Early-Warning Inverse Source Problem for the Elasto-Gravitational Equations

EARLY-WARNING INVERSE SOURCE PROBLEM FOR THE ELASTO-GRAVITATIONAL EQUATIONS*

L. BALDASSARI†, M. V. DE HOOP‡, E. FRANCINI§, AND S. VESSELLA§

Abstract. Through coupled physics, we study an early-warning inverse source problem for the constant-coefficient elasto-gravitational equations. It consists of a mixed hyperbolic-elliptic system of partial differential equations describing elastic wave displacement and gravity perturbations produced by a source in a homogeneous bounded medium. Within the Cowling approximation, we prove uniqueness and Lipschitz stability for the inverse problem of recovering the moment tensor and the location of the source from early-time measurements of the changes of the gravitational field. The setup studied in this paper is motivated by gravity-based earthquake early warning systems, which are gaining much attention recently.

Key words. Inverse Problems, Lipschitz stability, elastodynamic systems

MSC codes. 35R30, 35Q86, 35J05, 35L10.

1. Introduction. In this paper we study, through coupled physics, an early-warning inverse source problem for the elasto-gravitational equations motivated by seismology. It consists of a mixed hyperbolic-elliptic system of partial differential equations describing elastic wave displacement and gravity perturbations produced by a source in a homogeneous bounded medium.

Consider the following Cauchy problem with Neumann boundary condition for the elastic equation:

$$\begin{cases} \rho_0 \mathbf{u}_{tt} - \operatorname{div}(\mathbb{C}\nabla \mathbf{u}) = \mathbf{f} & (x,t) \in \Omega \times [0,\infty), \\ (\mathbb{C}\nabla \mathbf{u}) \cdot \nu = 0 & (x,t) \in \partial\Omega \times [0,\infty), \\ \mathbf{u}(x,0) = \mathbf{u}_t(x,0) = 0 & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^3$, $\mathbf{u} = \mathbf{u}(x,t)$ denotes the displacement, ν is the outward normal to $\partial\Omega$, $\rho_0 > 0$ denotes the constant density of the medium, and $\mathbb{C}$ is the isotropic stiffness tensor with constant Lamé parameters $\lambda_0, \mu_0 > 0$:

$$C_{ijk\ell} = \lambda_0 \delta_{ij}\delta_{k\ell} + \mu_0(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}).$$

The source term $\mathbf{f}$ is defined by

$$\mathbf{f} = -M\nabla Q(x-P)H(t-T),$$

where M is a constant 3×3 real valued matrix, $P \in \Omega$, $Q \in C_0^2(\Omega)$, H is the Heaviside function and $T \geq 0$. Here we study uniqueness and stability for the early-warning inverse source problem that consists in recovering the moment tensor M

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†Department of Mathematics and Computer Science, Universität Basel, Spiegelgasse 5, CH-4051 Basel, Switzerland (lorenzo.baldassari@unibas.ch).

‡Computational and Applied Mathematics and Earth Science, Rice University, Houston, TX 77005, USA.

§Dipartimento di Matematica e Informatica 'U. Dini', Università di Firenze, FI 50134, Italy.

and the location P of the source from early-time measurements of the changes of the gravitational field ∇S^+ generated according to the following transmission problem for the Newtonian Poisson's equation:

$$\begin{cases} \Delta S^- = -\rho_0 \mathrm{div}\mathbf{u} & (x,t) \in \Omega \times [0,\infty), \\ \Delta S^+ = 0 & (x,t) \in (\mathbb{R}^3 \setminus \overline{\Omega}) \times [0,\infty), \\ S^- = S^+ & (x,t) \in \partial\Omega \times [0,\infty), \\ (\nabla S^- + \rho_0 \mathbf{u}) \cdot \nu = \nabla S^+ \cdot \nu & (x,t) \in \partial\Omega \times [0,\infty), \\ S^+ \to 0 & |x| \to \infty, \end{cases} \tag{1.2}$$

where $\mathbf{u}$ solves (1.1). The early-time measurements of ∇S^+ are taken on an open ball. The coupling between the elastic equation and Poisson's equation is known in the literature as the elasto-gravitational coupling.

The setup above is motivated by gravity-based EEW (earthquake early warning) systems: Ω is meant to model the Earth, while $\mathbf{f}$ represents a source approximating an earthquake source with seismic moment tensor M and location P. In particular, the formulation that we follow here is based on the so-called non-self-gravitating model, which geophysicists have first used when they started studying gravity-based EEW a few years ago.

1.1. Context. EEW systems rapidly detect and estimate the magnitude of ongoing earthquakes in real time to provide advance warnings of impending ground motion [3]. Conventional EEW systems rely on detecting the P-elastic waves, whose finite speed of propagation imposes a minimum on the warning time. This is key, since EEW systems may fail to rapidly estimate the size of large offshore subduction earthquakes, like the 2011 Tohoku earthquake, due to the slowness of the elastic waves [16, 17, 18, 24]. Recently discovered PEGS (prompt elasto-gravity signals) have raised hopes that these limitations may be overcome [19, 22]. PEGS are earthquake-associated signals created by density-perturbation-induced gravity field, and by the associated elastic readjustment of the gravitationally perturbed Earth [20]. PEGS are readily present in the self-gravitating equations governing the earthquake-induced motion [6], but their observations have only been provided recently, in the occurrence of large earthquakes, by ground-based seismometers [23, 26]. Since these signals are transmitted at the speed of light everywhere on Earth, the earliest deformation signals are not expected to be carried by the fastest P-elastic wave, but by PEGS [11, 10]. This is why gravity-based EEW systems have been gaining a lot of attention in recent years [15, 4]. Including PEGS in early warning systems is expected to result in faster detection of large earthquakes, especially when compared to conventional P-elastic wave-based EEW systems [15, 4].

While the physics of PEGS is understood, their use lacks a mathematical justification. In other words, we still need to develop the theoretical framework of the PEGS inverse problem, with uniqueness and stability theorems and proofs. The main challenge here is that PEGS are present in the self-gravitating equations [6, Section 3.3.2], which differ from the ones studied in this paper by the presence of a non-local term,

$$S(x,t) = -\rho_0 \int_\Omega \frac{\mathrm{div}(\mathbf{u}(y,t))}{|x-y|}\,\mathrm{d}y,$$

added to the elastic equation:

$$\rho_0 \mathbf{u}_{tt}(x,t) - \mathrm{div}(\mathbb{C}\nabla\mathbf{u}(x,t)) + \rho_0 \nabla S(x,t) = \mathbf{f}(x,t), \qquad (x,t) \in \Omega \times [0,\infty).$$

However, as explained in [11, 22, 13], when measurements are not taken with a ground-based seismometer, it is not completely unrealistic to neglect the effects of the early-time ground-induced motion due to the non-locality of $\rho_0\nabla S$. For example, one might use future generation gravity strainmeters (such as torsion bars, superconducting gradiometers, or strainmeters based on atom interferometers) measuring the differential gravitational acceleration between two test masses, as discussed in [12, 21, 25]. Juhel et al. [12] argue that the high sensitivity of these instruments makes them ideal for measuring earthquake-induced gravity perturbations, particularly given their capability to partially reject the background noise through differential measurements.

Since our intention in this paper is to focus on the basic aspects of the inverse problem arising in gravity-based EEW systems, a way to go is to account for the effects of gravity within the Cowling approximation [5], that is by ignoring self-gravitation effects. The non-self-gravitating model proposed in this paper is still valuable for gaining insight into the role of the gravitational perturbations generated by Poisson's equation in the inverse problem. It also facilitates the mathematical analysis of the elastic and gravitational components of the system, making it possible to study the elastic equation independently from Poisson's equation.

1.2. Contribution and organization of the paper. In this paper we prove, under suitable a priori assumptions, uniqueness and Lipschitz stability of the early-warning inverse source problem. We make use of the following tools:

- Energy estimates for the elastic equation.
- Estimate of propagation of smallness for elliptic equations, see [2].

The main idea is to turn the elasto-gravitational coupling to our advantage, since the changes of the gravitational field ∇S^+ are generated instantaneously by Poisson's equation, and thus can be used to solve the inverse problem without having to wait for the elastic waves to reach the boundary of Ω.

The paper is structured as follows. In paragraph 1.4, we introduce notation that will be used throughout the work. Section 2 contains the description of the direct problem. In paragraph 2.1, we prove existence and uniqueness of the weak solutions to the elasto-gravitational equations. In paragraph 2.2 we give some energy estimates concerning the solutions to problem (1.1). In section 3, we formulate our inverse problem rigorously. We then prove the main results of this paper: uniqueness first, and then Lipschitz stability.

1.3. Notation. We shall denote by $B_r(x)$ the open ball in $\mathbb{R}^3$ of radius r and center x. We shall use the abbreviation B_r when the center is the origin.

Here and in the next sections we shall assume that the Earth is represented by a nonempty open bounded convex set Ω in $\mathbb{R}^3$ with C^2 boundary, i.e., locally it can be written as the graph of a C^2 function on $\mathbb{R}^2$.

For any $h > 0$, let us define the set

$$\Omega_h := \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > h\}.$$

Given a function $\mathbf{u} : \Omega \times [0, \infty) \to \mathbb{R}^3$, $\mathbf{u} = \mathbf{u}(x, t)$, we shall denote by $\partial_j u_i$ and $u_{i,t}$ the derivatives of the i-th component of $\mathbf{u}$ with respect to the x_j variable and to the time t, respectively, and similarly for higher order derivatives.

We shall also denote by $\mathbb{M}^m$ the space of $m \times m$ real valued matrices and by $\mathcal{L}(X, Y)$ the space of bounded linear operators between Banach spaces X and Y.

For every matrices $A, B \in \mathbb{M}^m$ and for every $\mathbb{L} \in \mathcal{L}(\mathbb{M}^m, \mathbb{M}^m)$, we shall use the following notation:

$$(\mathbb{L}A)_{ij} = L_{ijk\ell}A_{k\ell}, \qquad A \cdot B = A_{ij}B_{ij}, \qquad |A| = \sqrt{(A \cdot A)}.$$

Notice that here and in the sequel summation over repeated indexes is implied.

