REAL SEMISIMPLE LIE GROUPS AND BALANCED METRICS

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ABSTRACT. Given any non-compact real simple Lie group G_o of inner type and even dimension, we prove the existence of an invariant complex structure J and a Hermitian balanced metric with vanishing Chern scalar curvature on G_o and on any compact quotient $M = G_o/\Gamma$, with Γ a cocompact lattice. We also prove that (M, J) does not carry any pluriclosed metric, in contrast to the case of even dimensional compact Lie groups, which admit pluriclosed but not balanced metrics.

1. INTRODUCTION

Given a complex non-Kähler n-dimensional manifold (M, J) it is a natural and meaningful problem to find special Hermitian metrics which might help in understanding the geometry of M. Great effort has been spent in the last decades in this research topic and among special metrics the pluriclosed and the balanced conditions have shown to be highly significant.

The balanced condition can be defined saying that the fundamental form $\omega = h(\cdot, J\cdot)$ of a Hermitian metric h satisfies the non-linear condition $d\omega^{n-1} = 0$ or equivalently, $-J\theta = \delta\omega = 0$, where δ denotes the codifferential and θ the torsion 1-form (see e.g. [Ga2]). While this concept appears in [Ga1] under the name of semi-Kähler (see also [Gra]), in [Mi] the balanced condition was started to be thoroughly investigated, highlighting also the duality with the Kähler condition and establishing necessary and sufficient conditions for the existence of these metrics in terms of currents. While Kähler metrics are obviously balanced and share with these the important relation among Laplacians $\Delta_\partial = \Delta_{\overline{\partial}} = \frac{1}{2}\Delta$ (see [Ga1]), there are many examples of non-Kähler manifolds carrying balanced metrics. Basic examples are given by compact complex parallelizable manifolds, which are covered by complex unimodular Lie groups G and every left invariant Hermitian metric turns out to be balanced (see [AG][Ga1][Gr]). Further examples of balanced metrics are provided by any Hermitian invariant metric on a compact homogeneous flag manifold (see also [FGV] for a characterization of compact homogeneous complex manifolds carrying balanced metrics) as well as by twistor spaces of certain self-dual 4-manifolds ([Mi]) and more generally ([To]) by twistor spaces of compact hypercomplex manifolds (see also [Fo] for other examples on toric bundles over hyperkähler manifolds). Contrary to the Kählerness condition, being balanced is a birational invariant (see [AB1], so that e.g. Moishezon manifolds are balanced) and compact complex manifolds X which can be realized as the base of a holomorphic proper submersion $f : Y \rightarrow X$ inherit the balanced condition whenever Y has it ([Mi]), while the balanced property is not stable under small deformations of the complex structure (see [AB2],[FuY], [AU]). On the other hand, the balanced condition is obstructed, as on compact manifolds with balanced metrics no compact complex hypersurface is homologically trivial, so that for instance Calabi-Eckmann manifolds do not carry balanced metrics. This is in contrast with

the fact that Gauduchon metrics, which statisfy the weaker condition $\partial\bar{\partial}\omega^{n-1} = 0$, always exist on a compact complex manifold.

In more recent years, the rising interest in the Strominger System (see [GF] and [FeY] for the case of invariant solutions on complex Lie groups) has given balanced metrics a really central role in non-Kähler geometry, as the equivalence between the dilatino equation (i.e. one of the equations of the system) and the conformally balanced equation requires the solutions of the system to be necessarily balanced. We refer also to the work [FLY], where new examples of balanced metrics are constructed on some Calabi Yau non-Kähler threefolds, as well as to the results in [BV], where a new balanced flow is introduced and investigated.

The main goal of this paper is to search for invariant special Hermitian, in particular balanced, metrics in the class of semisimple real non-compact Lie groups and on their compact (non-Kähler) quotients by a cocompact lattice; actually it appears that, despite invariant complex structures on semisimple (reductive) Lie algebras being fully classified in [Sn] (after the special case of compact Lie algebras had been considered by Samelson ([Sam]) and later in [Pi]), they have never been deeply investigated from this point of view. In contrast, the case of K compact is fully understood, as in such a case it is very well known that every invariant complex structure can be deformed to an invariant one for which the opposite of the Cartan-Killing form is a pluriclosed Hermitian metric h, i.e. it satisfies $dd^c\omega_h = 0$. Moreover it has been proved in [FGV] that K does not carry *any* balanced metric at all, fueling the conjecture ([FV]) that a compact complex manifold carrying two Hermitian metrics, one balanced and the other pluriclosed, must be actually Kähler.

More specifically, in this work we focus on a large class of simple non-compact real Lie algebras $\mathfrak{g}_o$ of even dimension, namely those which are of inner type, i.e. when the maximal compactly embedded subalgebra $\mathfrak{k}$ in a Cartan decomposition of $\mathfrak{g}_o$ contains a Cartan subalgebra. In these algebras we construct standard invariant complex structures (regular in [Sn]) and write down the balanced condition for invariant Hermitian metrics. A careful analysis of the resulting equation together with some general argument on root systems allows us to show the existence of a suitable invariant complex structure and a corresponding Hermitian metric satisfying the balanced equation. By Borel's Theorem, every semisimple Lie group G_o admits a cocompact lattice Γ so that the compact quotient G_o/Γ inherits the invariant balanced structure from G_o. We note here that the resulting metrics come in families and moreover the same kind of arguments can be applied to show the existence of balanced structures on quotients G_o/S, where G_o is any simple non-compact Lie group of inner type of any dimension and S is a suitable abelian closed subgroup.

We are also able to prove that the compact quotients $M = G_o/\Gamma$, endowed with the invariant complex structure that allows the existence of balanced metrics, do not carry any pluriclosed metric. This result is in accordance with the conjecture by Fino and Vezzoni and in some sense reflects a kind of duality between the compact and non-compact case, switching the existence of balanced/pluriclosed Hermitian metrics. In the last section, we prove that these balanced manifolds M, despite having vanishing first Chern class, carry no non trivial holomorphic $(n, 0)$-forms; furthermore we prove that they have vanishing Chern scalar curvature. This last property may allow to better understand the geometry of these manifolds, according to some more recent results concerning the implications of vanishing Chern-scalar curvature on some geometric features (see [Y]).

