega F(x,u(x))dx\right|\le \tilde M\ \|u\|_{\mathbb{X}(\Omega)} \tag{2.10}$$

for all $u\in\mathbb{X}(\Omega)$.

Proof. By definition of F we have

$$\left|\int_\Omega F(x,u(x))dx\right| = \left|\int_\Omega\int_0^{u(x)} f(x,t)\,dt\,dx\right|\le M\int_\Omega |u(x)|dx$$

By the Hölder and Poincaré inequalities, we obtain

$$M\int_\Omega |u(x)|dx\le M|\Omega|^{\frac12}\|u\|_{L^2(\Omega)}\le \tilde M\|u\|_{\mathbb{X}(\Omega)}.$$

Hence we get (2.10), with $\tilde M$ depending on Ω. □

3. Proof of Theorem 1.2 and Theorem 1.3

The proof of Theorem 1.2 follows the classical streamlines in minimax theory. In particular, and as already mentioned, we will make use of the Saddle Point Theorem by Rabinowitz (see [32,33]), and therefore we have to check that its assumptions are satisfied.

3.1. Geometry of the functional $\mathcal{J}_\lambda$

Lemma 3.1. *Let f satisfy (f_{bc}) and $(F_{-\infty})$ and let $k \in \mathbb{N}$ be such that $\lambda_k < \lambda_{k+1}$. For every $K > 0$, there exists $r = r(K) > 0$ such that $\mathcal{J}_\lambda(u) \geq K$ for every $u \in \mathbb{P}_{k+1} \oplus H_k^0$ with $\|u\|_{\mathbb{X}(\Omega)} \geq r$.*

Proof. Since $u \in \mathbb{P}_{k+1} \oplus H_k^0$, we can write $u = u^+ + u^0$, where $u^+ \in \mathbb{P}_{k+1}$ and $u^0 \in H_k^0$. It now suffices to note that,

$$\begin{aligned}
\mathcal{J}_{\lambda_k}(u) &= \frac{1}{2}\mathcal{B}_\alpha(u,u) - \frac{\lambda_k}{2}\|u\|^2_{L^2(\Omega)} - \int_\Omega F(x,u)\,dx \\
&= \frac{1}{2}\mathcal{B}_\alpha(u^+,u^+) - \frac{\lambda_k}{2}\|u^+\|^2_{L^2(\Omega)} - \int_\Omega F(x,u)\,dx \\
&\geq \frac{\beta}{2}\|u^+\|^2_{\mathbb{X}(\Omega)} - \int_\Omega F(x,u)\,dx \quad \text{(by Lemma 2.6)} \\
&= \frac{\beta}{2}\|u^+\|^2_{\mathbb{X}(\Omega)} - \int_\Omega F(x,u^0)\,dx - \int_\Omega \big(F(x,u) - F(x,u^0)\big)\,dx \\
&= \frac{\beta}{2}\|u^+\|^2_{\mathbb{X}(\Omega)} - \int_\Omega F(x,u^0)\,dx - \int_\Omega \int_{u^0(x)}^{u(x)} f(x,t)dtdx \\
&\geq \frac{\beta}{2}\|u^+\|^2_{\mathbb{X}(\Omega)} - \int_\Omega F(x,u^0)\,dx - \tilde{M}\|u^+\|_{\mathbb{X}(\Omega)},
\end{aligned} \tag{3.1}$$

and the conclusion now easily follows from $(F_{-\infty})$. □

A similar statement holds when $(F_{+\infty})$ is in force, namely

Lemma 3.2. *Let f satisfy (f_{bc}) and let $k \in \mathbb{N}$ be such that $\lambda_k < \lambda_{k+1}$. For every $K > 0$, there exists $r = r(K) > 0$ such that $\mathcal{J}_\lambda(u) \geq K$ for every $u \in \mathbb{P}_{k+1}$ with $\|u\|_{\mathbb{X}(\Omega)} \geq r$.*

Proof. It suffices to note that, being $u \in \mathbb{P}_{k+1}$,

$$\begin{aligned}
\mathcal{J}_{\lambda_k}(u) &= \frac{1}{2}\mathcal{B}_\alpha(u,u) - \frac{\lambda_k}{2}\|u\|^2_{L^2(\Omega)} - \int_\Omega F(x,u)\,dx \\
&\geq \frac{\beta}{2}\|u\|^2_{\mathbb{X}(\Omega)} - \int_\Omega F(x,u)\,dx \quad \text{(by Lemma 2.6)} \\
&\geq \frac{\beta}{2}\|u\|^2_{\mathbb{X}(\Omega)} - \tilde{M}\|u\|_{\mathbb{X}(\Omega)}, \quad \text{(by Lemma 2.8)}.
\end{aligned} \tag{3.2}$$