2. The direct problem. We are now ready to present the direct or forward problem and prove its solvability. We first make some assumptions. We assume that the Earth is made of inhomogeneous linear elastic material. We denote by $\mathbb{C}(x) \in \mathcal{L}(\mathbb{M}^3, \mathbb{M}^3)$ the isotropic stiffness tensor with Lamé parameters $\lambda, \mu \in C^1(\overline{\Omega})$

$$C_{ijk\ell}(x) = \lambda(x)\delta_{ij}\delta_{k\ell} + \mu(x)(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}),$$

and by $\rho \in C^1(\overline{\Omega})$ the reference density. We assume that

$$\lambda(x) \geq \lambda_0 > 0, \quad \mu(x) \geq \mu_0 > 0, \quad \rho(x) \geq \rho_0 > 0 \quad \forall x \in \overline{\Omega}.$$

Before going further, we need to underline that, while the well-posedness of the direct problem can be proven with variable coefficients, the uniqueness and stability theorems for the inverse problem in Section 3 will require λ, μ and ρ to be constant. In such case, we will say that

$$\lambda = \lambda_0, \quad \mu = \mu_0, \quad \rho = \rho_0 \quad \text{in } \overline{\Omega}.$$

We model the source by a function $\mathbf{f}$ with at least $H^1(\Omega, \mathbb{R}^3)$ regularity, which approximates the body force defined in [6, 7] with origin time set to zero:

$$\mathbf{f} = -M\nabla Q(x - P)H(t),$$

where $M \in \mathbb{M}^3$ represents the moment tensor, P is the location of the source, $Q \in C_0^2(\Omega)$ and the Heaviside function H corresponds to an idealized time-rise function. We assume that there is no volume increase, i.e., the source does not model an earthquake caused by an explosion. Consequently, M is nonzero, symmetric and with vanishing trace [1, chapter 3]. We can regard $\mathbf{f}$ as a source that approximates an earthquake rupture, that is, slip on a geometrically flat fault; this entails a representation by double couples of equivalent forces, see [7, section 2.8] for details. The approximation is expected to be accurate in the "far field". Additionally, we assume that P is not on the boundary of Ω, that is

$$P \in \Omega_{d_0}, \tag{2.1}$$

for some positive constant d_0 such that $d_0 < \text{diam}(\Omega)/2$. Here and in the next sections, we also assume that

$$Q = q(|\cdot|),$$

where

$$\text{supp } q \subset \left[-\frac{d_0}{2}, \frac{d_0}{2}\right], \qquad \int_{\mathbb{R}^3} q(|x|) = 1. \tag{2.2}$$

Since we study our problem for $t \geq 0$, in what follows we will ignore the dependence on time in $\mathbf{f}$. However, it's worth noting that the main results of this paper extend to cases where the time dependency of the source is not described by a Heaviside function (for example, one might consider a class of time-dependent sources represented by $\mathbf{f} = -M\nabla Q(x-P)g(t)$, where g is not identically zero for early times).

The direct problem consists in finding the solution pair $(\mathbf{u}, S)$, where $\mathbf{u}$ solves the following Cauchy problem with Neumann boundary condition for the elastic equation

$$\begin{cases} \rho\mathbf{u}_{tt} - \operatorname{div}(\mathbb{C}\nabla\mathbf{u}) = \mathbf{f} & (x,t) \in \Omega \times [0,\infty), \\ (\mathbb{C}\nabla\mathbf{u}) \cdot \nu = 0 & (x,t) \in \partial\Omega \times [0,\infty), \\ \mathbf{u}(x,0) = \mathbf{u}_t(x,0) = 0 & x \in \Omega, \end{cases} \tag{2.3}$$

and S, written as

$$S(x,t) = \begin{cases} S^-(x,t) & (x,t) \in \Omega \times [0,\infty), \\ S^+(x,t) & (x,t) \in \mathbb{R}^3 \setminus \overline{\Omega} \times [0,\infty), \end{cases} \tag{2.4}$$

is the solution to the following transmission problem for Poisson's equation

$$\begin{cases} \Delta S^- = -\operatorname{div}(\rho\mathbf{u}) & (x,t) \in \Omega \times [0,\infty), \\ \Delta S^+ = 0 & (x,t) \in (\mathbb{R}^3 \setminus \overline{\Omega}) \times [0,\infty), \\ S^- = S^+ & (x,t) \in \partial\Omega \times [0,\infty), \\ (\nabla S^- + \rho\mathbf{u}) \cdot \nu = \nabla S^+ \cdot \nu & (x,t) \in \partial\Omega \times [0,\infty), \\ S^+ \to 0 & |x| \to \infty. \end{cases} \tag{2.5}$$

It is evident that we can solve first (2.3), and then (2.5).

2.1. Existence, uniqueness, and regularity of solutions. As anticipated, we will first show existence, uniqueness, and regularity of solutions to problem (2.3). The proofs are based on results of [14].

THEOREM 2.1. *Assume that* $\mathbf{f} \in L^2(\Omega, \mathbb{R}^3)$. *Then there is a unique weak solution* $\mathbf{u}$ *to* (2.3) *such that*

$$\mathbf{u} \in C([0,\infty); H^1(\Omega, \mathbb{R}^3)), \quad \mathbf{u}_t \in C([0,\infty); L^2(\Omega, \mathbb{R}^3)).$$

If $\mathbf{f} \in H^1(\Omega, \mathbb{R}^3)$, *then the solution to* (2.3) *is such that*

$$\mathbf{u} \in C([0,\infty); H^2(\Omega, \mathbb{R}^3)),$$

and satisfies (2.3) *in a pointwise sense. Also*

$$\mathbf{u}_t \in C([0,\infty); H^1(\Omega, \mathbb{R}^3)), \quad \mathbf{u}_{tt} \in C([0,\infty); L^2(\Omega, \mathbb{R}^3)).$$

Proof. To solve (2.3), we apply the Duhamel's principle. We consider the problem

$$\begin{cases} \rho\mathbf{w}_{tt} - \operatorname{div}(\mathbb{C}\nabla\mathbf{w}) = 0 & (x,t) \in \Omega \times (s,\infty), \\ (\mathbb{C}\nabla\mathbf{w}) \cdot \nu = 0 & (x,t) \in \partial\Omega \times [s,\infty), \\ \mathbf{w}(x,s;s) = 0 & x \in \Omega, \\ \mathbf{w}_t(x,s;s) = \mathbf{f}(x) & x \in \Omega. \end{cases}$$

By [14, Theorems 2.1 and 2.2], if $\mathbf{f} \in L^2(\Omega, \mathbb{R}^3)$, there exists a unique solution $\mathbf{w}$ such that

$$\mathbf{w} \in C([0,\infty); H^1(\Omega, \mathbb{R}^3)), \quad \mathbf{w}_t \in C([0,\infty); L^2(\Omega, \mathbb{R}^3)).$$

Additionally, if $\mathbf{f} \in H^1(\Omega, \mathbb{R}^3)$, we have that $\mathbf{w} \in C([0,\infty); H^2(\Omega, \mathbb{R}^3))$. Since, by the Duhamel's principle,

$$\mathbf{u}(x,t) := \int_0^t \mathbf{w}(x,t;s)\,\mathrm{d}s$$

solves (2.3), we can easily deduce existence, uniqueness, and regularity results for $\mathbf{u}$ from those of $\mathbf{w}$. ∎

We now consider the standard transmission problem for Poisson's equation of the gravitational potential

$$\begin{cases} \Delta S^- = -\mathrm{div}(\rho \mathbf{u}) & (x,t) \in \Omega \times [0,\infty), \\ \Delta S^+ = 0 & (x,t) \in (\mathbb{R}^3 \setminus \overline{\Omega}) \times [0,\infty), \\ S^- = S^+ & (x,t) \in \partial\Omega \times [0,\infty), \\ (\nabla S^- + \rho \mathbf{u}) \cdot \nu = \nabla S^+ \cdot \nu & (x,t) \in \partial\Omega \times [0,\infty), \\ S^+ \to 0 & |x| \to \infty. \end{cases} \tag{2.6}$$

By the definition of S given in (2.4), problem (2.6) can be written in weak formulation as

(2.7)

$$\text{for a.e. } t, \text{ find } S \in H^1(\mathbb{R}^3) \text{ such that: } \forall \phi \in H^1(\mathbb{R}^3), \int_{\mathbb{R}^3} \nabla S \cdot \nabla \phi = -\int_\Omega \rho \mathbf{u} \cdot \nabla \phi.$$

The Lax-Milgram theorem [9] implies the existence of a unique $S \in H^1(\mathbb{R}^3)$ solving (2.7).

2.2. Energy estimates. We will use the following energy estimates, whose proofs are contained in Appendix A. For the sake of simplicity we formulate such estimates for $\lambda = \lambda_0, \mu = \mu_0$ and $\rho = \rho_0$ in $\overline{\Omega}$, but they hold true also for λ, μ, and $\rho \in C^1(\overline{\Omega})$.

Proposition 2.2. *For any $\tau \geq 0$, the solution $\mathbf{u}$ to (2.3) satisfies the inequality*

$$\int_\Omega |\mathbf{u}(\cdot,\tau)|^2 \leq \tau^3 \frac{e^{\tau/\rho_0}}{\rho_0} \int_\Omega |\mathbf{f}|^2. \tag{2.8}$$

Proposition 2.3. *Let $\alpha > 0$, $x_0 \in \partial\Omega$ and $\tau > 0$. Denote by K_α the cone*

$$K_\alpha(x_0,\tau) := \{(x,t) \in \mathbb{R}^4 \text{ such that } 0 \leq t \leq \tau - \alpha|x - x_0|\},$$

and define

$$\widetilde{K}_\alpha(x_0,\tau) := K_\alpha(x_0,\tau) \cap (\Omega \times [0,\infty)).$$

For any $\gamma > 0$, and for

$$\alpha = \alpha_0 := \sqrt{\frac{\rho_0}{2(\lambda_0 + 2\mu_0)}},$$

the solution $\mathbf{u}$ to (2.3) satisfies the inequality

$$\int_{\widetilde{K}_{\alpha_0}(x_0,\tau)} |\mathbf{u}|^2 e^{-2\gamma t} \leq \left(\frac{\tau}{\rho_0 \gamma}\right)^2 \int_{\widetilde{K}_{\alpha_0}(x_0,\tau)} |\mathbf{f}|^2 e^{-2\gamma t}. \tag{2.9}$$

The following corollaries of Proposition 2.3 will be useful later. They show that at early times there exists a region close to $\partial\Omega$ where $\mathbf{u} = 0$ because the elastic waves have not arrived there yet. Figure 1 illustrates how to construct such region, exploiting the knowledge of Proposition 2.2 that, for any point of $\partial\Omega \times \{0\}$, $\mathbf{u}$ vanishes inside specified spacetime cones centred at that point.

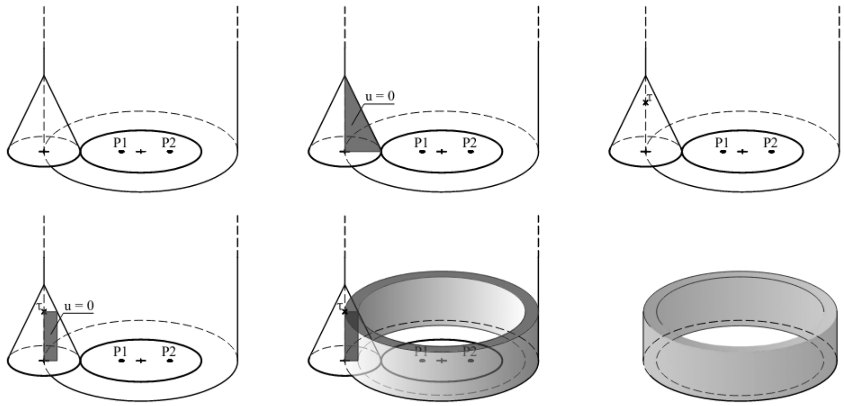

FIG. 1. *The above illustrations show step-by-step how to construct* $\Omega \setminus \overline{\Omega}_{d_0/2-\tau/\alpha_0} \times [0,\tau]$ *(cf. the last picture) such that it is contained in* $\widetilde{\mathcal{K}}_{\alpha_0}(\tau)$ *as is defined in Corollary 2.4. Notice that* $\boldsymbol{u} = 0$ *in the gray areas. By Proposition 2.3, for any point of* $\partial\Omega \times \{0\}$*,* $\boldsymbol{u}$ *vanishes inside specified spacetime cones centred at that point (cf. the second picture).*