The paper is structured as follows. In Section 2, we review basic facts on simple real non-compact Lie algebras with invariant complex structures and we consider a class of invariant Hermitian metrics for which we write down the balanced condition in terms of roots. In

section 3 we state our main result, namely Theorem 3.1, and we prove it by means of several steps. We first rewrite the balanced equation in terms of simple roots and then the key Lemma 3.3 allows us to select an invariant complex structure so that the relative balanced equation admits solutions. In section 4 we prove that the complex manifolds that we constructed in the previous section and that admit balanced metrics, do not carry *any* pluriclosed metric. In the last section, we show in Theorem 5.1 that these complex compact manifolds (M, J) have trivial first Chern class and that the balanced metrics we have constructed have vanishing Chern scalar curvature; as a consequence we show that the Kodaira dimension $\kappa(M) = -\infty$.

Aknowledgements.

The authors would like to thank Daniele Angella for valuable conversations.

2. PRELIMINARIES

Let $\mathfrak{g}_o$ be a real simple $2n$-dimensional Lie algebra. It is well known that either the complexification $\mathfrak{g}_o^c$ is a complex simple Lie algebra (and in this case $\mathfrak{g}_o$ is called absolutely simple) or $\mathfrak{g}_o$ is the realification $\mathfrak{g}_{\mathbb{R}}$ of a complex simple Lie algebra $\mathfrak{g}$ (see e.g. [He]).

When $\mathfrak{g}_o$ is even dimensional, it is known ([Mo], see also [Sas]) that $\mathfrak{g}_o$ admits an invariant complex structure, namely an endomorphism $J \in \mathrm{End}\,(\mathfrak{g}_o)$ with $J^2 = -\mathrm{Id}$ and vanishing Nijenhuis tensor or, equivalently, such that

$$\mathfrak{g}_o^c = \mathfrak{g}_o^{10} \oplus \mathfrak{g}_o^{01}, \qquad [\mathfrak{g}_o^{10}, \mathfrak{g}_o^{10}] \subseteq \mathfrak{g}_o^{10}.$$

If G_o is any Lie group with Lie algebra $\mathfrak{g}_o$, then the endomorphism J defines a (left)-invariant complex structure on G_o. Moreover, thanks to a result due to Borel ([Bo]), there exists a discrete, torsionfree cocompact lattice Γ so that $M := G_o/\Gamma$ is compact and the left-invariant complex structure J on G_o descends to a complex structure J on M.

We recall that when G_o is compact and even-dimensional, i.e. $\mathfrak{g}_o$ is of compact type, the existence of an invariant complex structure was already established by Samelson ([Sam]), while in [Pi] it was shown that every invariant complex structure on G_o is obtained by means of Samelson's construction.

If we now consider an even-dimensional G_o and a compact quotient M endowed with an invariant complex structure J, we are interested in the existence of special Hermitian metrics h. The following proposition states a known fact, namely the non-existence of (invariant) Kähler structures.

Proposition 2.1. *The group* G_o *does not admit any invariant Kähler metric and the compact quotient* $M = G_o/\Gamma$ *is not Kähler.*

Proof. The first assertion is contained in [Ch], but we give here an elementary proof. If ω is an invariant symplectic form on $\mathfrak{g}_o$, then the closedness condition $d\omega = 0$ can be written as follows for $x, y, z \in \mathfrak{g}_o$

(2.1) $$\omega([x, y], z) + \omega([z, x], y) + \omega([y, z], x) = 0.$$

If B denotes the non-degenerate Cartan-Killing form of $\mathfrak{g}_o$, then we can define the endomorphism $F \in \mathrm{End}\,(\mathfrak{g}_o)$ by $B(Fx, y) = \omega(x, y)$ $(x, y \in \mathfrak{g}_o)$ so that F turns out to be a derivation by (2.1). As $\mathfrak{g}_o$ is semisimple, there exists a unique $z \in \mathfrak{g}_o$ with $F = \mathrm{ad}(z)$, so that $z \in \ker \omega$, a contradiction.

We now suppose that the compact manifold M has a Kähler metric with Kähler form ω. Using ω and a symmetrization procedure that goes back to [Be], we now construct an *invariant* Kähler form on G_o, obtaining a contradiction. We fix a basis $x_1, ..., x_{2n}$ of $\mathfrak{g}_o$ and we extend each vector as a left invariant vector fields on G_o; these vector fields can be projected down to M as vector fields $x_1^*, \ldots, x_{2n}^*$ that span the tangent space TM at each point. As G_o is semisimple, we can find a biinvariant volume form $d\mu$, that also descends to a volume form on M. We now define a left-invariant non-degenerate 2-form ϕ on G_o by setting

$$\phi_e(x_i, x_j) := \int_M \omega(x_i^*, x_j^*) \, d\mu.$$

As $\mathcal{L}_{x_k^*} d\mu = 0$ for every k, we have for every $i, j, k = 1, \ldots, 2n$

$$\int_M x_k^* \omega(x_i^*, x_j^*) \, d\mu = \int_M \mathcal{L}_{x_k^*}(\omega(x_i^*, x_j^*) \, d\mu) = 0$$

by Stokes' theorem and therefore we obtain that

$$d\phi(x_i, x_j, x_k) = \int_M d\omega(x_i^*, x_j^*, x_k^*) \, d\mu = 0.$$

This implies that ϕ is a symplectic form and the proof is concluded. □

Therefore we are interested in the existence of special Hermitian metrics on the complex manifold (M, J), in particular balanced and pluriclosed metrics, when the group G_o is of non-compact type.

The case of a simple Lie algebra $\mathfrak{g}_o$ which is the realification of a complex simple Lie algebra $\mathfrak{g}$ can be easily treated and will be dealt with in subsection 2.3.

