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Wigner function and kinetic theory for massive spin-1/2 particles

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Abstract

We calculate the Wigner function for massive spin-1/2 particles in an inhomogeneous electromagnetic field to leading order in the Planck constant $\hbar$. Going beyond leading order in $\hbar$ we then derive a generalized Boltzmann equation in which the force exerted by an inhomogeneous electromagnetic field on the particle dipole moment arises naturally. Furthermore, a kinetic equation for this dipole moment is derived. Carefully taking the massless limit we find agreement with previous results. The case of global equilibrium with rotation is also studied. Our framework can be used to study polarization effects induced by vorticity and magnetic field in relativistic heavy-ion collisions.

Keywords: Relativistic heavy-ion collisions, kinetic theory, Wigner function, polarization.

In the recent years, there has been an intense theoretical activity which has led to a deeper understanding of the transport properties of chiral matter, see e.g. Ref. [1] for a review. However, only few attempts have been made to derive a covariant kinetic theory for *massive* particles using Wigner functions [2, 3]. The aim of our work is to fill this gap. Here we report on a recent paper [4] where the kinetic theory for massive spin-1/2 particles in an inhomogeneous electromagnetic field was derived. This framework provides a basis to study polarization effects in relativistic heavy-ion collisions [5]. Our starting point is the covariant Wigner function [6, 7, 8]. In order to solve the equations of motion for the Wigner function, we employ an expansion in the Planck constant $\hbar$ and truncate at the lowest non-trivial order.

1. Wigner function for massive spin-1/2 particles

The Wigner function is defined as the Fourier transform of the two-point correlation function [9],

$$W_{\alpha\beta}(x, p) = \int \frac{d^4y}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar}p \cdot y} \langle : \bar{\psi}_\beta(x_1) U(x_1, x_2) \psi_\alpha(x_2) : \rangle. \quad (1)$$

Here, x_1 and x_2 are the space-time coordinates of two different points, with $y^\mu \equiv x_1^\mu - x_2^\mu$ and $x^\mu \equiv (x_1^\mu + x_2^\mu)/2$ and $U(x_1, x_2)$ is a gauge link. In this paper, the electromagnetic field $\mathbb{A}_\mu$ will be treated as an external, classical field. Under this assumption one can derive the exact kinetic equation for the Wigner function [9]:

$$(\gamma \cdot K - m)W(x, p) = 0. \quad (2)$$

Here one has defined the operator $K^\mu \equiv \Pi^\mu + \frac{1}{2}i\hbar\nabla^\mu$, with the generalized space-time derivative and momentum operators $\nabla^\mu \equiv \partial_x^\mu - j_0(\Delta)F^{\mu\nu}\partial_{p\nu}$ and $\Pi^\mu \equiv p^\mu - \frac{\hbar}{2}j_1(\Delta)F^{\mu\nu}\partial_{p\nu}$, where $\Delta \equiv \frac{\hbar}{2}\partial_p \cdot \partial_x$ and $F^{\mu\nu} = \partial_x^\mu \mathbb{A}^\nu - \partial_x^\nu \mathbb{A}^\mu$ is the electromagnetic field-strength tensor. We should emphasize that in Eq. (2) the space-time derivative ∂_x contained in Δ only acts on $F^{\mu\nu}$, but not on the Wigner function. The functions $j_0(x) = \sin x/x$ and $j_1(x) = (\sin x - x \cos x)/x^2$ are spherical Bessel functions.

In order to derive a kinetic equation for massive spin-1/2 particles, it is advantageous to decompose the Wigner function in terms of a basis formed by the 16 independent generators of the Clifford algebra,

$$W = \frac{1}{4} \left(\mathcal{F} + i\gamma^5 \mathcal{P} + \gamma^\mu \mathcal{V}_\mu + \gamma^5 \gamma^\mu \mathcal{A}_\mu + \frac{1}{2} \sigma^{\mu\nu} \mathcal{S}_{\mu\nu} \right). \quad (3)$$

The coefficients $\mathcal{F}, \mathcal{P}, \mathcal{V}^\mu, \mathcal{A}^\mu$, and $\mathcal{S}^{\mu\nu}$ correspond to the scalar, pseudo-scalar, vector, axial-vector, and tensor part of the Wigner function, respectively. They will be determined by solving Eq. (2) employing an expansion in $\hbar$.