COROLLARY 2.4. *Define*

$$\widetilde{\mathcal{K}}_{\alpha_0}(\tau) := \bigcup_{x_0 \in \partial\Omega} \widetilde{K}_{\alpha_0}(x_0, \tau).$$

There exists $\tau_0 > 0$ *such that, for any* $\tau \in (0, \tau_0)$*,* $\mathbf{u} = 0$ *in* $\widetilde{\mathcal{K}}_{\alpha_0}(\tau)$*.*

Proof. Since supp $\mathbf{f} \subset \Omega_{d_0/2}$ by (2.1), it suffices to take $\tau_0 = (\alpha_0 d_0)/2$ to have that, for any $\tau \in (0, \tau_0)$, supp $\mathbf{f} \cap \widetilde{K}_{\alpha_0}(x_0, \tau) = \emptyset$, independently from the choice of $x_0 \in \partial\Omega$. Using (2.9), we obtain that $\mathbf{u} = 0$ in $\widetilde{\mathcal{K}}_{\alpha_0}(\tau)$. ∎

COROLLARY 2.5. *Let* τ_0 *be the same as in Corollary 2.4. We have* $\mathbf{u} = 0$ *on* $\partial\Omega \times [0, \tau_0)$*. Also, for any* $\tau \in (0, \tau_0)$*, we have*

$$\Omega \setminus \overline{\Omega}_{\frac{d_0}{2} - \frac{\tau}{\alpha_0}} \times [0, \tau] \subseteq \widetilde{\mathcal{K}}_{\alpha_0}(\tau),$$

hence $\mathbf{u} = 0$ *there.*

3. The inverse problem. In this section, we will investigate the following inverse problem: *given early-time measurements of the changes of the gravitational field* ∇S^+ *generated by the source, can we determine uniquely and in a stable way the moment tensor* M *and the location* P *of the source?*

We suppose to have measured ∇S^+ on $B_{r_0}(\bar{x})$ contained in $\mathbb{R}^3 \setminus \overline{\Omega}$, for times in the range $t \in [0, t_0]$. To prove uniqueness and stability for the early-warning inverse problem we need to assume that the Lamé parameters and the density are positive constants, i.e. $\lambda = \lambda_0$, $\mu = \mu_0$, $\rho = \rho_0$. In addition, to prove the Lipschitz stability estimate, we need to assume that all the admissible moment tensors M satisfy the condition

$$m_0 \leq |M| \leq M_0, \quad \text{given } m_0, M_0 > 0. \tag{3.1}$$

Before going further, it is noteworthy to point out that the early-warning inverse problem we propose in this paper can be solved for any $t_0 > 0$, hence the adjective "early-warning". This is an advantage with respect to conventional inverse seismic problems. In fact, since the elastic waves propagate at finite speed, one has to wait for them to reach the boundary of Ω (see, for example, Theorem 4.2 in [8]). The changes of the gravitational field, instead, are generated instantaneously by Poisson's equation, and thus can be used to solve the inverse problem without having to impose a minimum on the time needed to determine uniquely the source.

3.1. Statement of the main results. The main result of this paper is the Lipschitz stability of the inverse problem:

THEOREM 3.1 (Lipschitz stability). *Let $B_{r_0}(\bar{x}) \subset \mathbb{R}^3 \setminus \overline{\Omega}$, $t_0 > 0$. Consider two sources, $\mathbf{f}^{(1)}$ and $\mathbf{f}^{(2)}$, such that*

$$\mathbf{f}^{(j)} = -M^{(j)}\nabla(q(|x - P^{(j)}|)), \quad j = 1, 2,$$

where $M^{(1)}, M^{(2)} \in \mathbb{M}^3$ are nonzero, symmetric, with vanishing trace, satisfy (3.1), *$P^{(1)}, P^{(2)} \in \Omega_{d_0}$, and $q \in C^2_0(\Omega)$ satisfies* (2.2).

Let $(\mathbf{u}^{(1)}, S^{(1)})$ and $(\mathbf{u}^{(2)}, S^{(2)})$ be (weak) solutions to (2.3)-(2.5) *associated to $\mathbf{f}^{(1)}$ and $\mathbf{f}^{(2)}$, respectively, when $\lambda = \lambda_0$, $\mu = \mu_0$ and $\rho = \rho_0$. If*

$$\|\nabla S^{(2)}(\cdot, t) - \nabla S^{(1)}(\cdot, t)\|_{L^2(B_{r_0}(\bar{x}))} \leq \varepsilon \quad \text{in } [0, t_0],$$

then we have

$$\left|P^{(2)} - P^{(1)}\right| + \left|M^{(2)} - M^{(1)}\right| \leq C\varepsilon,$$

where C is a positive constant depending on M_0, m_0, r_0, d_0, Ω, λ_0, μ_0, ρ_0 and t_0.

Uniqueness follows from Theorem 3.1 by letting $\varepsilon \to 0$. However, for the sake of the reader's understanding, in the sequel we first give the proof of uniqueness, since it clearly presents the role played by the elasto-gravitational coupling effect in solving the inverse problem. Also, focusing first on uniqueness will make the proof of the desired Lipschitz stability a bit lighter, since we will reuse some of the calculations.

Our uniqueness result can be summarized as follows:

THEOREM 3.2 (uniqueness). *Let $B_{r_0}(\bar{x}) \subset \mathbb{R}^3 \setminus \overline{\Omega}$, $t_0 > 0$. Under the same hypothesis of Theorem 3.1 for $\mathbf{f}^{(1)}$, $\mathbf{f}^{(2)}$, $(\mathbf{u}^{(1)}, S^{(1)})$, $(\mathbf{u}^{(2)}, S^{(2)})$, λ, μ and ρ, if*

$$\nabla S^{(1)} = \nabla S^{(2)} \quad \text{in } B_{r_0}(\bar{x}) \times [0, t_0],$$

then

$$M^{(1)} = M^{(2)} \quad \text{and} \quad P^{(1)} = P^{(2)}.$$

3.2. Proof of Theorem 3.2. Before going through the proof of uniqueness, we shall need the following technical result:

LEMMA 3.3. *We have*

$$\int_{B_{\frac{d_0}{2}}} \frac{q'(|x|)}{|x|} x_j^2 = -1, \qquad \int_{B_{\frac{d_0}{2}}} \frac{q'(|x|)}{|x|} x_j = \int_{B_{\frac{d_0}{2}}} \frac{q'(|x|)}{|x|} x_j x_k x_\ell = 0.$$

Proof. Define

$$L_k := \int_{B_{\frac{d_0}{2}}} \frac{q'(|x|)}{|x|} x_j^2.$$

By symmetry, $L := L_1 = L_2 = L_3$. We have

$$3L = L_1 + L_2 + L_3 = \int_{B_{\frac{d_0}{2}}} \frac{q'(|x|)}{|x|}(x_1^2 + x_2^2 + x_3^2) = \int_{B_{\frac{d_0}{2}}} q'(|x|)|x|$$
$$= 4\pi \int_0^{\frac{d_0}{2}} q'(s)s^3 = -12\pi \int_0^{\frac{d_0}{2}} q(s)s^2 = -3 \int_{B_{\frac{d_0}{2}}} q(|x|) = -3,$$

which implies $L = -1$.

To prove the other results, it suffices to notice that we integrate an odd function over a spherically symmetric domain around the origin, hence the integral is zero. □

We are now in the position of proving our uniqueness result. We proceed as follows. First, we apply the unique continuation property to (2.6): since ∇S vanishes in a ball outside Ω, the objective is to show that it vanishes everywhere outside Ω and in a neighborhood of $\partial\Omega$. Then, after selecting a specified set of test functions, we resort to integration by parts to prove the desired uniqueness result. It is worth noticing that applying the unique continuation property inside Ω is nontrivial, since $\Delta S = -\rho_0 \mathrm{div}\mathbf{u}$ there and, in principle, the right-hand side may be nonzero. As we will see, this is precisely where the energy estimates of Subsection 2.2 and the hypothesis that we are taking early-time measurements of ∇S will come into play.

Proof of Theorem 3.2. Define

$$\mathbf{u} := \mathbf{u}^{(2)} - \mathbf{u}^{(1)}, \quad S := S^{(2)} - S^{(1)}, \quad \mathbf{f} := \mathbf{f}^{(2)} - \mathbf{f}^{(1)}.$$

($\mathbf{u}$, S) solves (in a weak sense) the following system of elasto-gravitational equations:

$$\begin{cases} \rho_0 \mathbf{u}_{tt} - \mathrm{div}(\mathbb{C}\nabla\mathbf{u}) = \mathbf{f} & (x,t) \in \Omega \times [0,\infty), \\ \Delta S^- = -\rho_0 \mathrm{div}\mathbf{u} & (x,t) \in \Omega \times [0,\infty), \\ \Delta S^+ = 0 & (x,t) \in (\mathbb{R}^3 \setminus \overline{\Omega}) \times [0,\infty), \\ (\mathbb{C}\nabla\mathbf{u}) \cdot \nu = 0 & (x,t) \in \partial\Omega \times [0,\infty), \\ S^- = S^+ & (x,t) \in \partial\Omega \times [0,\infty), \\ (\nabla S^- + \rho_0\mathbf{u}) \cdot \nu = \nabla S^+ \cdot \nu & (x,t) \in \partial\Omega \times [0,\infty), \\ \mathbf{u}(x,0) = \mathbf{u}_t(x,0) = 0 & x \in \Omega, \\ S^+ \to 0 & |x| \to \infty. \end{cases} \tag{3.2}$$

Also, since $\mathbf{f} \in H^1(\Omega, \mathbb{R}^3)$, the regularity results given in Section 2 hold true. In particular, the estimate of Proposition 2.3 and its corollaries can be applied to $\mathbf{u}$.