We will now focus on some subclasses of simple real algebras, namely those which are absolutely simple and of inner type.

2.1. **Simple Lie algebras of inner type.** Let $\mathfrak{g}_o$ be an absolutely simple real algebra of non-compact type. It is well-known that $\mathfrak{g}_o$ admits a Cartan decompositon

$$\mathfrak{g}_o = \mathfrak{k} + \mathfrak{p},$$

where $\mathfrak{k}$ is a maximal compactly embedded subalgebra and

$$[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k},$$

so that $(\mathfrak{g}_o, \mathfrak{k})$ is a symmetric pair. Moreover the algebra $\mathfrak{g}_o$ is said to be *of inner type* when the symmetric pair $(\mathfrak{g}_o, \mathfrak{k})$ is of inner type, i.e. when a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ is a Cartan subalgebra of $\mathfrak{g}_o$, i.e. its complexification $\mathfrak{t}^c$ is a Cartan subalgebra of $\mathfrak{g}_o^c$. Using the notation as in [He], p. 126, we obtain the list of all inner symmetric pairs $(\mathfrak{g}_o, \mathfrak{k})$ of non-compact type with $\mathfrak{g}_o$ simple and even dimensional (Table 1).

2.2. **Invariant complex structures.** In this section we will describe how to construct invariant complex structures on even-dimensional absolutely simple non-compact Lie algebras $\mathfrak{g}_o$.

We fix a maximal abelian subalgebra $\mathfrak{t} \subseteq \mathfrak{k}$, so that $\mathfrak{h} := \mathfrak{t}^c$ is a Cartan subalgebra of $\mathfrak{g} := \mathfrak{g}_o^c$. Note that if $\mathfrak{g}_o$ is even dimensional , the same holds for $\mathfrak{t}$. The corresponding root system is denoted by R and we have the following decompositions

$$\mathfrak{k}^c = \mathfrak{t}^c \oplus \bigoplus_{\alpha \in R_\mathfrak{k}} \mathfrak{g}_\alpha, \quad \mathfrak{p}^c = \bigoplus_{\alpha \in R_\mathfrak{p}} \mathfrak{g}_\alpha,$$

where a root α will be called *compact* (resp. *non-compact*), when $\mathfrak{g}_\alpha \subseteq \mathfrak{k}^c$ (resp. $\mathfrak{g}_\alpha \subseteq \mathfrak{p}^c$) and the set of all compact (resp. non-compact) roots is denoted by $R_\mathfrak{k}$ (resp. $R_\mathfrak{p}$). It is a standard

Type	$\mathfrak{g}$	$\mathfrak{k}$	conditions
A	$\mathfrak{su}(p,q)$	$\mathfrak{su}(p) + \mathfrak{su}(q) + \mathbb{R}$	$p \geq q \geq 1,\ p+q$ odd
B	$\mathfrak{so}(2p+1, 2q)$	$\mathfrak{so}(2p+1) + \mathfrak{so}(2q)$	$p \geq 0, q \geq 1,\ p+q$ even
C	$\mathfrak{sp}(2n, \mathbb{R})$	$\mathfrak{su}(2n) + \mathbb{R}$	$n \geq 1$
C	$\mathfrak{sp}(p,q)$	$\mathfrak{sp}(p) + \mathfrak{sp}(q)$	$p, q \geq 1,\ p+q$ even
D	$\mathfrak{so}(4n)^*$	$\mathfrak{su}(2n) + \mathbb{R}$	$n \geq 2$
D	$\mathfrak{so}(2p, 2q)$	$\mathfrak{so}(2p) + \mathfrak{so}(2q)$	$p, q \geq 1, p+q$ even ≥ 4
G	$\mathfrak{g}_{2(2)}$	$\mathfrak{su}(2) + \mathfrak{su}(2)$	
F	$\mathfrak{f}_{4(-20)}$	$\mathfrak{so}(9)$	
F	$\mathfrak{f}_{4(4)}$	$\mathfrak{su}(2) + \mathfrak{sp}(3)$	
E	$\mathfrak{e}_{6(2)}$	$\mathfrak{su}(2) + \mathfrak{su}(6)$	
E	$\mathfrak{e}_{6(-14)}$	$\mathfrak{so}(10) + \mathbb{R}$	
E	$\mathfrak{e}_{8(8)}$	$\mathfrak{so}(16)$	
E	$\mathfrak{e}_{8(-24)}$	$\mathfrak{su}(2) + \mathfrak{e}_7$	

TABLE 1. Inner symmetric pairs $(\mathfrak{g}, \mathfrak{k})$ of non-compact type with $\mathfrak{g}$ simple and even dimensional.

fact that $\mathfrak{u} := \mathfrak{k} + i\mathfrak{p} \subseteq \mathfrak{g}$ is a compact real form of $\mathfrak{g}$ and that we can choose a basis $\{E_\alpha\}_{\alpha \in R}$ of root spaces so that

$$\tau(E_\alpha) = -E_{-\alpha}, \qquad B(E_\alpha, E_{-\alpha}) = 1, \qquad [E_\alpha, E_{-\alpha}] = H_\alpha$$

where τ denotes the anticomplex involution defining $\mathfrak{u}$, B is the Cartan Killing form of $\mathfrak{g}$ and H_α is the B-dual of α (see e.g. [He]). If σ is the involutive anticomplex map defining $\mathfrak{g}_o$, we then have that

$$\sigma(E_\alpha) = -E_{-\alpha}, \quad \alpha \in R_\mathfrak{k},$$
$$\sigma(E_\alpha) = E_{-\alpha}, \quad \alpha \in R_\mathfrak{p}.$$

If we fix an ordering , namely a splitting $R = R^+ \cup R^-$ with $R^- = -R^+$ and $(R^+ + R^+) \cap R \subseteq R^+$, we can define a subalgebra

$$\mathfrak{q} := \mathfrak{h}_1 \oplus \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha,$$

where $\mathfrak{h}_1 \subset \mathfrak{h}$ is a subspace so that $\mathfrak{h}_1 \oplus \sigma(\mathfrak{h}_1) = \mathfrak{h}$. The so defined subalgebra $\mathfrak{q} \subset \mathfrak{g}$ satisfies

$$\mathfrak{g} = \mathfrak{q} \oplus \sigma(\mathfrak{q})$$

and therefore it defines a complex structure J on $\mathfrak{g}_o$ with the property that $\mathfrak{q} = \mathfrak{g}_o^{10}$. This complex structure depends on the arbitrary choice of $\mathfrak{h}_1$, i.e. on the arbitrary choice of a complex structure on $\mathfrak{t}$.