The conclusion now easily follows. □

Remark 3.3. An immediate consequence of Lemma 3.1 or of Lemma 3.2 is that

$$\liminf_{\substack{u\in\mathbb{P}_{k+1}\\ \|u\|_{\mathbb{X}(\Omega)}\to+\infty}} \frac{\mathcal{J}_{\lambda_k}(u)}{\|u\|^2_{\mathbb{X}(\Omega)}} > 0. \tag{3.3}$$

Proposition 3.4. *Let f satisfy (f_{bc}) and let $\lambda_k < \lambda_{k+1}$ for some $k \in \mathbb{N}$. If $(F_{+\infty})$ is in force, then we have*

$$\lim_{\substack{u\in H_k\\ \|u\|_{\mathbb{X}(\Omega)}\to+\infty}} J_{\lambda_k}(u) = -\infty, \tag{3.4}$$

while if $(F_{-\infty})$ holds, then

$$\lim_{\substack{u\in H_k^-\\ \|u\|_{\mathbb{X}(\Omega)}\to+\infty}} J_{\lambda_k}(u) = -\infty, \tag{3.5}$$

Proof. Let us start with the case in which $(F_{+\infty})$ holds. Since $u \in H_k$ we can write $u = u^- + u^0$ with $u^- \in H_k^-$ and $u^0 \in H_k^0$. Then we have

$$\begin{aligned}\mathcal{J}_{\lambda_k}(u) =& \frac{1}{2}\mathcal{B}_\alpha(u,u) - \frac{\lambda_k}{2}\int_\Omega |u(x)|^2 dx - \int_\Omega (F(x,u^-(x)+u^0(x)) - F(x,u^0(x)))dx\\ &- \int_\Omega F(x,u^0(x))dx\end{aligned}$$

Notice that, as in proof of Lemma 2.8, we have

$$\begin{aligned}\left|\int_\Omega (F(x,u^-(x)+u^0(x)) - F(x,u^0(x)))dx\right| &\le \left|\int_\Omega \int_{u^0(x)}^{u^-(x)+u^0(x)} f(x,t)\,dtdx\right|\\ &\le M\int_\Omega |u^-(x)|dx\\ &\le \tilde{M}\|u^-\|_{\mathbb{X}(\Omega)}.\end{aligned}$$

Thus, by Lemma 2.7 and the previous inequality, we obtain

$$\mathcal{J}_{\lambda_k}(u) \le -\frac{\gamma}{2}\|u^-\|^2_{\mathbb{X}(\Omega)} + \tilde{M}\|u^-\|_{\mathbb{X}(\Omega)} - \int_\Omega F(x,u^0(x))dx. \tag{3.6}$$

Moreover, by Proposition 2.4(c) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\|u\|^2_{\mathbb{X}(\Omega)} &= \|u^0\|^2_{\mathbb{X}(\Omega)} + \|u^-\|^2_{\mathbb{X}(\Omega)} + 2\langle u^0, u^-\rangle_{\mathbb{X}(\Omega)}\\ &\le \|u^0\|^2_{\mathbb{X}(\Omega)} + \|u^-\|^2_{\mathbb{X}(\Omega)} + 2\,\|u^0\|_{\mathbb{X}(\Omega)}\,\|u^-\|_{\mathbb{X}(\Omega)}.\end{aligned}$$

Thus, since $\|u\|_{\mathbb{X}(\Omega)}$ diverges at $+\infty$ we have that at least one of the two norms, either $\|u^0\|_{\mathbb{X}(\Omega)}$ or $\|u^-\|_{\mathbb{X}(\Omega)}$, goes to infinity, as well. Assume that $\|u^0\|_{\mathbb{X}(\Omega)} \to +\infty$, then $\|u^-\|_{\mathbb{X}(\Omega)}$ can be finite or infinite. By $(F_{+\infty})$ and by (3.6) we get $\mathcal{J}_\lambda(u) \to -\infty$. Otherwise, suppose that $\|u^0\|_{\mathbb{X}(\Omega)}$ is finite, then $\|u^-\|_{\mathbb{X}(\Omega)}$ diverges to $+\infty$ and by Lemma 2.8 the last term in (3.6) has a linear growth. Hence $\mathcal{J}_{\lambda_k}(u) \to -\infty$. This closes the first part.