2. General solution and kinetic equations up to order $\hbar$

The components of the Wigner function are not all independent. We choose to express $\mathcal{P}, \mathcal{V}^\mu$, and $\mathcal{A}^\mu$ in terms of $\mathcal{F}$ and $\mathcal{S}^{\mu\nu}$. From the equations for the Wigner-function components we obtain the modified on-shell conditions for the scalar and tensor components,

$$\begin{aligned} (\Pi \cdot \Pi - m^2)\mathcal{F} &= \frac{\hbar}{2}\Pi^\mu \nabla^\nu \mathcal{S}_{\nu\mu}, \\ (\Pi \cdot \Pi - m^2)\mathcal{S}_{\mu\nu} &= -\Pi^\alpha \Pi_{[\mu} \mathcal{S}_{\nu]\alpha} - \frac{\hbar}{2}\nabla_{[\mu} \Pi_{\nu]}\mathcal{F} - \frac{\hbar^2}{4}\nabla_{[\mu} \nabla^\alpha \mathcal{S}_{\nu]\alpha} + \frac{\hbar}{2}\epsilon_{\mu\nu\alpha\beta}\Pi^\alpha \nabla^\beta \mathcal{P}, \end{aligned} \quad (4)$$

where $A_{[\mu} B_{\nu]} \equiv A_\mu B_\nu - A_\nu B_\mu$. For the zeroth order we use the solution from Ref. [10]. The first-order solution is obtained by finding general solutions of the on-shell conditions and plugging the zeroth-order solution into the first-order equations. Eventually the Wigner function to first order in $\hbar$ can be written as

$$\begin{aligned} \mathcal{F} &= m \left[V \delta(p^2 - m^2) - \frac{\hbar}{2} F^{\mu\nu} \bar{\Sigma}_{\mu\nu} \delta'(p^2 - m^2) \right] + \mathcal{O}(\hbar^2), \\ \mathcal{P} &= \frac{\hbar}{4m} \epsilon^{\mu\nu\alpha\beta} \nabla_\mu^{(0)} \left[p_\nu \bar{\Sigma}_{\alpha\beta} \delta(p^2 - m^2) \right] + \mathcal{O}(\hbar^2), \\ \mathcal{V}_\mu &= p_\mu \left[V \delta(p^2 - m^2) - \frac{\hbar}{2} F^{\alpha\beta} \bar{\Sigma}_{\alpha\beta} \delta'(p^2 - m^2) \right] + \frac{\hbar}{2} \nabla^{(0)\nu} \left[\bar{\Sigma}_{\mu\nu} \delta(p^2 - m^2) \right] + \mathcal{O}(\hbar^2), \\ \mathcal{A}_\mu &= -\frac{1}{2} \epsilon_{\mu\nu\alpha\beta} p^\nu \left[\bar{\Sigma}^{\alpha\beta} \delta(p^2 - m^2) - \hbar F^{\alpha\beta} V \delta'(p^2 - m^2) \right] + \mathcal{O}(\hbar^2), \\ \mathcal{S}_{\mu\nu} &= m \left[\bar{\Sigma}_{\mu\nu} \delta(p^2 - m^2) - \hbar F_{\mu\nu} V \delta'(p^2 - m^2) \right] + \mathcal{O}(\hbar^2). \end{aligned} \quad (5)$$