We remark that the complex structure J enjoys the further property of being $\mathrm{ad}(\mathfrak{t})$-invariant, namely

$$[\mathrm{ad}(x), J] = 0, \quad x \in \mathfrak{t}.$$

Therefore if G_o is a Lie group with Lie algebra $\mathfrak{g}_o$, then J extends to a left-invariant complex structure on G_o and it will be also right-invariant with respect to right translations by elements $h \in T := \exp(\mathfrak{t})$ (note that T might be non-compact, unless G_o has finite center).

We will call such an invariant complex structure *standard*.

Remark 2.2. In [Sn] the class of (simple) real Lie algebras of inner type is called "Class I" and it is then proved that *every* invariant complex structure in these algebras are standard, with respect to a suitable choice of a Cartan subalgebra (such complex structures are called regular in [Sn]).

2.3. **Invariant metrics and the balanced condition.** Let M be a compact complex manifold of the form G_o/Γ, endowed with a complex structure J which is induced by a standard invariant complex structure J on G_o, as in the previous section. It is clear that any left invariant J-Hermitian metric h on G_o induces an Hermitian metric $\bar{h}$ on M and $\bar{h}$ is balanced or pluriclosed if and only if h is so. For the converse, we prove the following

Proposition 2.3. *If* (M, J) *admits a balanced (pluriclosed) Hermitian metric, there exists a left invariant and right* T*-invariant Hermitian metric on* G_o *which is balanced (pluriclosed resp.).*

Proof. Suppose we have a balanced metric h on M with associated fundamental form ω. Then using the same notation and arguments as in the proof of Prop.2.1, we define a left-invariant positive $(n-1, n-1)$-form ϕ on G_o as follows

$$\phi_e(x_{i_1}, \ldots, x_{i_{2n-2}}) := \int_M \omega^{n-1}(x_{i_1}^*, \ldots, x_{i_{2n-2}}^*) \, d\mu.$$

As $d\omega^{n-1} = 0$, we obtain that also $d\phi = 0$. Therefore, we can find an unique $(1,1)$-form $\hat{\omega}$ so that $\hat{\omega}^{n-1} = \phi$ (see [Mi]) and the metric given by $\hat{\omega}$ is balanced. As ϕ is left invariant, so is $\hat{\omega}$ by uniqueness. Now, the group $\mathrm{Ad}(T)$ is compact and using a standard avaraging process we can make ϕ_e also $\mathrm{Ad}(T)$-invariant. This means that ϕ is also invariant under right T-translations. Again, by the uniqueness, the same will hold true for $\hat{\omega}$.

As for the pluriclosed condition, the lifted metric from M to G_o is clearly pluriclosed and can be made T-invariant by a standard averaging.   □

Remark 2.4. We can now deal with the case when $\mathfrak{g}_o$ is the realification of a simple Lie algebra $\mathfrak{g}$. In this case the complex structure J commutes with $\mathrm{ad}(\mathfrak{g}_o)$ and $\mathfrak{g}_o = \mathfrak{u} + i\mathfrak{u}$ is a Cartan decomposition, where $\mathfrak{u}$ is a compact real form of $\mathfrak{g}$. Let G_o be a real group with algebra $\mathfrak{g}_o$ and let U be the compact subgroup with algebra $\mathfrak{u}$. Then the metric h which coincides with $-B$ on $\mathfrak{u}$, with B on $i\mathfrak{u}$ and such that $h(\mathfrak{u}, i\mathfrak{u}) = 0$ is a Hermitian metric which is balanced. Indeed, h is $\mathrm{Ad}(U)$-invariant and therefore the corresponding $\delta\omega$ is $\mathrm{Ad}(U)$-invariant 1-form, hence it vanishes identically. This is consistent with the fact that complex parallelizable manifolds carry balanced metrics as they carry Chern-flat metrics, as noted in [Ga1], p. 121 (see also [AG],[Gr]).

On the other hand, G_o admits no invariant pluriclosed metric. Indeed, any such metric h can be avaraged to produce an $\mathrm{Ad}(U)$-invariant pluriclosed metric, which would be balanced by the previous argument. This is not possible, as a metric which is balanced and pluriclosed at the same time has to be Kähler (see e.g. [AI]), contrary to Prop 2.1.

We now focus on the case where $\mathfrak{g}_o$ is absolutely simple of inner type, endowed with an invariant complex structure. We fix a Cartan subalgebra $\mathfrak{t} \subseteq \mathfrak{k}$ with corresponding root system $R = R_{\mathfrak{k}} \cup R_{\mathfrak{p}}$ as in section 2.1 and we consider an ordering $R = R^+ \cup R^-$ giving an invariant complex structure J_o on $\mathfrak{g}_o/\mathfrak{t}$. We extend J_o to an invariant complex structure J on $\mathfrak{g}_o$.