The case in which $(F_{-\infty})$ holds is rather simpler. Indeed, keeping the notation u^- for functions in H_k^- and reasoning as above, we have that

$$\mathcal{J}_{\lambda_k}(u^-) \leq -\frac{\gamma}{2}\|u^-\|^2_{\mathbb{X}(\Omega)} + \bar{M}\|u^-\|_{\mathbb{X}(\Omega)},$$

this completes the proof. □

3.2. Palais-Smale condition

Let us start recalling the following notion.

Definition 3.5. We say that $\{u_j\}_{j\in\mathbb{N}}$ is a Palais-Smale sequence for $\mathcal{J}_\lambda$ at level $c \in \mathbb{R}$ if $\mathcal{J}_\lambda(u_j) \to c$ as $j \to \infty$ and

$$\mathcal{J}'_\lambda(u_j) \to 0, \quad \text{as } j \to +\infty \tag{3.7}$$

holds true.

Proposition 3.6. *Let f satisfy (f_{bc}) and $(F_{\pm\infty})$. Suppose further that $\lambda_k < \lambda_{k+1}$ for some $k \in \mathbb{N}$. If $\{u_j\}_{j\in\mathbb{N}}$ is a Palais-Smale sequence for $\mathcal{J}_{\lambda_k}$, then $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $\mathbb{X}(\Omega)$.*

Proof. Let $u_j = u_j^0 + u_j^- + u_j^+$ where $u_j^0 \in H_k^0$, $u_j^- \in H_k^-$ and $u_j^+ \in \mathbb{P}_{k+1}$. We will show that all sequence u_j^0, u_j^-, u_j^+ are bounded.

Let us start noticing that by (3.7), we have

$$\begin{aligned}\varepsilon(1)\|u_j^\pm\|_{\mathbb{X}(\Omega)} &\geq \left|\langle \mathcal{J}'_{\lambda_k}(u_j), u_j^\pm\rangle\right| \\ &= \left|\mathcal{B}_\alpha(u_j, u_j^\pm) - \lambda_k \int_\Omega u_j(x)u_j^\pm(x)\,dx - \int_\Omega f(x, u_j(x))u_j^\pm(x)dx\right|,\end{aligned} \tag{3.8}$$

where $\varepsilon(1) \to 0$ as $j \to \infty$. Since f is bounded, similarly to Lemma 2.8 we have

$$\left|\int_\Omega f(x, u_j(x))u_j^\pm(x)\,dx\right| \leq \tilde{M}\|u_j^\pm\|_{\mathbb{X}(\Omega)}. \tag{3.9}$$

Thanks to Proposition 2.4 (c), we have

$$\langle \mathcal{J}'_{\lambda_k}(u_j), u_j^\pm\rangle = \mathcal{B}_\alpha(u_j^\pm, u_j^\pm) - \lambda_k \int_\Omega |u_j^\pm(x)|^2\,dx - \int_\Omega f(x, u_j(x))u_j^\pm(x)dx. \tag{3.10}$$

Since u_j^+ belongs to $\mathbb{P}_{k+1}$, by Lemma 2.6 and (3.9) we get

$$\varepsilon(1)\|u_j^+\|_{\mathbb{X}(\Omega)} \geq \beta\|u_j^+\|^2_{\mathbb{X}(\Omega)} - \tilde{M}\|u_j^+\|_{\mathbb{X}(\Omega)},$$

so that the sequence $\{u_j^+\}_{j\in\mathbb{N}}$ is bounded in $\mathbb{X}(\Omega)$. Furthermore, again by (3.8), (3.10), Lemma 2.7 and (3.9) we get

$$\varepsilon(1)\|u_j^-\|_{\mathbb{X}(\Omega)} \geq -\langle \mathcal{J}'_{\lambda_k}(u_j), u_j^-\rangle \geq \gamma\|u^-\|^2_{\mathbb{X}(\Omega)} - \tilde{M}\|u^-\|_{\mathbb{X}(\Omega)}.$$

Then the sequence $\{u_j^-\}_{j\in\mathbb{N}}$ is bounded in $\mathbb{X}(\Omega)$, as well.