The undetermined functions V and $\bar{\Sigma}_{\mu\nu}$ satisfy one constraint equation,

$$p^\nu \bar{\Sigma}_{\mu\nu} \delta(p^2 - m^2) = \frac{\hbar}{2} \delta(p^2 - m^2) \nabla_\mu^{(0)} V + \mathcal{O}(\hbar^2), \quad (6)$$

and two kinetic equations

$$\begin{aligned} 0 &= \delta(p^2 - m^2) \left[p \cdot \nabla^{(0)} V + \frac{\hbar}{4} (\partial_x^\alpha F^{\mu\nu}) \partial_{p\alpha} \bar{\Sigma}_{\mu\nu} \right] + \mathcal{O}(\hbar^2), \\ 0 &= \delta(p^2 - m^2) \left[p \cdot \nabla^{(0)} \bar{\Sigma}_{\mu\nu} - F_{[\mu}^\alpha \bar{\Sigma}_{\nu]\alpha} + \frac{\hbar}{2} (\partial_{x\alpha} F_{\mu\nu}) \partial_p^\alpha V \right] + \mathcal{O}(\hbar^2). \end{aligned} \quad (7)$$

From the results (5) and (7) it is possible to derive fluid-dynamical equations of motion with spin degrees of freedom using the canonical definitions of the energy-momentum and spin tensors [4]. In accordance with previous works [11, 12], the conservation of the total angular momentum is promoted as an additional fluid-dynamical equation, where the divergence of the spin tensor is related to the antisymmetric part of the energy-momentum tensor.

3. Comparison to the massless and classical case

The on-shell dipole moment $\bar{\Sigma}_{\mu\nu}$ can be written at the zeroth order as $\bar{\Sigma}^{\mu\nu} = \Sigma^{\mu\nu}A$ where

$$\Sigma^{\mu\nu} = -\frac{1}{m}\epsilon^{\mu\nu\alpha\beta}p_\alpha n_\beta \quad (8)$$

is the dipole-moment tensor, n_β is the polarization vector and A is the spin-antisymmetric part of the distribution function. On the classical level, $\Sigma^{\mu\nu}$ is the intrinsic angular-momentum tensor about the center of mass. In a relativistic theory, the center of mass of a particle is frame-dependent. In order to have a frame-independent definition of $\Sigma^{\mu\nu}$, one requires $p_\mu \Sigma^{\mu\nu} = 0$ as a gauge condition. This requirement identifies the dipole-moment tensor as the intrinsic angular-momentum tensor about the center of mass in the rest frame of the particle [13]. For massless particles there is no rest frame, thus both the position (in the classical case the center of momentum) and the dipole-moment tensor can at first be defined in an arbitrary frame characterized by a time-like four-vector u^μ , which means that we choose the gauge condition $u^\mu \Sigma_{\mu\nu} = 0$ [14]. Consequently, the frame vector u^μ must assume the role of p^μ in Eq. (8). Moreover, since n^μ and p^μ are parallel for massless particles, the momentum p^μ can assume the role of n^μ in Eq. (8). Finally, in order to obtain the massless case we need to replace the normalization factor $1/m$ in Eq. (8). The energy of a massive on-shell particle in its rest frame is $p_{rf}^0 = \sqrt{p^2} = m$. The energy of a massless particle in the rest frame of u^μ , however, is $p_u^0 = p \cdot u$. Thus, it is natural to replace the normalization $1/m$ in Eq. (8) by $1/(p \cdot u)$. We emphasize that this replacement can only be done in the presence of a δ -function which sets the rest-frame energy equal to the mass m . With these replacements we can show the agreement of our results for the vector and axial-vector current given in Eqs. (5) in the massless limit with the previously known massless solution [15, 16, 17].