We also fix a basis of a complement of $\mathfrak{t}$ in $\mathfrak{g}_o$

$$v_\alpha := \frac{1}{\sqrt{2}}(E_\alpha - E_{-\alpha}), \ w_\alpha := \frac{i}{\sqrt{2}}(E_\alpha + E_{-\alpha}), \ \alpha \in R_{\mathfrak{k}}^+,$$

$$v_\alpha := \frac{1}{\sqrt{2}}(E_\alpha + E_{-\alpha}), \ w_\alpha := \frac{i}{\sqrt{2}}(E_\alpha - E_{-\alpha}), \ \alpha \in R_\mathfrak{p}^+,$$

so that $v_\alpha, w_\alpha \in \mathfrak{g}_o$ for every $\alpha \in R^+$ and moreover

$$Jv_\alpha = w_\alpha, \ Jw_\alpha = -v_\alpha,$$

$$[H, v_\alpha] = -i\alpha(H)w_\alpha, \ H \in \mathfrak{h},$$

$$[v_\alpha, w_\alpha] = iH_\alpha, \ \alpha \in R_\mathfrak{k}^+,$$

$$[v_\alpha, w_\alpha] = -iH_\alpha, \ \alpha \in R_\mathfrak{p}^+.$$

We now construct invariant Hermitian metrics h on $\mathfrak{g}_o$. First, we define h on $\mathfrak{t}$ by choosing a J-Hermitian metric $h_\mathfrak{t}$ on $\mathfrak{t}$. If we set $\mathfrak{m}_\alpha := \mathrm{Span}\{v_\alpha, w_\alpha\}_{\alpha \in R^+}$, we define for $\alpha \neq \beta \in R^+$

$$h(\mathfrak{t}, \mathfrak{m}_\alpha) = 0, \quad h(\mathfrak{m}_\alpha, \mathfrak{m}_\beta) = 0,$$

$$h(v_\alpha, v_\alpha) = h(w_\alpha, w_\alpha) = h_\alpha^2, \quad h(v_\alpha, w_\alpha) = 0$$

for $h_a \in \mathbb{R}^+$.

In particular we are interested in constructing balanced Hermitian metrics, namely Hermitian metrics whose associated $(1,1)$-form $\omega = h(\cdot, J\cdot)$ satisfies $d\omega^{n-1} = 0$ or equivalently $\delta\omega = 0$, where δ denotes the codifferential.

We use the expression

$$\delta\omega(x) = -\mathrm{Tr}\nabla.\omega(\cdot, x) = -\sum_i^{2n} \nabla_{e_i}\omega(e_i, x) =$$

$$= \sum_i \omega(\nabla_{e_i}e_i, x) + \omega(e_i, \nabla_{e_i}x),$$

where ∇ denotes the Levi Civita connection of h and $\{e_i\}$ is an orthonormal basis of $\mathfrak{g}_o$ w.r.t. h. Note that both h and J are $\mathrm{ad}(\mathfrak{t})$-invariant and therefore $\delta\omega$, which is also $\mathrm{ad}(\mathfrak{t})$-invariant, does not vanish only when evaluated on elements $x \in \mathfrak{t}$.

We have the following expression for the Levi Civita connection, namely for $x, y, z \in \mathfrak{g}_o$

$$2h(\nabla_x y, z) = h([x, y], z) + h([z, x], y) + h([z, y], x).$$

Then for every $x \in \mathfrak{t}, y \in \mathfrak{g}_o$

$$h(\nabla_y y, x) = h([x, y], y) = 0.$$

Therefore for $x \in \mathfrak{t}$ we have

(2.2) $$\delta\omega(x) = \sum_i \omega(e_i, \nabla_{e_i}x) = -\sum_i h(Je_i, \nabla_{e_i}x) =$$

$$= -\frac{1}{2}\left(h([e_i, x], Je_i) + h([Je_i, e_i], x) + h([Je_i, x], e_i)\right).$$

We now observe that J is $\mathrm{ad}(\mathfrak{t})$-invariant and therefore $h([Je_i, x], e_i) = -h([e_i, x], Je_i)$ for every $i = 1, \ldots, 2n$, so that (2.2) can be written as

$$-\delta\omega(x) = \frac{1}{2}\sum_i h([Je_i, e_i], x) =$$

$$= \frac{1}{2} \cdot 2\left(\sum_{\alpha \in R_\mathfrak{k}^+} \frac{1}{h_\alpha^2}h([w_\alpha, v_\alpha], x) + \sum_{\alpha \in R_\mathfrak{p}^+} \frac{1}{h_\alpha^2}h([w_\alpha, v_\alpha], x)\right) =$$

$$= \sum_{\alpha \in R_{\mathfrak{k}}^+} \frac{1}{h_\alpha^2} h(-iH_\alpha, x) + \sum_{\alpha \in R_{\mathfrak{p}}^+} \frac{1}{h_\alpha^2} h(iH_\alpha, x),$$

so that $\delta\omega|_{\mathfrak{t}} = 0$ if and only if

$$-\sum_{\alpha \in R_{\mathfrak{k}}^+} \frac{1}{h_\alpha^2} H_\alpha + \sum_{\alpha \in R_{\mathfrak{p}}^+} \frac{1}{h_\alpha^2} H_\alpha = 0.$$

Summing up, the metric h is balanced when the following equation is satisfied

$$(2.3) \qquad\qquad \sum_{\alpha \in R_{\mathfrak{k}}^+} \frac{1}{h_\alpha^2} \alpha = \sum_{\alpha \in R_{\mathfrak{p}}^+} \frac{1}{h_\alpha^2} \alpha.$$

Note that this does *not* depend on the choice of the metric along the toral part $\mathfrak{t}$.

3. MAIN RESULT

In this section we will prove our main result

Theorem 3.1. *Every non-compact simple Lie group G_o of even dimension and of inner type admits an invariant complex structure J and an invariant balanced J-Hermitian metric.*

Note that by Borel's Theorem, we can use a cocompact latice $\Gamma \subset G_o$ to obtain compact quotients $M = G_o/\Gamma$, which will inherit the same balanced structure.