We finally prove that u_j^0 is bounded in $\mathbb{X}(\Omega)$. First of all, we recall that u_j^0 is an eigenfunction associated to λ_k, namely

$$\mathcal{B}_\alpha(u_j^0, u_j^0) = \lambda_k \int_\Omega |u_j^0(x)|^2 dx. \tag{3.11}$$

By the Palais-Smale condition, the equation (3.11) and Proposition 2.4 (c), we gain

$$\begin{aligned} c \leftarrow J_{\lambda_k}(u_j) =& \frac{1}{2}\mathcal{B}_\alpha(u_j^+, u_j^+) + \frac{1}{2}\mathcal{B}_\alpha(u_j^-, u_j^-) - \frac{\lambda_k}{2}\int_\Omega \left(|u_j^+(x)|^2 + |u_j^-(x)|^2\right) dx \\ & - \int_\Omega \left(F(x, u_j(x)) - F(x, u_j^0(x))\right) dx - \int_\Omega F(x, u_j^0(x)) dx. \end{aligned}$$

Thus, we have that

$$\begin{aligned} \left|\int_\Omega F(x, u_j^0(x)) dx\right| \leq |J_{\lambda_k}(u_j)| + \Bigg|& \frac{1}{2}\mathcal{B}_\alpha(u_j^+, u_j^+) + \frac{1}{2}\mathcal{B}_\alpha(u_j^-, u_j^-) \\ & - \frac{\lambda_k}{2}\int_\Omega \left(|u_j^+(x)|^2 + |u_j^-(x)|^2\right) dx \; - \int_\Omega \left(F(x, u_j(x)) - F(x, u_j^0(x))\right) dx\Bigg| \end{aligned} \tag{3.12}$$

By the Poincaré inequality and the bound on u_j^+ and u_j^- we gain

$$\left|\frac{\lambda_k}{2}\int_\Omega \left(|u_j^+(x)|^2 + |u_j^-(x)|^2\right) dx\right| \leq C\left(\|u_j^+\|^2_{\mathbb{X}(\Omega)} + \|u_j^-\|^2_{\mathbb{X}(\Omega)}\right) \leq \tilde{C}$$

for some $\tilde{C} > 0$ and all $j \in \mathbb{N}$. Moreover,

$$\begin{aligned} \left|\int_\Omega \left(F(x, u_j(x)) - F(x, u_j^0(x))\right)\right| \leq & \int_\Omega \left| \int_{u_j^0(x)}^{u_j^0(x)+u_j^+(x)+u_j^-(x)} f(x,t) dt \right| \\ \leq & M \int_\Omega \left(|u_j^-| + |u_j^+|\right) dx \\ \leq & \tilde{M}\left(\|u_j^-\|_{\mathbb{X}(\Omega)} + \|u_j^+\|_{\mathbb{X}(\Omega)}\right) \leq C_1 \end{aligned}$$

for some C_1 and all $j \in \mathbb{N}$. Therefore, from (3.12), recalling that $u_j^\pm$ are bounded, we obtain

$$\left|\int_\Omega F(x, u_j^0(x)) dx\right| \leq |c| + o(1) + \left|\frac{1}{2}\mathcal{B}_\alpha(u_j^+, u_j^+) + \frac{1}{2}\mathcal{B}_\alpha(u_j^-, u_j^-)\right| + \tilde{C} + C_1 \leq C_2,$$

where $C_2 > 0$ is a constant independent of j and $o(1) \to 0$ as $j \to \infty$. Hence the sequence of integrals $\int_\Omega F(x, u_j^0(x)) dx$ is bounded. Finally, since u_j^0 belongs to H_k^0, by $(F_{\pm\infty})$ we get that u_j^0 is bounded in $\mathbb{X}(\Omega)$. $\square$

We establish the validity of the Palais-Smale condition thanks to the following result.

Proposition 3.7. *Let f satisfy (f_{bc}) and $(F_{\pm\infty})$. Suppose further that $\lambda_k < \lambda_{k+1}$ for some $k \in \mathbb{N}$. If $\{u_j\}_{j\in\mathbb{N}}$ is a Palais-Smale sequence for $\mathcal{J}_{\lambda_k}$, then there exists u_∞ in $\mathbb{X}(\Omega)$ such that u_j strongly converges to u_∞ in $\mathbb{X}(\Omega)$.*

Proof. By Proposition 3.6 u_j is bounded and $\mathbb{X}(\Omega)$ is reflexive, since $\mathbb{X}(\Omega)$ is an Hilbert space. Then there exists $u_\infty \in \mathbb{X}(\Omega)$ such that, up to a subsequence, u_j weakly converges to u_∞ in $\mathbb{X}(\Omega)$. Since $\mathbb{X}(\Omega)$ is compactly embedded in $H_0^s(\Omega)$ (and so in $L^2(\Omega)$), then, up to a subsequence, $u_j \to u_\infty$ in $H_0^s(\Omega)$ (and so in $L^2(\Omega)$) and $u_j \to u_\infty$ a.e. in Ω. This implies that

$$\mathcal{B}_\alpha(u_j, \varphi) \to \mathcal{B}_\alpha(u_\infty, \varphi) \tag{3.13}$$

for all $\varphi \in \mathbb{X}(\Omega)$, as $j \to +\infty$.