Let τ_0 be the same as in Corollary 2.4. In what follows, we assume that $t_0 < \tau_0$. We can do this, since our intention is to prove the uniqueness result for small times. We begin by recalling that, from Corollary 2.5, $\mathbf{u} = 0$ in $\Omega \setminus \overline{\Omega}_{\frac{d_0}{2} - \frac{t_0}{\alpha_0}} \times [0, t_0]$, hence

$$\begin{cases} \Delta S^- = -\rho_0 \mathrm{div}\mathbf{u} & (x,t) \in \Omega_{\frac{d_0}{2} - \frac{t_0}{\alpha_0}} \times [0,t_0], \\ \Delta S^- = 0 & (x,t) \in \Omega \setminus \overline{\Omega}_{\frac{d_0}{2} - \frac{t_0}{\alpha_0}} \times [0,t_0], \\ \Delta S^+ = 0 & (x,t) \in (\mathbb{R}^3 \setminus \overline{\Omega}) \times [0,t_0], \\ S^- = S^+ & (x,t) \in \partial\Omega \times [0,t_0], \\ \nabla S^- \cdot \nu = \nabla S^+ \cdot \nu & (x,t) \in \partial\Omega \times [0,t_0], \\ S^+ \to 0 & |x| \to \infty. \end{cases}$$

The continuity of the transmission conditions on $\partial\Omega$ assures that S defined in (2.4) belongs to $H^2_{\mathrm{loc}}(\mathbb{R}^3 \setminus \overline{\Omega}_{\frac{d_0}{2}-\frac{t_0}{\alpha_0}})$ for every $t \in [0, t_0]$. We apply the unique continuation property. Since $\nabla S = 0$ in $B_{r_0}(\bar{x}) \times [0, t_0]$ and $S \to 0$ as $|x| \to \infty$, we have that

$$S(x,t) = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega}_{\frac{d_0}{2}-\frac{t_0}{\alpha_0}} \quad \text{for every } t \in [0, t_0]. \tag{3.3}$$

Now the hypothesis that $\lambda = \lambda_0$, $\mu = \mu_0$ and $\rho = \rho_0$ are constants comes into play. Denote by ϕ a smooth function such that $\Delta\phi = 0$ in Ω. Recall that $\mathbf{u}$ solves

$$\begin{cases} \rho_0 \mathbf{u}_{tt} - \operatorname{div}(\mathbb{C}\nabla\mathbf{u}) = \mathbf{f} & (x,t) \in \Omega \times [0, t_0], \\ (\mathbb{C}\nabla\mathbf{u}) \cdot \nu = 0 & (x,t) \in \partial\Omega \times [0, t_0], \\ \mathbf{u}(x,0) = \mathbf{u}_t(x,0) = 0 & x \in \Omega. \end{cases}$$

Multiplying the first equation of the system above by $\nabla\phi$ and integrating over Ω yield

$$\rho_0 \int_\Omega \mathbf{u}_{tt} \cdot \nabla\phi - \int_\Omega \partial_j (C_{ijkl} \partial_k u_l) \partial_i \phi = \int_\Omega \mathbf{f} \cdot \nabla\phi. \tag{3.4}$$

We notice that

$$\int_\Omega \partial_j (C_{ijkl} \partial_k u_l) \partial_i \phi = \int_\Omega [\partial_j (C_{ijkl} \partial_k u_l \partial_i \phi) - C_{ijkl} \partial_k u_l \partial^2_{ji} \phi] = - \int_\Omega C_{ijkl} \partial_k u_l \partial^2_{ji} \phi,$$

where the last equality follows from the Neumann boundary condition for $\mathbf{u}$. Since $\Delta\phi = 0$, we have

$$\begin{aligned} \int_\Omega C_{ijkl} \partial_k u_l \partial^2_{ji} \phi &= \lambda_0 \int_\Omega \Delta\phi \operatorname{div}\mathbf{u} + 2\mu_0 \int_\Omega \partial_i u_j \partial^2_{ij} \phi \\ &= 2\mu_0 \int_\Omega \partial_i (u_j \partial^2_{ij} \phi) - 2\mu_0 \int_\Omega u_j \partial_j \Delta\phi = 2\mu_0 \int_{\partial\Omega} u_j \partial^2_{ij} \phi \nu_i = 0, \end{aligned}$$

where the last equality follows from the fact that $\mathbf{u} = 0$ on $\partial\Omega \times [0, t_0]$. Thus equation (3.4) becomes

$$\rho_0 \int_\Omega \mathbf{u}_{tt} \cdot \nabla\phi = \int_\Omega \mathbf{f} \cdot \nabla\phi. \tag{3.5}$$

Define

$$z := \rho_0 \int_\Omega \mathbf{u} \cdot \nabla\phi. \tag{3.6}$$

From [9, Section 5.9.2] we have

$$z_{tt} = \rho_0 \int_\Omega \mathbf{u}_{tt} \cdot \nabla\phi. \tag{3.7}$$

To prove the uniqueness of both the moment tensor and location of the source, we first need to verify that $z(t) = 0$ in $[0, t_0]$. Since $\mathbf{u} = 0$ on $\partial\Omega \times [0, t_0]$, integrating by parts yields:

$$\begin{aligned} z &= \rho_0 \int_\Omega \partial_j (u_j \phi) - \operatorname{div}\mathbf{u}\, \phi = \rho_0 \int_{\partial\Omega} u_j \nu_j \phi + \int_\Omega \Delta S \phi \\ &= \int_{\partial\Omega} \frac{\partial S}{\partial\nu} \phi - S \frac{\partial\phi}{\partial\nu}. \end{aligned} \tag{3.8}$$

Then the fact that $z = 0$ follows from (3.3). Thus equation (3.5) gives

$$\int_\Omega \mathbf{f} \cdot \nabla\phi = 0,$$

for every ϕ harmonic function in Ω. Since $\mathbf{f} = \mathbf{f}^{(2)} - \mathbf{f}^{(1)}$, the equality above yields:

$$\begin{aligned}\int_{B_{\frac{d_0}{2}}(P^{(1)})} \frac{q'(|x-P^{(1)}|)}{|x-P^{(1)}|} M^{(1)}(x-P^{(1)}) \cdot \nabla\phi \\ = \int_{B_{\frac{d_0}{2}}(P^{(2)})} \frac{q'(|x-P^{(2)}|)}{|x-P^{(2)}|} M^{(2)}(x-P^{(2)}) \cdot \nabla\phi.\end{aligned} \tag{3.9}$$

To prove that $M^{(1)} = M^{(2)}$ and $P^{(1)} = P^{(2)}$, we have to make a specific choice for ϕ.

Step 1: Moment tensor. We first consider $\phi(x) = x_1x_2$. Trivially, such a function is harmonic in Ω. Also, after a change of variables, we get

$$\begin{aligned}&\int_{B_{\frac{d_0}{2}}(P^{(1)})} \frac{q'(|x-P^{(1)}|)}{|x-P^{(1)}|} M^{(1)}_{kl}(x_l - (P^{(1)})_l)\partial_k\phi \\ &= \int_{B_{\frac{d_0}{2}}} \frac{q'(|y|)}{|y|} \left(M^{(1)}_{1l} y_l y_2 + M^{(1)}_{2l} y_l y_1 + M^{(1)}_{1l} y_l (P^{(1)})_2 + M^{(1)}_{2l} y_l (P^{(1)})_1 \right).\end{aligned}$$

Since $M^{(1)}$ is symmetric, Lemma 3.3 gives

$$\int_{B_{\frac{d_0}{2}}(P^{(1)})} \frac{q'(|x-P^{(1)}|)}{|x-P^{(1)}|} M^{(1)}_{kl}(x_l - (P^{(1)})_l)\partial_k\phi = M^{(1)}_{12} \int_{B_{\frac{d_0}{2}}} \frac{q'(|y|)}{|y|} \left(y_1^2 + y_2^2\right) = -2M^{(1)}_{12}.$$

Repeating the same calculations for $M^{(2)}$ yields

$$\int_{B_{\frac{d_0}{2}}(P^{(2)})} \frac{q'(|x-P^{(2)}|)}{|x-P^{(2)}|} M^{(2)}(x-P^{(2)}) \cdot \nabla\phi = -2M^{(2)}_{12}.$$

From (3.9), we have $M^{(1)}_{12} = M^{(2)}_{12}$. By taking $\phi = x_ix_j$, with $i \neq j$, we finally get

$$M^{(1)}_{ij} = M^{(2)}_{ij}, \quad i,j = 1,2,3, \quad i \neq j.$$

We now repeat the calculations above for $\phi = (x_1^2 - x_2^2)/2$, and exploit the fact that both $M^{(1)}$ and $M^{(2)}$ have vanishing traces. After a change of variables, we get

$$\begin{aligned}&\int_{B_{\frac{d_0}{2}}(P^{(1)})} \frac{q'(|x-P^{(1)}|)}{|x-P^{(1)}|} M^{(1)}_{kl}(x_l - (P^{(1)})_l)\partial_k\phi \\ &= \int_{B_{\frac{d_0}{2}}} \frac{q'(|y|)}{|y|} \left(M^{(1)}_{1l} y_l y_1 - M^{(1)}_{2l} y_l y_2 + M^{(1)}_{1l} y_l (P^{(1)})_2 - M^{(1)}_{2l} y_l (P^{(1)})_1 \right) \\ &= \int_{B_{\frac{d_0}{2}}} \frac{q'(|y|)}{|y|} \left(M^{(1)}_{11} y_1^2 - M^{(1)}_{22} y_2^2 \right) = M^{(1)}_{22} - M^{(1)}_{11}.\end{aligned}$$

Repeating the same calculations for $M^{(2)}$ yields

$$\int_{B_{\frac{d_0}{2}}(P^{(2)})} \frac{q'(|x-P^{(2)}|)}{|x-P^{(2)}|} M^{(2)}(x-P^{(2)}) \cdot \nabla\phi = M^{(2)}_{22} - M^{(2)}_{11}.$$

From (3.9), we have

$$M^{(1)}_{11} - M^{(2)}_{11} = M^{(1)}_{22} - M^{(2)}_{22}.$$

By taking $\phi = (x_1^2 - x_3^2)/2$ and $\phi = (x_2^2 - x_3^2)/2$, we get also

$$M^{(1)}_{11} - M^{(2)}_{11} = M^{(1)}_{33} - M^{(2)}_{33}, \qquad M^{(1)}_{22} - M^{(2)}_{22} = M^{(1)}_{33} - M^{(2)}_{33}.$$

Using the results above, together with $\mathrm{tr}M^{(1)} = \mathrm{tr}M^{(2)} = 0$:

$$\left(M^{(1)}_{11} - M^{(2)}_{11}\right) + \left(M^{(1)}_{22} - M^{(2)}_{22}\right) + \left(M^{(1)}_{33} - M^{(2)}_{33}\right) = 0,$$

we find

$$M^{(1)}_{11} = M^{(2)}_{11}, \qquad M^{(1)}_{22} = M^{(2)}_{22}, \qquad M^{(1)}_{33} = M^{(2)}_{33}.$$

Putting everything together, we finally get $M^{(1)} = M^{(2)}$.

Step 2: Location of the source. Denote now by M the moment tensor. We begin by noticing that (3.9) can be rewritten as follows:

$$\begin{aligned} &\int_{B_{\frac{d_0}{2}}(P^{(1)})} \frac{q'(|x-P^{(1)}|)}{|x-P^{(1)}|} M(x-P^{(1)}) \cdot \nabla\phi \\ &\qquad = \int_{B_{\frac{d_0}{2}}(P^{(2)})} \frac{q'(|x-P^{(2)}|)}{|x-P^{(2)}|} M(x-P^{(2)}) \cdot \nabla\phi. \end{aligned} \tag{3.10}$$

We now consider $\phi = x_1^3 - 3x_2^2 x_1$. Again, such a function is harmonic in Ω. Also, after a change of variable, we get

$$\begin{aligned} &\int_{B_{\frac{d_0}{2}}(P^{(1)})} \frac{q'(|x-P^{(1)}|)}{|x-P^{(1)}|} M_{kl}(x_l - (P^{(1)})_l)\partial_k\phi \\ &= \int_{B_{\frac{d_0}{2}}} \frac{q'(|y|)}{|y|} \Big(M_{11}y_1(3(y_1+(P^{(1)})_1)^2 - 3(y_2+(P^{(1)})_2)^2) + M_{12}y_2(3(y_1+(P^{(1)})_1)^2 \\ &\quad -3(y_2+(P^{(1)})_2)^2) + M_{13}y_3(3(y_1+(P^{(1)})_1)^2 - 3(y_2+(P^{(1)})_2)^2) \\ &\quad +M_{21}y_1(-6(y_2+(P^{(1)})_2)(y_1+(P^{(1)})_1)) + M_{22}y_2(-6(y_2+(P^{(1)})_2)(y_1+(P^{(1)})_1)) \\ &\quad +M_{23}y_3(-6(y_2+(P^{(1)})_2)(y_1+(P^{(1)})_1))\Big). \end{aligned}$$