We start noting that equation (2.3) involves the unknowns $\{h_\alpha\}_{\alpha \in R^+}$ and also a choice of positive roots, i.e. an ordering or equivalenty a complex structure on $\mathfrak{g}_o$. We will always fix a complex structure on $\mathfrak{t}$ once for all. It is known that giving an ordering on the root system R is equivalent to the choice of a system of simple roots Π and that two systems of simple roots are conjugate under the action of the Weyl group W. We may fix a system of simple roots $\Pi = \{\alpha_1, \dots, \alpha_r\}$ and put $\Pi = \Pi_c \cup \Pi_{nc}$, where $\Pi_{c/nc}$ denotes the set of simple roots which are compact or noncompact. We set $\Pi_c = \{\phi_1, \dots, \phi_k\}$, $\Pi_{nc} = \{\psi_1, \dots, \psi_l\}$, $k + l = r = \text{rank}(\mathfrak{g}_o)$. Each root $\alpha \in R^+$ can be written as

$$\alpha = \sum_{i=1}^k n_i(\alpha)\phi_i + \sum_{j=1}^l m_j(\alpha)\psi_j$$

for $n_i(\alpha), m_j(\alpha) \in \mathbb{N}$ nonnegative integers. If we set $g_\alpha := \frac{1}{h_\alpha^2}$ and $g_j := g_{\phi_j}, h_j := g_{\psi_j}$, equation (2.3) can be written as

$$\sum_{\alpha \in R_{\mathfrak{k}}^+, \alpha \notin \Pi} g_\alpha \left(\sum_j n_j(\alpha)\phi_j + \sum_j m_j(\alpha)\psi_j \right) + \sum_j g_j \phi_j =$$

$$= \sum_{\alpha \in R_{\mathfrak{p}}^+, \alpha \notin \Pi} g_\alpha \left(\sum_j n_j(\alpha)\phi_j + \sum_j m_j(\alpha)\psi_j \right) + \sum_j h_j \psi_j,$$

and therefore

$$(3.4) \qquad \begin{cases} g_j = \displaystyle\sum_{\alpha \in R_{\mathfrak{p}}^+,\, \alpha \notin \Pi} g_\alpha n_j(\alpha) \quad - \sum_{\alpha \in R_{\mathfrak{k}}^+,\, \alpha \notin \Pi} g_\alpha n_j(a), \;\; j = 1, \dots, k, \\[2em] h_j = \displaystyle\sum_{\alpha \in R_{\mathfrak{k}}^+,\, \alpha \notin \Pi} g_\alpha m_j(\alpha) \quad - \sum_{\alpha \in R_{\mathfrak{p}}^+,\, \alpha \notin \Pi} g_\alpha m_j(a), \;\; j = 1, \dots, l. \end{cases}$$

Remark 3.2. If we consider for instance the case $\mathfrak{g}_o = \mathfrak{su}(p, q)$ ($p + q$ even, $p, q \geq 2$) and the standard system of simple roots $\Pi = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \ldots, \epsilon_{p-1} - \epsilon_p, \epsilon_p - \epsilon_{p+1}, \ldots, \epsilon_{p+q-1} - \epsilon_{p+q}\}$ of $\mathfrak{sl}(p+q, \mathbb{C})$, then $\Pi_{nc} = \{\epsilon_p - \epsilon_{p+1}\}$ and Π_c gives a system of simple roots for the semisimple part $\mathfrak{k}_{ss}$ of $\mathfrak{k}$. This means that every root $\alpha \in R_{\mathfrak{k}}^+, \alpha \notin \Pi$ is a linear combination of roots in Π_c and therefore the righthandside of the last equation in (3.4) is non-positive, so that (3.4) has no solution. This shows that the choice of the invariant complex structure might not be straightforward.

The following lemmata provide key tools in our argument.

Lemma 3.3. *For each symmetric pair $(\mathfrak{g}_o, \mathfrak{k})$ as in Table 1, $(\mathfrak{g}_o, \mathfrak{k}) \ncong (\mathfrak{so}(1, 2n), \mathfrak{so}(2n))$ and given a Cartan subalgebra $\mathfrak{t} \subseteq \mathfrak{k}$ with corresponding root system R, there exists an ordering of the roots, hence a system of simple roots Π, such that*

$$(3.5) \qquad \forall \psi \in \Pi_{nc} \, \exists \psi' \in \Pi_{nc} \text{ with } \psi + \psi' \in R.$$

This implies that, if $\Pi_{nc} = \{\psi_1, \ldots, \psi_l\}$, then for every $\psi_j \in \Pi_{nc}$ there exists $\alpha \in R_{\mathfrak{k}}^+$ with $m_j(\alpha) \neq 0$ and $\alpha \in \mathrm{Span}\{\Pi_{nc}\}$.

Remark Note that $\mathfrak{sp}(1, 1) \cong \mathfrak{so}(1, 4)$ is also not admissible in the above Lemma. In general, for $\mathfrak{g}_o = \mathfrak{so}(1, 2n)$ we have the standard system $\Pi = \{\epsilon_i - \epsilon_{i+1}, \epsilon_n, \; i = 1, \ldots, n - 1\}$ with $\Pi_{nc} = \{\epsilon_n\}$. As R_c consists precisely of all the short roots, it is clear that for any element σ of the Weyl group $W \cong \mathbb{Z}_2^n \rtimes \mathcal{S}_n$ we have that $\sigma(\Pi)_{nc}$ consists of one element. We will deal with this case later on.

Proof. We first deal with the classical case. We start with the standard system of simple roots Π, following the notation as in [He]. It is immediate to check that in this case Π_{nc} consists of a single root ψ.