Since u_j is a Palais-Smale sequence, we have

$$\begin{aligned} 0 \leftarrow \langle \mathcal{J}'_{\lambda_k}(u_j), u_j - u_\infty \rangle =& \mathcal{B}_\alpha(u_j, u_j) - \mathcal{B}_\alpha(u_j, u_\infty) \\ & - \lambda_k \int_\Omega u_j(x)(u_j(x) - u_\infty(x))dx \\ & - \int_\Omega f(x, u_j(x))(u_j(x) - u_\infty(x))dx. \end{aligned} \tag{3.14}$$

Now, by the Hölder inequality and the bound on f we get

$$\begin{aligned} &\left| \lambda_k \int_\Omega u_j(x)(u_j(x) - u_\infty(x))dx + \int_\Omega f(x, u_j(x))(u_j(x) - u_\infty(x))dx \right| \\ &\leq \left(\lambda_k \|u_j\|_{L^2(\Omega)} + M|\Omega|^{\frac{1}{2}} \right) \|u_j - u_\infty\|_{L^2(\Omega)} \to 0, \end{aligned}$$

as $j \to +\infty$. Therefore, passing to the limit in (3.14) and taking into account (3.13) we get

$$\mathcal{B}_\alpha(u_j, u_j) \to \mathcal{B}_\alpha(u_\infty, u_\infty).$$

Since $u_j \to u$ in $H_0^s(\Omega)$, we conclude that $\|u_j\|_{\mathbb{X}(\Omega)} \to \|u_\infty\|_{\mathbb{X}(\Omega)}$. $\mathbb{X}(\Omega)$ being uniformly convex, we conclude that $u_j \to u_\infty$ strongly in $\mathbb{X}(\Omega)$. □

By combining Propositions 3.6 and 3.7 we have the proof of the following compactness property.

Proposition 3.8. *Let f satisfy (f_{bc}) and $(F_{\pm\infty})$. Suppose further that $\lambda_k < \lambda_{k+1}$ for some $k \in \mathbb{N}$. Then $\mathcal{J}_{\lambda_k}$ satisfies the Palais-Smale condition at level c for any $c \in \mathbb{R}$, namely every Palais-Smale sequence at level c admits a strongly convergent subsequence.*

We are now ready to conclude with the

Proof of Theorem 1.2. Let us start fixing some notation. Since λ is an eigenvalue, there exists $k \in \mathbb{N}$ such that $\lambda = \lambda_k < \lambda_{k+1}$. Once k has been found, we fix the decomposition $\mathbb{X}(\Omega) = H_k \oplus \mathbb{P}_{k+1}$, with H_k having finite dimension.

Let us start with the case in which $(F_{+\infty})$ is in force. From (3.3) for any $H > 0$ there exist $R > 0$ such that, if $u \in \mathbb{P}_{k+1}$ and $\|u\|_{\mathbb{X}(\Omega)} \geq R$, then

$$J_{\lambda_k}(u) > H.$$

When $u \in \mathbb{P}_{k+1}$ and $\|u\|_{\mathbb{X}(\Omega)} \leq R$, by (2.4), the Rellich-Kondrachov Theorem, the Hölder inequality and (2.4) we have

$$\begin{aligned} J_{\lambda_k}(u) &\geq \frac{\lambda_{k+1} - \lambda_k}{2} \int_\Omega |u(x)|^2 dx - \int_\Omega F(x, u(x))dx \\ &\geq -M \int_\Omega |u(x)|dx \geq -\tilde{M}\|u(x)\|_{\mathbb{X}(\Omega)} \geq -\tilde{M}R =: -C_R, \end{aligned}$$

where $\tilde{M}$ is a positive constant. Therefore, we obtain

$$J_{\lambda_k}(u) \geq -C_R \quad \text{for any} \quad u \in \mathbb{P}_{k+1}. \tag{3.15}$$