Since $M_{12} = M_{21}$ and $M_{11} = -M_{22} - M_{22}$, Lemma (3.3) implies that

$$\begin{aligned} &\int_{B_{\frac{d_0}{2}}(P^{(1)})} \frac{q'(|x-P^{(1)}|)}{|x-P^{(1)}|} M_{kl}(x_l - (P^{(1)})_l)\partial_k\phi \\ &= \int_{B_{\frac{d_0}{2}}} \frac{q'(|y|)}{|y|} \Big(6M_{11}y_1^2(P^{(1)})_1 - 6M_{12}y_2^2(P^{(1)})_2 - 6M_{21}y_1^2(P^{(1)})_2 - 6M_{22}y_2^2(P^{(1)})_1 \Big) \\ &= 6(2M_{22} + M_{33})(P^{(1)})_1 + 12M_{12}(P^{(1)})_2. \end{aligned}$$

Repeating the same calculations for $P^{(2)}$ yields

$$\int_{B_{\frac{d_0}{2}}(P^{(2)})} \frac{q'(|x-P^{(2)}|)}{|x-P^{(2)}|} M_{kl}(x_l-(P^{(2)})_l)\partial_k\phi = 6(2M_{22}+M_{33})(P^{(2)})_1+12M_{12}(P^{(2)})_2.$$

From (3.10), we have

$$(2M_{22}+M_{33})((P^{(2)})_1-(P^{(1)})_1)+2M_{12}((P^{(2)})_2-(P^{(1)})_2)=0. \tag{3.11}$$

We now consider $\phi = x_2^3 - 3x_1^2x_2$. For such a choice of ϕ, we obtain

$$(2M_{22}+M_{33})((P^{(2)})_2-(P^{(1)})_2)+2M_{12}((P^{(1)})_1-(P^{(2)})_1)=0. \tag{3.12}$$

We multiply (3.11) by $2M_{22}+M_{33}$:

$$(2M_{22}+M_{33})^2((P^{(1)})_1-(P^{(2)})_1)+2M_{12}(2M_{22}+M_{33})((P^{(1)})_2-(P^{(2)})_2)=0$$

and (3.12) by $2M_{12}$:

$$2M_{12}(2M_{22}+M_{33})((P^{(2)})_2-(P^{(1)})_2)+4M_{12}^2((P^{(1)})_1-(P^{(2)})_1)=0$$

After summing both equations, we get

$$((2M_{22}+M_{33})^2+4M_{12}^2)((P^{(1)})_1-(P^{(2)})_1)=0. \tag{3.13}$$

We now consider $\phi = x_1^3 - 3x_3^2x_1$ and $\phi = x_3^2 - 3x_3x_1^2$. We follow the same procedure as the one described above. We get

$$((M_{22}+2M_{33})^2+4M_{31}^2)((P^{(1)})_1-(P^{(2)})_1)=0. \tag{3.14}$$

Finally, we consider $\phi = x_1x_2x_3$. We get

$$M_{32}((P^{(1)})_1-(P^{(2)})_1)=0. \tag{3.15}$$

Assume now that $(P^{(1)})_1 \neq (P^{(2)})_1$. Then

$$M_{32}=M_{12}=M_{31}=M_{22}=M_{33}=0.$$

Since M is symmetric and $\mathrm{tr}M=0$, the assumption $(P^{(1)})_1 \neq (P^{(2)})_1$ gives $M=0$, which is absurd.

Finally, to prove that $(P^{(1)})_2=(P^{(2)})_2$ and $(P^{(1)})_3=(P^{(2)})_3$, it will suffice to consider also the following harmonic functions:

$$\phi = x_2^3 - 3x_2x_3^2, \quad \text{and } \phi = x_3^3 - 3x_3x_2^2. \qquad \square$$

3.3. Proof of Theorem 3.1. The proof of stability is quite technical and longer than the proof of uniqueness. We proceed as follows. First, we multiply the elastic equation by a set of harmonic test functions to be specified at a later stage, as we did for the uniqueness proof. The result is equation (3.5). Now the objective is to propagate the smallness of the data directly into the right-hand side of (3.5), which encodes the information on both the moment tensor and location of the source. In doing so, we exploit the fact that S is harmonic in a neighborhood of $\partial\Omega$ (as we have seen, this is a consequence of Corollary 2.4) and use a theorem from [2]. Then, we retrace some calculations done in the uniqueness proof and choose harmonic polynomials of degree three as test functions.

Proof. Define $\mathbf{u}$, S, and $\mathbf{f}$ as at the beginning of the uniqueness proof of Theorem 3.2. To prove the desired Lipschitz stability estimate, we first need to propagate the smallness of the data

$$\|\nabla S(\cdot,t)\|_{L^2(B_{r_0}(\bar{x}))} \leq \varepsilon \quad \text{in } [0,t_0],$$

into the integral

$$\left|\int_\Omega \mathbf{f}\cdot\nabla\phi\right|,$$

since $\mathbf{f}$, by definition, encodes the information on both $|M^{(2)}-M^{(1)}|$ and $|P^{(2)}-P^{(1)}|$.

Let τ_0 be the same as in Corollary 2.4. As in the uniqueness proof of Theorem 3.2, we consider, without loss of generality, $t_0<\tau_0$. Moreover, we recall that

$$z=\rho_0\int_\Omega \mathbf{u}\cdot\nabla\phi, \quad z_{tt}=\int_\Omega \mathbf{f}\cdot\nabla\phi,$$

Throughout the proof, we shall fix $t_1\leq\frac{t_0}{2}$. Since

$$z(0)=\rho_0\int_\Omega \mathbf{u}(x,0)\cdot\nabla\phi=0, \quad z_t(0)=\rho_0\int_\Omega \mathbf{u}_t(x,0)\cdot\nabla\phi=0,$$

we have

$$\begin{aligned} z(t_1) &= \int_0^{t_1} z_t(s)\,\mathrm{d}s = \int_0^{t_1}\int_0^s z_{tt}(\eta)\,\mathrm{d}\eta\,\mathrm{d}s \\ &= \int_0^{t_1}\left(\int_\eta^{t_1}\mathrm{d}s\right) z_{tt}(\eta)\,\mathrm{d}\eta = \int_0^{t_1}(t_1-\eta)\,z_{tt}(\eta)\,\mathrm{d}\eta = \frac{t_1^2}{2}\int_\Omega \mathbf{f}\cdot\nabla\phi, \end{aligned}$$

hence

$$\left|\int_\Omega \mathbf{f}\cdot\nabla\phi\right| \leq 2\frac{|z(t_1)|}{t_1^2}. \tag{3.16}$$

It is now apparent that, in the first part of the proof, our main efforts shall be devoted to proving that the smallness of the data propagates into $|z(t_1)|$.

From (3.8), we write

$$\begin{aligned} |z(t_1)| &= \left|\int_{\partial\Omega}\frac{\partial S(\cdot,t_1)}{\partial\nu}\phi - S(\cdot,t_1)\frac{\partial\phi}{\partial\nu}\right| \\ &\leq \int_{\partial\Omega}\left|\frac{\partial S(\cdot,t_1)}{\partial\nu}\right||\phi| + \left|\int_{\partial\Omega} S(\cdot,t_1)\frac{\partial\phi}{\partial\nu}\right|. \end{aligned}$$

We first notice that

$$\int_{\partial\Omega}\left|\frac{\partial S(\cdot,t_1)}{\partial\nu}\right||\phi| \leq \underbrace{\|\nabla S(\cdot,t_1)\|_{L^\infty(\partial\Omega)}}_{\omega_1}\int_{\partial\Omega}|\phi|.$$

Secondly, for $x_0\in B_{R_0}\setminus\overline{\Omega}$, where B_{R_0} is such that

$$\overline{B_{r_0}(\bar{x})\cup\Omega}\subset B_{R_0},$$

we have

$$\left|\int_{\partial\Omega} S(\cdot,t_1)\frac{\partial\phi(x)}{\partial\nu}\right| \le \left|\int_{\partial\Omega}(S(\cdot,t_1)-S(x_0,t_1))\frac{\partial\phi}{\partial\nu}\right|$$
$$\le \underbrace{\|S(\cdot,t_1)-S(x_0,t_1)\|_{L^\infty(\partial\Omega)}}_{\omega_2}\int_{\partial\Omega}\left|\frac{\partial\phi}{\partial\nu}\right|.$$

Putting everything together, we obtain

$$|z(t_1)| \le \omega_1\int_{\partial\Omega}|\phi| + \omega_2\int_{\partial\Omega}\left|\frac{\partial\phi}{\partial\nu}\right| \le |\partial\Omega|\,(\omega_1+\omega_2)\,\|\phi\|_{C^1(\partial\Omega)}.$$

We thus begin to quantify the smallness of $|z(t_1)|$ by estimating

$$\omega_1 := \|\nabla S(\cdot,t_1)\|_{L^\infty(\partial\Omega)}.$$

We shall define d_1 such that $S(\cdot,t_1)$ is harmonic in $B_{\frac{d_1}{4}}(x)$, for $x\in\partial\Omega$. By Corollary 2.5, we set

$$d_1 = \frac{d_0}{2} - \frac{t_1}{\alpha_0}.$$

By the mean value property of harmonic functions, for any $x\in\partial\Omega$, we have

$$|\nabla S(x,t_1)| \le \frac{1}{\left|B_{\frac{d_1}{4}}(x)\right|}\int_{B_{\frac{d_1}{4}}(x)}|\nabla S(\cdot,t_1)| \le \frac{c}{d_1^{\frac{3}{2}}}\|\nabla S(\cdot,t_1)\|_{L^2(B_{R_0}\setminus\overline{\Omega}_{\frac{d_1}{2}})}, \tag{3.17}$$

where $c = \left(\frac{48}{\pi}\right)^{\frac{1}{2}}$. To estimate ω_1, we exploit the following estimate of propagation of smallness [2, Theorem 5.1]:

$$\|\nabla S(\cdot,t_1)\|_{L^2(B_{R_0}\setminus\overline{\Omega}_{\frac{d_1}{2}})} \le C_0\|\nabla S(\cdot,t_1)\|^{\theta}_{L^2(B_{r_0}(\bar{x}))}\|\nabla S(\cdot,t_1)\|^{1-\theta}_{L^2(B_{2R_0}\setminus\overline{\Omega}_{d_1})}. \tag{3.18}$$

for some $\theta\in(0,1)$ and $C_0>0$ depending on λ_0, μ_0, ρ_0, r_0, d_0 and Ω.