We first deal with the case where ψ is a short root. Let Λ be the set of all simple roots which are connected to ψ in the Dynkin diagram relative to Π. If $s \in W$ denotes the reflection around ψ, then s leaves every element $\Pi \setminus \Lambda$ pointwise fixed. We observe that Λ consists of either at most three short roots or it contains a long root. In the first case, $s(\Lambda) = \{\psi + \lambda | \lambda \in \Lambda\} \subseteq R_{\mathfrak{p}}$ so that $s(\Pi)_{nc} = \{-\psi, s(\Lambda)\}$ and therefore the system of simple roots $s(\Pi)$ satisfies (3.5). If Λ contains a long root, then it also contains a short root, unless $(\mathfrak{g}_o, \mathfrak{k}) = (\mathfrak{so}(2, 3), \mathbb{R} + \mathfrak{so}(3))$, that is isomorphic to $(\mathfrak{sp}(2), \mathfrak{u}(2))$; this case will be dealt with in the second part of the proof. Therefore $\Lambda = \{\phi_1, \phi_2\}$ with ϕ_1 short and ϕ_2 long. Again the reflection s around ψ gives $s(\phi_1) = \psi + \phi_1$ and $s(\phi_2) = \phi_2 + 2\psi \in R_{\mathfrak{k}}$ or $s(\phi_2) = \psi + \phi_2 \in R_{\mathfrak{p}}$. This implies that the system of simple roots $s(\Pi)$ has $s(\Pi)_{nc} = \{-\psi, \psi + \phi_1\}$ or $\{-\psi, \psi + \phi_1, \psi + \phi_2\}$ and in both cases it satisfies (3.5).

We are left with the case where ψ is a long root, namely the case where $\mathfrak{g}_o = \mathfrak{sp}(2n, \mathbb{R})$ and $\mathfrak{k} = \mathfrak{u}(2n)$. A standard system of simple roots is given by $\Pi = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \ldots, \epsilon_{2n-1} - \epsilon_{2n}, 2\epsilon_{2n}\}$ and $\Pi_{nc} = \{\psi = 2\epsilon_{2n}\}$. Again using s_β, we see that $s_\beta(\Pi)_{nc} = \{-2\epsilon_{2n}, \epsilon_{2n-1} + \epsilon_{2n}\}$ so that condition (3.5) is satisfied.

We may now deal with the exceptional cases. Starting with the standard system of simple roots Π, we list the set Π_{nc}, that turns out to consist of a single root β. For each case, using the symmetry s_β we obtain the system of simple roots $\Pi' := s_\beta(\Pi)$ that satisfies condition (3.5).
(1) $(\mathfrak{g}_o, \mathfrak{k}) = (\mathfrak{g}_2, \mathfrak{su}(2) + \mathfrak{su}(2))$. Here $\Pi = \{\alpha, \beta\}$, with β long. We have $\Pi_{nc} = \{\beta\}$ and $\Pi' = \{-\beta, \alpha + \beta\}$.

(2) $(\mathfrak{g}_o, \mathfrak{k}) = (\mathfrak{f}_{4(-20)}, \mathfrak{so}(9))$. According to [He], the standard system of simple roots is $\Pi = \{\alpha_1 = \epsilon_2 - \epsilon_3, \alpha_2 = \epsilon_3 - \epsilon_4, \alpha_3 = \epsilon_4, \alpha_4 = \frac{1}{2}(e_1 - \epsilon_2 - \epsilon_3 - \epsilon_4)\}$ so that $\Pi_{nc} = \{\alpha_4\}$ and therefore $\Pi'_{nc} = \{-\alpha_4, \alpha_4 + \alpha_3\}$.

(3) $(\mathfrak{g}_o, \mathfrak{k}) = (\mathfrak{f}_{4(4)}, \mathfrak{su}(2) + \mathfrak{sp}(3))$. In this case $\Pi_{nc} = \{\alpha_1\}$ and therefore $\Pi'_{nc} = \{-\alpha_1, \alpha_1 + \alpha_2\}$.

(4) $(\mathfrak{g}_o, \mathfrak{k}) = (\mathfrak{e}_{8(8)}, \mathfrak{so}(16))$. For $\mathfrak{e}_8$ we have the standard system of simple roots

$$\alpha_1 = \frac{1}{2}(\epsilon_1 + \epsilon_8) - \frac{1}{2}(\epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 + \epsilon_6 + \epsilon_7), \alpha_2 = \epsilon_1 + \epsilon_2,$$

$$\alpha_j = \epsilon_{j-1} - \epsilon_{j-2}, \ j = 3, \ldots, 8.$$

Then $\Pi_{nc} = \{\alpha_1\}$ and $\Pi'_{nc} = \{-\alpha_1, \alpha_1 + \alpha_3\}$.

(5) $(\mathfrak{g}_o, \mathfrak{k}) = (\mathfrak{e}_{8(-24)}, \mathfrak{su}(2) + \mathfrak{e}_7)$. Keeping the same notation for simple roots as above, we have $\Pi_{nc} = \{\alpha_8\}$ and $\Pi'_{nc} = \{-\alpha_8, \alpha_8 + \alpha_7\}$.

(6) $(\mathfrak{g}_o, \mathfrak{k}) = (\mathfrak{e}_{6(2)}, \mathfrak{su}(2) + \mathfrak{su}(6))$. As the system root system Π can be taken to be composed of the simple roots $\{\alpha_1, \ldots, \alpha_6\}$ of $\mathfrak{e}_8$, we have $\Pi_{nc} = \{\alpha_2\}$ and $\Pi'_{nc} = \{-\alpha_2, \alpha_2 + \alpha_4\}$.

(7) $(\mathfrak{g}_o, \mathfrak{k}) = (\mathfrak{e}_{6(-14)}, \mathbb{R} + \mathfrak{so}(10))$. We have $\Pi_{nc} = \{\alpha_1\}$ and $\Pi'_{nc} = \{-\alpha_1, \alpha_1 + \alpha_3\}$. $\qquad\square$

Lemma 3.4. *For every system of simple roots $\Pi = \Pi_c \cup \Pi_{nc}$ with $\Pi_c = \{\phi_1, \ldots, \phi_k\}$ we have*

$$\forall \, j = 1, \ldots, k, \ \exists \, \alpha \in R_{\mathfrak{p}}^+, \ \alpha \notin \Pi : \ n_j(\alpha) \neq 0,$$

where $n_j(\alpha)$ denotes the coordinate of α along the root ϕ_j.