Furthermore, by (3.4) in Proposition 3.4, there exists $T > 0$ such that, for any $u \in H_k$ with $\|u\|_{\mathbb{X}(\Omega)} \geq T$, we have

$$J_{\lambda_k}(u) < -C_R. \tag{3.16}$$

Hence, by (3.15) and (3.16) we conclude that

$$\sup_{\substack{u \in H_k \\ \|u\|_{\mathbb{X}(\Omega)} = T}} J_{\lambda_k}(u) < -C_R \leq \inf_{u \in \mathbb{P}_{k+1}} J_{\lambda_k}(u),$$

so the functional J_{λ_k} satisfies the geometric assumption (I_3) and (I_4) of [33, Theorem 4.6]. Moreover, by Proposition 3.8 J_{λ_k} satisfies the Palais-Smale condition. Then the Saddle Point Theorem ([33, Theorem 4.6]) provides the existence of a critical point $u \in \mathbb{X}(\Omega)$ for the functional J_{λ_k} with

$$\mathcal{J}_{\lambda_k}(u) \leq \max_{\substack{v \in H_k \\ \|u\|_{\mathbb{X}(\Omega)} \leq T}} \mathcal{J}_{\lambda_k}(v).$$

The case $(F_{-\infty})$ can be treated similarly considering the following decomposition:

$$\mathbb{X}(\Omega) = H_k^- \oplus \left(H_k^0 \oplus \mathbb{P}_{k+1}\right),$$

where H_k^- is the finite dimensional subspace while $\mathbb{P}_{k+1} \oplus H_k^0$ is the infinite dimensional one. Reasoning as above, by using (3.5) in place of (3.4) from Proposition 3.4, we conclude the proof of the theorem. □

Remark 3.9. Assumption (f_{bc}) covers the case $f(x, 0) \neq 0$. This implies that the trivial solution is not allowed for this type of nonlinearities, like $f(x,t) = e^{-t^2}\text{sign}(t)$.

Concerning the multiplicity result stated in Theorem 1.3, its proof is an easy corollary of Theorem A.1 below.

Proof of Theorem 1.3. We consider first the following decomposition

$$\mathbb{X}(\Omega) = H_k^- \oplus \left(H_k^0 \oplus \mathbb{P}_{k+1}\right).$$

As before, we can assume that $\lambda = \lambda_k < \lambda_{k+1}$ for some $k \in \mathbb{N}$. We now consider the sphere of radius $r > 0$ in the finite dimensional subspace H_k, namely

$$S := \left\{u \in H_k : \|u\|_{\mathbb{X}(\Omega)} = r\right\}.$$

By Lemma 2.7 (since $u \in S \subset H_k$) and (F_{pos}), if $r > 0$ is small enough, being the norms in $L^\infty(\Omega)$ and in $\mathbb{X}(\Omega)$ equivalent, as H_k is finite dimensional, we get

$$\sup_{u \in S} J_{\lambda_k}(u) < 0.$$

This fact, coupled with the lower bound on $J_{\lambda_k}(u)$ for $u \in H_k^0 \oplus \mathbb{P}_{k+1}$ established in (3.15), allows to apply Theorem A.1 with $E = \mathbb{X}(\Omega)$ and $\tilde{E} = H_k^0 \oplus \mathbb{P}_{k+1}$, which yields the desired conclusion, since $\gamma(S) = \dim(H_k)$ (see [30, Remark 5.62]). □

Appendix A

We recall some basic facts about the Krasnoselskii genus and an abstract result due to Rabinowitz.

Let $A \subset \mathbb{R}^N$ be a closed and symmetric set. The genus $\gamma(A)$ of A is defined as the least integer n (if it exists) such that there is an odd function $f \in C(A, \mathbb{R}^n \setminus \{0\})$.

Set $\Sigma := \{A \subset \mathbb{R}^N : A \text{ is closed and symmetric}\}$.

Theorem A.1 *(Theorem 1.9, [32]). Let E be a real Banach space and let $I \in C^1(E, \mathbb{R})$ be even with $I(0) = 0$ and satisfy the Palais-Smale condition at any level. Suppose further that*

1. *there exists a closed subspace $\tilde{E} \subset E$ of codimension j and a constant b such that $I_{|\tilde{E}} \geq b$, and*
2. *there exists $A \in \Sigma$ with $\gamma(A) = m > j$ and $\sup_A I < 0$.*

Then I possesses at least $m - j$ distinct pairs of nontrivial critical points.

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