We first estimate $\|\nabla S(\cdot,t_1)\|_{L^2(B_{2R_0}\setminus\overline{\Omega}_{d_1})}$. Notice that

$$\mathrm{div}(\nabla S(x,t_1)) = -\rho_0\mathrm{div}\left(\mathbf{u}(x,t_1)\chi_{\Omega_{d_1}}(x)\right).$$

From [9, Section 5.9.1], we obtain

$$\begin{aligned}\|S(\cdot,t_1)\|_{H^1(\mathbb{R}^3)} &\le C\rho_0\|\mathrm{div}\left(\mathbf{u}(\cdot,t_1)\chi_{\Omega_{d_1}}\right)\|_{H^{-1}(\mathbb{R}^3)}\\ &\le C\rho_0\left(\int_{\Omega_{d_1}}|\mathbf{u}(\cdot,t_1)|^2\right)^{\frac{1}{2}}.\end{aligned} \tag{3.19}$$

We then use the energy estimate of Proposition 2.2 for $\mathbf{u}(\cdot,t_1)$. We have

$$\int_\Omega|\mathbf{u}(\cdot,t_1)|^2 \le \frac{t_1^3e^{t_1/\rho_0}}{\rho_0}\int_\Omega|\mathbf{f}|^2. \tag{3.20}$$

We now focus on the right-hand side of the inequality above. We write

$$\begin{aligned}\mathbf{f}(x) =& -(M^{(2)}-M^{(1)})q'(|x-P^{(2)}|)\frac{(x-P^{(2)})}{|x-P^{(2)}|}\\ &- M^{(1)}(\nabla(q(|x-P^{(2)}|)) - \nabla(q(|x-P^{(1)}|))).\end{aligned}$$

Since $|M^{(1)}|, |M^{(2)}| \leq M_0$, we get

$$\begin{aligned}\int_\Omega |\mathbf{f}|^2 \leq 2|M^{(2)} - M^{(1)}|^2 \int_\Omega (q'(|x-P^{(2)}|))^2 \\ + 2M_0^2 \int_\Omega \left|\nabla(q(|x-P^{(2)}|)) - \nabla(q(|x-P^{(1)}|))\right|^2 .\end{aligned} \tag{3.21}$$

Notice that

$$\begin{aligned}&\left|\partial_j(q(|x-P^{(2)}|)) - \partial_j(q(|x-P^{(1)}|))\right| \\ &= \left(\int_0^1 \left|\nabla\partial_j(q(|x-(P^{(1)}+(P^{(2)}-P^{(1)})\eta)|))\right| \mathrm{d}\eta\right) |P^{(2)}-P^{(1)}|,\end{aligned}$$

hence

$$\begin{aligned}&\int_\Omega \left|\partial_j(q(|x-P^{(2)}|)) - \partial_j(q(|x-P^{(1)}|))\right|^2 \\ &\quad \leq |P^{(2)}-P^{(1)}|^2 \int_0^1 \int_\Omega \left|\nabla\partial_j(q(|x-(P^{(1)}+(P^{(2)}-P^{(1)})\eta)|))\right|^2 .\end{aligned}$$

From (3.21), we have

$$\left(\int_\Omega |\mathbf{f}|^2\right)^{\frac{1}{2}} \leq C_1|M^{(2)} - M^{(1)}| + C_2|P^{(2)} - P^{(1)}|,$$

where

$$C_1 := \left(\int_{B_{\frac{d_0}{2}}} (q'(|x-P^{(2)}|))^2\right)^{\frac{1}{2}},$$

$$C_2 := \left(M_0^2 \int_0^1 \int_\Omega \left|\nabla\partial_j(q(|x-(P^{(1)}+(P^{(2)}-P^{(1)})\eta)|))\right|^2\right)^{\frac{1}{2}}.$$

From (3.20), we obtain

$$\left(\int_\Omega |\mathbf{u}(\cdot,t_1)|^2\right)^{\frac{1}{2}} \leq \sqrt{\frac{t_1^3 e^{t_1/\rho_0}}{\rho_0}} \left(C_1|M^{(2)} - M^{(1)}| + C_2|P^{(2)} - P^{(1)}|\right),$$

and, from (3.19), we have

$$\|\nabla S(\cdot,t_1)\|_{L^2(B_{2R_0}\setminus\overline{\Omega}_{d_1})} \leq C_3\left(|M^{(2)} - M^{(1)}| + |P^{(2)} - P^{(1)}|\right),$$

where

$$C_3 := C\sqrt{\rho_0 t_1^3 e^{t_1/\rho_0}} \max\{C_1, C_2\}.$$

Estimate (3.18) then gives

$$\|\nabla S(\cdot,t_1)\|_{L^2(B_{R_0}\setminus\overline{\Omega}_{\frac{d_1}{2}})} \leq C_0 C_3 \varepsilon^\theta \left(|M^{(2)} - M^{(1)}| + |P^{(2)} - P^{(1)}|\right)^{1-\theta}.$$

Using (3.17), we finally get

$$\omega_1 \leq \frac{cC_0C_3}{d_1^{\frac{3}{2}}}\varepsilon^\theta \left(|M^{(2)} - M^{(1)}| + |P^{(2)} - P^{(1)}|\right)^{1-\theta}. \tag{3.22}$$

In fact, the same inequality holds for

$$\|\nabla S(\cdot, t_1)\|_{L^\infty(B_{R_0}\setminus\Omega)} \leq \frac{cC_0C_3}{d_1^{\frac{3}{2}}}\varepsilon^\theta \left(|M^{(2)} - M^{(1)}| + |P^{(2)} - P^{(1)}|\right)^{1-\theta}. \tag{3.23}$$

We now estimate

$$\omega_2 := \|S(\cdot, t_1) - S(x_0, t_1)\|_{L^\infty(\partial\Omega)}, \quad \text{fixed some } x_0 \in B_{R_0} \setminus \overline{\Omega}.$$

For any $x \in \partial\Omega$, we notice that

$$S(x, t_1) - S(x_0, t_1) = \int_0^{s_0} \frac{\mathrm{d}}{\mathrm{d}s} S(\gamma(s), t_1) = \int_0^{s_0} \nabla S(\gamma(s), t_1)\gamma'(s),$$

hence, by (3.23),

$$\begin{aligned} |S(x, t_1) - S(x_0, t_1)| &\leq \|\nabla S(\cdot, t_1)\|_{L^\infty(B_{R_0}\setminus\Omega)} \left|\int_0^{s_0} \gamma'(s)\right| \\ &\leq s_0 \frac{cC_0C_3}{d_1^{\frac{3}{2}}}\varepsilon^\theta \left(|M^{(2)} - M^{(1)}| + |P^{(2)} - P^{(1)}|\right)^{1-\theta}, \end{aligned}$$

where γ is an arc-length parameterized curve such that $\gamma(0) = x_0$ and $\gamma(s_0) = x$. Since Ω is convex, there exists $K > 0$ depending only on the diameter of Ω and R_0 such that

$$s_0 \leq K, \quad \forall x_0 \in B_{R_0} \setminus \overline{\Omega}, \quad \forall x \in \partial\Omega,$$

hence

$$\omega_2 \leq K\frac{cC_0C_3}{d_1^{\frac{3}{2}}}\varepsilon^\theta \left(|M^{(2)} - M^{(1)}| + |P^{(2)} - P^{(1)}|\right)^{1-\theta}. \tag{3.24}$$

Finally, putting (3.16), (3.22) and (3.24) together, we find that

$$\left|\int_\Omega \mathbf{f} \cdot \nabla\phi\right| \leq \varepsilon_1 \|\phi\|_{C^1(\partial\Omega)}, \tag{3.25}$$

where

$$\varepsilon_1 := \frac{2C_4}{t_1^2}\varepsilon^\theta (|M^{(2)} - M^{(1)}| + |P^{(2)} - P^{(1)}|)^{1-\theta}, \tag{3.26}$$

and

$$C_4 := \frac{cC_0C_3}{d_1^{\frac{3}{2}}}|\partial\Omega| \max\{1, K\}.$$

We are now able to propagate the smallness of the data directly into $|M^{(2)} - M^{(1)}|$ and $|P^{(2)} - P^{(1)}|$ by using the definition of $\mathbf{f}$:

$$\begin{aligned} \int_\Omega \mathbf{f} \cdot \nabla\phi = &\int_{B_{\frac{d_0}{2}}(P^{(1)})} \frac{q'(|x - P^{(1)}|)}{|x - P^{(1)}|} M^{(1)}(x - P^{(1)}) \cdot \nabla\phi \\ &- \int_{B_{\frac{d_0}{2}}(P^{(2)})} \frac{q'(|x - P^{(2)}|)}{|x - P^{(2)}|} M^{(2)}(x - P^{(2)}) \cdot \nabla\phi. \end{aligned}$$

In what follows, we retrace some calculations done in the uniqueness proof of Theorem 3.2, starting from equation (3.9) up to (3.15).

Step 1: Moment tensor. We begin by recalling that, for $\phi = x_i x_j$, $i \neq j$, we have:

$$\int_\Omega \mathbf{f} \cdot \nabla \phi = -2M_{ij}^{(2)} + 2M_{ij}^{(1)}.$$

By (3.25), we thus find

$$\left| M_{ij}^{(2)} - M_{ij}^{(1)} \right| \leq C_5 \frac{\varepsilon_1}{2}, \quad i,j = 1,2,3, \quad i \neq j, \tag{3.27}$$

where C_5 refers to a constant greater than $\|\phi\|_{\partial\Omega}$ for any choices of ϕ we will make throughout the remainder of the proof. By taking $\phi = (x_1^2 - x_2^2)/2$, $\phi = (x_1^2 - x_3^2)/2$, and $\phi = (x_2^2 - x_3^2)/2$, we also get

$$\left| (M_{11}^{(2)} - M_{11}^{(1)}) - (M_{22}^{(2)} - M_{22}^{(1)}) \right| \leq C_5 \varepsilon_1,$$

$$\left| (M_{11}^{(2)} - M_{11}^{(1)}) - (M_{33}^{(2)} - M_{33}^{(1)}) \right| \leq C_5 \varepsilon_1,$$

and

$$\left| (M_{22}^{(2)} - M_{22}^{(1)}) - (M_{33}^{(2)} - M_{33}^{(1)}) \right| \leq C_5 \varepsilon_1.$$

We can write

$$M_{11}^{(2)} - M_{11}^{(1)} = M_{22}^{(2)} - M_{22}^{(1)} + \sigma_1, \qquad M_{22}^{(2)} - M_{22}^{(1)} = M_{33}^{(2)} - M_{33}^{(1)} + \sigma_2,$$

and

$$M_{33}^{(2)} - M_{33}^{(1)} = M_{11}^{(2)} - M_{11}^{(1)} + \sigma_3,$$

where $|\sigma_i| \leq C_5 \varepsilon_1$, $i = 1,2,3$. Using the results above, together with $\mathrm{tr} M^{(1)} = \mathrm{tr} M^{(2)} = 0$:

$$\left(M_{11}^{(2)} - M_{11}^{(1)} \right) + \left(M_{22}^{(2)} - M_{22}^{(1)} \right) + \left(M_{33}^{(2)} - M_{33}^{(1)} \right) = 0,$$

we find

$$|M_{ii}^{(2)} - M_{ii}^{(1)}| \leq C_5 \frac{\varepsilon_1}{3}, \quad i = 1,2,3. \tag{3.28}$$

Putting (3.27) and (3.28) together, we finally get

$$|M^{(2)} - M^{(1)}| \leq C_6 \varepsilon_1, \quad C_6 = \frac{5\sqrt{3}}{6} C_5. \tag{3.29}$$