We show that Eq. (7) gives rise to the first and second Mathisson–Papapetrou–Dixon (MPD) equations [18, 19] as well as to the Bargmann–Michel–Telegdi (BMT) equation [20], which were derived for classical, extended, spinning particles with non-vanishing dipole moment. Comparing Eq. (7) to the generic form of the collisionless relativistic Boltzmann–Vlasov equation [19, 21] we find that in our case the external force F_s^μ acting on a particle with spin up/down for $s = \pm$ is given as the sum of the Lorentz force and the Mathisson force.

$$F_s^\mu = \frac{1}{m} \left[F^{\mu\nu} p_\nu + s \frac{\hbar}{4} (\partial_x^\mu F^{\nu\rho}) \Sigma_{\nu\rho}^{(0)} \right], \quad (9)$$

In Refs. [18, 19], the first MPD equation for particles with classical dipole moment $m^{\mu\nu}$ was derived. Our results agree with this, setting $m_{\mu\nu} \rightarrow g\mu_B \frac{s}{2} \Sigma_{\mu\nu}^{(0)}$, with Bohr's magneton $\mu_B \equiv e\hbar/(2m)$, where e is the electric charge, and the gyromagnetic ratio $g = 2$, as expected for Dirac particles with spin $1/2$.

The evolution of the zeroth-order dipole-moment tensor is given by the second equation in (7):

$$m\dot{\Sigma}_{\mu\nu}^{(0)} \equiv m(\dot{x}^\alpha \partial_{x^\alpha} + \dot{p}^\alpha \partial_{p^\alpha}) \Sigma_{\mu\nu}^{(0)} = F_{[\mu}^\alpha \Sigma_{\nu]\alpha}^{(0)}, \quad (10)$$

with F_s^μ given by Eq. (9) to zeroth order. Equation (10) is identical to the second MPD equation [18, 19].

4. Global equilibrium

In this section we will consider a special solution of Eqs. (7) obtained in global equilibrium with rigid rotation. Assuming the standard form of the collision term, the distribution function in equilibrium must have the form [14, 19] $V^{eq} = \sum_s (e^{g_s} + 1)^{-1}$, with g_s being a linear combination of the collisional invariants,

namely, charge, kinetic momentum, and total angular momentum. Requiring that the Boltzmann equation (7) is fulfilled, we obtain from the third equation (5)

$$\begin{aligned} \mathcal{V}^\mu = & \frac{2}{(2\pi\hbar)^3} \sum_s \left[\delta(p^2 - m^2) \left(p^\mu - ms \frac{\hbar}{2} \tilde{\omega}^{\mu\nu} n_\nu \frac{\partial}{\partial g_s^{(0)}} \right) + ms \hbar \tilde{F}^{\mu\nu} n_\nu \delta'(p^2 - m^2) \right. \\ & \left. - \frac{s\hbar}{2m} \delta(p^2 - m^2) \epsilon^{\mu\alpha\beta} p_\alpha \left(\nabla_\nu^{(0)} n_\beta \right) \right] \left[\theta(p^0) f_s^{(0)+} + \theta(-p^0) f_s^{(0)-} \right] + \mathcal{O}(\hbar^2). \end{aligned} \quad (11)$$

The term containing $\tilde{F}^{\mu\nu} = (1/2)\epsilon^{\mu\alpha\beta} F_{\alpha\beta}$ in Eq. (11) is caused by off-shell effects and describes the vector current induced by electromagnetic fields, which yields the analogue of the chiral magnetic effect in the case of non-zero mass. On the other hand, the term containing the dual of the thermal vorticity $\tilde{\omega}^{\mu\nu}$ describes the current induced by vorticity and thus gives the analogue of the chiral vortical effect.

The axial-vector current in global equilibrium can be obtained in a similar way. One identifies three contributions to the axial-vector current in the massive case. One term describes the spin precession in the presence of an electromagnetic field according to the BMT equation. The second term gives rise to the axial current in the direction of the vorticity, which is the analogue of the axial chiral vortical effect. Finally, the last term describes the axial current along the magnetic field, which is the analogue of the chiral separation effect. These terms are analogous to those found in Refs. [2, 22, 23].

After completion of our work [4], we became aware of related studies [24, 25].

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