Step 2: Location of the source. We begin by noticing that estimate (3.29) implies

$$M^{(2)} = M^{(1)} + M^\sigma,$$

where $M^\sigma \in \mathbb{M}^3$ and $|M^\sigma| \leq C_6\varepsilon_1$. Repeating the same calculations done in the uniqueness proof of Theorem 3.2 yields

$$\begin{aligned}&6(2M^{(1)}_{22} + M^{(1)}_{33})(P^{(1)} - P^{(2)})_1 + 12M^{(1)}_{12}(P^{(1)} - P^{(2)})_2\\&= 6(2M^\sigma_{22} + M^\sigma_{33})(P^{(2)})_1 + 12M^\sigma_{12}(P^{(2)})_2 + \int_\Omega \mathbf{f}\cdot\nabla\phi\end{aligned}$$

for $\phi = x_1^3 - 3x_2^2x_1$. Hence

$$\left|(2M^{(1)}_{22} + M^{(1)}_{33})(P^{(1)} - P^{(2)})_1 + 2M^{(1)}_{12}(P^{(1)} - P^{(2)})_2\right| \leq C_7\varepsilon_1,$$

where C_7 depends on the diameter of Ω. Analogously, by taking $\phi = x_2^3 - 3x_1^2x_2$, we obtain

$$\left|(2M^{(1)}_{22} + M^{(1)}_{33})(P^{(1)} - P^{(2)})_2 + 2M^{(1)}_{12}(P^{(1)} - P^{(2)})_1\right| \leq C_7\varepsilon_1.$$

We can then write the following system:

$$\begin{cases}(2M^{(1)}_{22} + M^{(1)}_{33})(P^{(1)} - P^{(2)})_1 + 2M^{(1)}_{12}(P^{(1)} - P^{(2)})_2 = \tilde{\sigma}_1,\\ 2M^{(1)}_{12}(P^{(1)} - P^{(2)})_2 - (2M^{(1)}_{22} + M^{(1)}_{33})(P^{(1)} - P^{(2)})_2 = \tilde{\sigma}_2,\end{cases}$$

where $|\tilde{\sigma}_1|, |\tilde{\sigma}_2| \leq C_7\varepsilon_1$. Hence

$$(P^{(1)} - P^{(2)})_1 = \frac{\begin{vmatrix}\tilde{\sigma}_1 & 2M^{(1)}_{12}\\ \tilde{\sigma}_2 & 2M^{(1)}_{22} + M^{(1)}_{33}\end{vmatrix}}{-\left((2M^{(1)}_{22} + M^{(1)}_{33})^2 + 4(M^{(1)}_{12})^2\right)}.$$

We have

$$\left|\left((2M^{(1)}_{22} + M^{(1)}_{33})^2 + 4(M^{(1)}_{12})^2\right)(P^{(1)} - P^{(2)})_1\right| \leq 5C_7M_0^2\varepsilon_1.$$

Repeating the same calculations done above for $\phi = x_1^3 - 3x_3^2x_1$, $\phi = x_3^3 - 3x_3x_1^2$ and $\phi = x_1x_2x_3$ yields

$$\left|\left((M^{(1)}_{22} + 2M^{(1)}_{33})^2 + 4(M^{(1)}_{13})^2\right)(P^{(1)} - P^{(2)})_1\right| \leq 5C_7M_0^2\varepsilon_1,$$

and

$$\left|(M^{(1)}_{32})^2(P^{(1)} - P^{(2)})_1\right| \leq C_7M_0^2\varepsilon_1.$$

By putting everything together and using $\mathrm{tr}M^{(1)} = 0$, we obtain:

$$\begin{cases}\left|\left((M^{(1)}_{11} - M^{(1)}_{22})^2 + 4(M^{(1)}_{12})^2\right)(P^{(1)} - P^{(2)})_1\right| \leq C_8\varepsilon_1,\\ \left|\left((M^{(1)}_{33} - M^{(1)}_{11})^2 + 4(M^{(1)}_{13})^2\right)(P^{(1)} - P^{(2)})_1\right| \leq C_8\varepsilon_1,\\ \left|M^{(1)}_{32}(P^{(1)} - P^{(2)})_1\right| \leq C_8\varepsilon_1.\end{cases} \tag{3.30}$$

where $C_8 = 5C_7M_0^2$. We set $\lambda > 0$ such that

$$\begin{cases} (M_{11}^{(1)} - M_{22}^{(1)})^2 + 4(M_{12}^{(1)})^2 \leq \lambda^2, \\ (M_{33}^{(1)} - M_{11}^{(1)})^2 + 4(M_{13}^{(1)})^2 \leq \lambda^2, \\ (M_{32}^{(1)})^2 \leq \lambda^2. \end{cases} \tag{3.31}$$

We first want to show that

$$|M_{ij}^{(1)}| \leq \lambda, \quad i, j = 1, 2, 3. \tag{3.32}$$

System (3.31) gives

$$|M_{12}^{(1)}| \leq \frac{\lambda}{2}, \quad |M_{13}^{(1)}| \leq \frac{\lambda}{2}, \quad |M_{32}^{(1)}| \leq \lambda,$$

and

$$|M_{11}^{(1)} - M_{22}^{(1)}| \leq \lambda, \quad |M_{33}^{(1)} - M_{11}^{(1)}| \leq \lambda,$$

that is

$$M_{22}^{(1)} = -\tilde{\lambda}_1 + M_{11}^{(1)}, \quad M_{33}^{(1)} = -\tilde{\lambda}_2 + M_{11}^{(1)},$$

with $|\tilde{\lambda}_1|, |\tilde{\lambda}_2| \leq \lambda$. Using $\mathrm{tr} M^{(1)} = 0$, the equations above yield

$$3M_{11}^{(1)} - \tilde{\lambda}_1 - \tilde{\lambda}_2 = 0,$$

hence $|M_{11}^{(1)}| \leq \frac{2}{3}\lambda$. Since the same calculations can be done for $M_{22}^{(1)}$ and $M_{33}^{(1)}$, we proved (3.32), hence

$$|M|^2 \leq 9\lambda^2.$$

We recall now that (3.1) implies $|M|^2 \geq m_0^2$, hence

$$\lambda \geq \frac{m_0}{3}.$$

This means that if we set

$$\lambda = \frac{m_0}{6}$$

in (3.31), at least one of the coefficients of the system must be greater than $\frac{m_0}{6}$. By (3.30), we have

$$\left|(P^{(1)} - P^{(2)})_1\right| \leq \frac{6C_8}{m_0}\varepsilon_1.$$

By repeating the same calculations for $(P^{(1)} - P^{(2)})_2$, we finally obtain

$$\left|P^{(1)} - P^{(2)}\right| \leq \frac{6C_8}{m_0}\varepsilon_1.$$

Putting everything together yields

$$\left|P^{(1)} - P^{(2)}\right| + \left|M^{(1)} - M^{(2)}\right| \leq \frac{C_9}{m_0}\varepsilon_1, \quad C_9 = m_0C_6 + \frac{6C_8}{m_0}.$$

We now recall from (3.26) that

$$\varepsilon_1 = \frac{2C_4}{t_1^2}\varepsilon^\theta(|M^{(2)} - M^{(1)}| + |P^{(2)} - P^{(1)}|)^{1-\theta},$$

hence

$$\left|P^{(1)}-P^{(2)}\right|+\left|M^{(1)}-M^{(2)}\right| \leq \frac{C_{10}}{m_0}\varepsilon^{\theta}(|M^{(2)}-M^{(1)}|+|P^{(2)}-P^{(1)}|)^{1-\theta}, \tag{3.33}$$

where

$$C_{10}=\frac{2C_4}{t_1^2}C_9.$$

From (3.33), we finally get the desired Lipschitz stability estimate

$$\left|P^{(1)}-P^{(2)}\right|+\left|M^{(1)}-M^{(2)}\right| \leq \left(\frac{C_{10}}{m_0}\right)^{\frac{1}{\theta}}\varepsilon.$$

∎

4. Concluding remarks. In this paper we have studied an early-warning inverse source problem for the elasto-gravitational equations that is motivated by seismology. The problem involves a mixed hyperbolic-elliptic system of partial differential equations describing elastic wave displacement and gravity perturbations produced by a source in a homogeneous bounded medium. Within the Cowling approximation, we have shown how to turn the so-called elasto-gravitational coupling to our advantage: the changes of the gravitational field are generated instantaneously by Poisson's equation and thus can be used to solve the early-warning inverse problem without having to impose a minimum on the time needed to determine the source.

This paper is the first step towards developing the theoretical framework of the PEGS inverse problem. Proving uniqueness and stability theorems for the self-gravitating elastic equations will be the object of future research.

Appendix A. Energy estimates.

A.1. Proof of proposition 2.2.

Proof. We begin by multiplying both sides of the first equation of the system (2.3) by $\mathbf{u}_t$. We get

$$\rho_0\mathbf{u}_{tt}\cdot\mathbf{u}_t-\operatorname{div}(\mathbb{C}\nabla\mathbf{u})\cdot\mathbf{u}_t=\mathbf{f}\cdot\mathbf{u}_t,$$

hence

$$\frac{\rho_0}{2}\partial_t|\mathbf{u}_t|^2+\frac{1}{2}\partial_t(\mathbb{C}\nabla\mathbf{u}\cdot\nabla\mathbf{u})-\operatorname{div}(\mathbb{C}\nabla\mathbf{u}\cdot\mathbf{u}_t)=\mathbf{f}\cdot\mathbf{u}_t.$$

We integrate the equation above over $\Omega\times[0,s]$. Since $\mathbb{C}\nabla\mathbf{u}\cdot\nu=0$ on $\partial\Omega$, we obtain

$$\frac{1}{2}\int_\Omega\left(\rho_0|\mathbf{u}_t(x,s)|^2+\mathbb{C}\nabla\mathbf{u}(x,s)\cdot\nabla\mathbf{u}(x,s)\right)\,\mathrm{d}x=\int_0^s\int_\Omega\mathbf{f}(x)\cdot\mathbf{u}_t(x,t)\,\mathrm{d}x\mathrm{d}t,$$

hence

$$\int_\Omega|\mathbf{u}_t(x,s)|^2\,\mathrm{d}x\leq\frac{1}{\rho_0}\left(\int_0^s\int_\Omega|\mathbf{f}(x)|^2\,\mathrm{d}x\mathrm{d}t+\int_0^s\int_\Omega|\mathbf{u}_t(x,t)|^2\,\mathrm{d}x\mathrm{d}t\right).$$

By Grönwall's inequality, we find

$$\int_\Omega|\mathbf{u}_t(x,s)|^2\,\mathrm{d}x\leq\frac{1}{\rho_0}\left(\int_0^s\int_\Omega|\mathbf{f}(x)|^2\,\mathrm{d}x\mathrm{d}t\right)e^{s/\rho_0}. \tag{A.1}$$

Since $\mathbf{u}(x,0)=0$, we have

$$|\mathbf{u}(x,\tau)|\leq\int_0^\tau|\mathbf{u}_t(x,s)|\,\mathrm{d}s\leq\sqrt{\tau}\left(\int_0^\tau|\mathbf{u}_t(x,s)|^2\,\mathrm{d}s\right)^{\frac{1}{2}},$$

hence, by (A.1), we get

$$\begin{aligned}\int_\Omega |\mathbf{u}(x,\tau)|^2\,\mathrm{d}x &\le \tau\int_0^\tau\int_\Omega |\mathbf{u}_t(x,s)|^2\,\mathrm{d}x\mathrm{d}s\\ &\le \tau\int_0^\tau \frac{1}{\rho_0}\left(\int_0^s\int_\Omega |\mathbf{f}(x)|^2\,\mathrm{d}x\mathrm{d}t\right)e^{s/\rho_0}\,\mathrm{d}s\\ &\le \frac{\tau e^{\tau/\rho_0}}{\rho_0}\int_0^\tau\left(\int_0^s\int_\Omega |\mathbf{f}(x)|^2\,\mathrm{d}x\mathrm{d}t\right)\mathrm{d}s\end{aligned}$$

hence

$$\int_\Omega |\mathbf{u}(x,\tau)|^2\,\mathrm{d}x \le \frac{\tau^3 e^{\tau/\rho_0}}{\rho_0}\int_\Omega |\mathbf{f}(x)|^2\,\mathrm{d}x.$$

□

A.2. Proof of proposition 2.3.

Proof. Define $\mathbf{v} := \mathbf{u}\,e^{-\gamma t}$, $\gamma > 0$. It is easy to check that

$$\mathbf{u}_t = (\mathbf{v}_t + \gamma\mathbf{v})e^{\gamma t},$$

$$\mathbf{u}_{tt} = (\mathbf{v}_{tt} + 2\gamma\mathbf{v}_t + \gamma^2\mathbf{v})e^{\gamma t},$$

hence, if $\mathbf{u}$ solves (2.3), we have that $\mathbf{v}$ solves

$$\rho_0\left(\mathbf{v}_{tt} + 2\gamma\mathbf{v}_t + \gamma^2\mathbf{v}\right) - \mathrm{div}(\mathbb{C}\nabla\mathbf{v}) = \mathbf{f}\,e^{-\gamma t}.$$

After multiplying both sides of the above equation by $\mathbf{v}_t$, we obtain:

$$\frac{\rho_0}{2}\left(\partial_t|\mathbf{v}_t|^2 + 4\gamma|\mathbf{v}_t|^2 + \gamma^2\partial_t|\mathbf{v}|^2\right) - \partial_j(C_{ijkl}\partial_k v_l)v_{i,t} = \mathbf{f}\cdot\mathbf{v}_t\,e^{-\gamma t}.$$

We integrate the equation above over

$$\widetilde{K}_\alpha(x_0,\tau) := K_\alpha(x_0,\tau)\cap(\Omega\times[0,\infty)),$$

where

$$K_\alpha(x_0,\tau) := \{(x,t)\in\mathbb{R}^4 \text{ such that } 0\le t\le \tau-\alpha|x-x_0|\}.$$

We write

$$\begin{aligned}&\int_{\widetilde{K}_\alpha(x_0,\tau)} \partial_t\left(\frac{\rho_0}{2}|\mathbf{v}_t|^2 + \rho_0\frac{\gamma^2}{2}|\mathbf{v}|^2\right) - \partial_j(C_{ijkl}\partial_k v_l)v_{i,t}\\ &+\int_{\widetilde{K}_\alpha(x_0,\tau)} 2\rho_0\gamma|\mathbf{v}_t|^2 = \int_{\widetilde{K}_\alpha(x_0,\tau)} \mathbf{f}\cdot\mathbf{v}_t\,e^{-\gamma t}.\end{aligned}\tag{A.2}$$

We want to prove that, if we set

$$I_1(\alpha) := \int_{\widetilde{K}_\alpha(x_0,\tau)} \partial_t\left(\frac{\rho_0}{2}|\mathbf{v}_t|^2 + \rho_0\frac{\gamma^2}{2}|\mathbf{v}|^2\right) - \partial_j(C_{ijkl}\partial_k v_l)v_{i,t},$$

then $I_1(\alpha)\ge 0$ for a precise choice of the parameter α. We first observe that

$$\begin{aligned}\int_{\widetilde{K}_\alpha(x_0,\tau)} \partial_j(C_{ijkl}\partial_k v_l)v_{i,t} &= \int_{\widetilde{K}_\alpha(x_0,\tau)} \partial_j(C_{ijkl}\partial_k v_l v_{i,t}) - C_{ijkl}\partial_k v_l\partial_j v_{i,t}\\ &= \int_{\widetilde{K}_\alpha(x_0,\tau)} \partial_j(C_{ijkl}\partial_k v_l v_{i,t}) - \frac{1}{2}\partial_t(C_{ijkl}\partial_k v_l\partial_j v_i).\end{aligned}$$

Let us set

$$\Gamma_1 := (\Omega \times \{0\}) \cap K_\alpha(x_0,\tau), \quad \Gamma_2 := (\Omega \times (0,\infty)) \cap \partial K_\alpha(x_0,\tau),$$
$$\Gamma_3 := (\partial\Omega \times (0,\infty)) \cap K_\alpha(x_0,\tau).$$

Notice that

$$\partial \widetilde{K}_\alpha(x_0,\tau) = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3.$$

Thanks to the regularity provided by Theorem 2.1, we employ the divergence theorem and obtain

$$\int_{\partial \widetilde{K}_\alpha(x_0,\tau)} \left(\frac{\rho_0}{2}|\mathbf{v}_t|^2 + \rho_0 \frac{\gamma^2}{2}|\mathbf{v}|^2 + \frac{1}{2} C_{ijkl}\partial_k v_l \partial_j v_i \right) (\mathbf{N}\cdot\mathbf{e}_t) - (C_{ijkl}\partial_k v_l v_{i,t})(\mathbf{N}\cdot\mathbf{e}_j),$$

where $\mathbf{N}$ is the outward normal vector to $\partial\widetilde{K}_\alpha(x_0,\tau)$. It is defined as follows:

- on Γ_1: $\mathbf{N} = (0,0,0,-1)$.
- on Γ_2: $\mathbf{N} = \frac{1}{\sqrt{1+\alpha^2}}\left(\frac{\alpha(x-x_0)}{|x-x_0|}, 1\right)$.
- on Γ_3: $\mathbf{N} = (\nu, 0)$, where ν is the outward normal vector to $\partial\Omega$.

Since $\mathbf{v} = \mathbf{v}_t = 0$ on Γ_1 and $(\mathbb{C}\nabla\mathbf{v})\cdot\nu = 0$ on Γ_3, we can write

$$I_1(\alpha) = \frac{1}{2\sqrt{1+\alpha^2}} \int_{\Gamma_2} \underbrace{\rho_0|\mathbf{v}_t|^2 + \rho_0\gamma^2|\mathbf{v}|^2 + C_{ijkl}\partial_k v_l \partial_j v_i - 2\alpha\,(C_{ijkl}\partial_k v_l v_{i,t})\,\frac{(x-x_0)_j}{|x-x_0|}}_{I_2(\alpha)}.$$

We now define

$$\xi := \mathbf{v}_t, \qquad A := \frac{\nabla\mathbf{v} + (\nabla\mathbf{v})^\top}{2}, \quad \eta := \frac{(x-x_0)}{|x-x_0|}.$$

Since

$$\frac{\rho_0\gamma^2}{2\sqrt{1+\alpha^2}} \int_{\Gamma_2} |\mathbf{v}|^2 \geq 0,$$

we have

$$I_2(\alpha) \geq \rho_0|\xi|^2 + \lambda_0(\mathrm{tr}A)^2 + 2\mu_0|A|^2 - 2\alpha(\lambda(\mathrm{tr}A)\xi\cdot\eta + 2\mu_0 A\xi\cdot\eta).$$

Since $|\eta| = 1$ and ρ_0, λ_0 and $\mu_0 > 0$, we have, for every $\epsilon > 0$,

$$I_2(\alpha) \geq |\xi|^2(\rho_0 - \epsilon\alpha(\lambda_0 + 2\mu_0)) + \left(1 - \frac{\alpha}{\epsilon}\right)(\lambda_0(\mathrm{tr}A)^2 + 2\mu_0|A|^2).$$

Finally, for $\alpha = \alpha_0$ and $\epsilon = \alpha_0$, we obtain

$$I_2(\alpha_0) \geq |\xi|^2(\rho_0 - \alpha_0^2(\lambda_0 + 2\mu_0)) \geq 0,$$

which implies that

$$I_1(\alpha_0) \geq 0.$$

This means that, by (A.2), we have

$$\text{(A.3)} \quad 2\rho_0\gamma \int_{\widetilde{K}_{\alpha_0}(x_0,\tau)} |\mathbf{v}_t|^2 \leq I_1(\alpha_0) + 2\rho_0\gamma \int_{\widetilde{K}_{\alpha_0}(x_0,\tau)} |\mathbf{v}_t|^2 = \int_{\widetilde{K}_{\alpha_0}(x_0,\tau)} \mathbf{f}\cdot\mathbf{v}_t\, e^{-\gamma t}.$$

By the Cauchy-Schwarz inequality, we get

$$\int_{\widetilde{K}_{\alpha_0}(x_0,\tau)} \mathbf{f}\cdot\mathbf{v}_t\, e^{-\gamma t} \le \left(\int_{\widetilde{K}_{\alpha_0}(x_0,\tau)} |\mathbf{f}|^2 e^{-2\gamma t}\right)^{\frac{1}{2}} \left(\int_{\widetilde{K}_{\alpha_0}(x_0,\tau)} |\mathbf{v}_t|^2\right)^{\frac{1}{2}},$$

hence (A.3) becomes:

$$(2\rho_0\gamma)^2 \int_{\widetilde{K}_{\alpha_0}(x_0,\tau)} |\mathbf{v}_t|^2 \le \int_{\widetilde{K}_{\alpha_0}(x_0,\tau)} |\mathbf{f}|^2 e^{-2\gamma t}. \tag{A.4}$$

Observe that, since $\mathbf{v}(x,0)=0$ and using again the Cauchy-Schwarz inequality, we have

$$|\mathbf{v}(x,t)|^2 \le t\int_0^t |\mathbf{v}_s(x,s)|^2\,\mathrm{d}s \le \tau\int_0^t |\mathbf{v}_s(x,s)|^2\,\mathrm{d}s$$

since $t\le\tau$. Hence

$$\int_{\widetilde{K}_{\alpha_0}(x_0,\tau)} |\mathbf{v}(x,t)|^2\,\mathrm{d}x\mathrm{d}t \le \tau\int_{B_{\tau/\alpha_0}(x_0)\cap\Omega} \left(\int_0^{\tau-\alpha_0|x-x_0|}\left(\int_0^t |\mathbf{v}_s(x,s)|^2\,\mathrm{d}s\right)\mathrm{d}t\right)\mathrm{d}x.$$

Since $t\le\tau-\alpha_0|x-x_0|$, we write

(A.5)

$$\begin{aligned}\int_{\widetilde{K}_{\alpha_0}(x_0,\tau)} |\mathbf{v}(x,t)|^2\,\mathrm{d}x\mathrm{d}t &\le \tau\int_{B_{\tau/\alpha_0}(x_0)\cap\Omega} (\tau-\alpha_0|x-x_0|)\left(\int_0^{\tau-\alpha_0|x-x_0|} |\mathbf{v}_s(x,s)|^2\,\mathrm{d}s\right)\mathrm{d}x\\ &\le \tau^2\int_{\widetilde{K}_{\alpha_0}(x_0,\tau)} |\mathbf{v}_s(x,s)|^2\,\mathrm{d}x\mathrm{d}s.\end{aligned}$$

Finally, putting together (A.4) and (A.5), since $\mathbf{v}=\mathbf{u}\,e^{-\gamma t}$, we obtain (2.9). ∎

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