Proof. We start noting that the centralizer $C_{\mathfrak{k}^c}(\mathfrak{p}^c) = C_{\mathfrak{k}}(\mathfrak{p})^c = \{0\}$. It then follows that $[E_{\phi_j}, \mathfrak{p}^c] \neq \{0\}$, hence there exists $\gamma \in R_{\mathfrak{p}}$ with $[E_{\phi_j}, E_\gamma] \neq 0$, i.e. $\phi_j + \gamma \in R_{\mathfrak{p}}$. Now, if $\gamma > 0$, then $\alpha := \phi_j + \gamma \in R_{\mathfrak{p}}^+ \setminus \Pi$ and $n_j(\alpha) \geq 1$. Suppose now $\gamma < 0$. We write $\gamma = c_j\phi_j + \sum_{\theta \in \Pi \setminus \phi_j} c_\theta \theta$ for some nonpositive integers c_j, c_θ. As $\gamma \neq -\phi_j$, there exists at least one negative coefficient $c_\theta < 0$, for some $\theta \in \Pi, \theta \neq \phi_j$. Therefore the root $\gamma + \phi_j$ must be negative and $1 + c_j \leq 0$, i.e. $\alpha := -\gamma \in R_{\mathfrak{p}}^+ \setminus \Pi$ and $n_j(\alpha) = -c_j \geq 1$. $\qquad\square$

We now fix a system of simple roots Π as in Lemma 3.3. In order to solve the corresponding system of equations (3.4) for the positive unknowns $\{g_i, h_j, g_\alpha\}$, we will show how to choose the positive values $\{g_\alpha\}_{\alpha \in R^+ \setminus \Pi}$ in such a way to guarantee that the constants $\{g_i, h_j\}$, defined to satisfy (3.4), are positive.

We set

$$\Sigma_{\mathfrak{k}} := \{\alpha \in R_{\mathfrak{k}}^+ \,|\, \alpha \notin \Pi, \alpha \in \mathrm{Span}\{\Pi_{nc}\}\}, \qquad A_{\mathfrak{k}} = (R_{\mathfrak{k}} \setminus \Pi_c) \setminus \Sigma_{\mathfrak{k}}.$$

Then the system of equations (3.4) can be written as

(3.6)
$$\begin{cases} g_j = \displaystyle\sum_{\alpha \in R_{\mathfrak{p}}^+, \, \alpha \notin \Pi} g_\alpha n_j(\alpha) - \sum_{\alpha \in A_{\mathfrak{k}}} g_\alpha n_j(a), & j = 1, \ldots, k, \ (1) \\[2ex] h_j = \displaystyle\sum_{\alpha \in R_{\mathfrak{k}}^+, \, \alpha \notin \Pi} g_\alpha m_j(\alpha) - \sum_{\alpha \in R_{\mathfrak{p}}^+, \, \alpha \notin \Pi} g_\alpha m_j(a), & j = 1, \ldots, l. \ (2) \end{cases}$$

We start assigning $g_\alpha = 1$ for every $\alpha \in A_{\mathfrak{k}}$.

Then, for every $j = 1, \ldots, k$, we use Lemma 3.4 selecting a root $\alpha \in R_{\mathfrak{p}}^+$ with $n_j(\alpha) \neq 0$, $\alpha \notin \Pi$. This root α, which depends on j, contributes to the first sum in the righthandside of equation (1) in (3.6) and the value g_α can be chosen big enough so that g_j is strictly positive. Summing up, we can assign values $\{g_\alpha\}_{\alpha \in R_{\mathfrak{p}}^+ \setminus \Pi_{nc}}$ so that all $g_j, j = 1, \ldots, k$ can be defined as in (3.6), (1), and are strictly positive.

We now turn to equation (3.6)-(2), which can now be written as

$$(3.7) \qquad h_j = \sum_{\alpha \in \Sigma_{\mathfrak{k}}} g_\alpha m_j(\alpha) + \sum_{\alpha \in A_{\mathfrak{k}}} m_j(\alpha) - \sum_{\alpha \in R_{\mathfrak{p}}^+, \ \alpha \notin \Pi} g_\alpha m_j(a),$$

where in the righthandside the last two sums have a fixed value. Now, by Lemma 3.3, we know that for every $j = 1, \ldots, l$, we can find $\alpha \in \Sigma_{\mathfrak{k}}$ with $m_j(\alpha) \neq 0$. These roots can be used to choose the coefficients g_α big enough to guarantee that $h_j > 0$, when defined to satisfy (3.7), is strictly positive.

In order to complete the proof of our main result Theorem (3.1), we are left with the case $(\mathfrak{g}_o, \mathfrak{k}) = (\mathfrak{so}(1, 2n), \mathfrak{so}(2n))$ with standard system of simple roots $\Pi = \{\epsilon_i - \epsilon_{i+1}, \epsilon_n, \ i = 1, \ldots, n-1\}$, $\Pi_{nc} = \{\epsilon_n\}$. We see that

$$R_{\mathfrak{k}}^+ = \{\epsilon_i \pm \epsilon_j, \ i < j\}, \quad R_{\mathfrak{p}} = \{\epsilon_1, \ldots, \epsilon_n\}.$$

Now, we use equation (2.3) and search for positive real numbers $\{x, y, z_i, \ i = 1, \ldots, n\}$ so that

$$x \cdot \sum_{i < y} \epsilon_i - \epsilon_j + y \cdot \sum_{i < j} \epsilon_i + e_j = \sum_{i=1}^n z_i \epsilon_i,$$

i.e.

$$\sum_{i=1}^n [(x+y)(n-i) + (x-y)(i-1)] \epsilon_i = \sum_{i=1}^n z_i \epsilon_i.$$

It is clear that the above equation has positive solutions by simply choosing $x > y > 0$.

Remark 3.5. We can consider the metric h_o which coincides with $-B$ on the compact part $\mathfrak{k}$, with B on $\mathfrak{p}$ and such that $h_o(\mathfrak{k}, \mathfrak{p}) = 0$. This metric is easily seen to depend only on $\mathfrak{g}_o$ and *not* on the Cartan decomposition $\mathfrak{g}_o = \mathfrak{k} + \mathfrak{p}$. We could then ask whether there exists a suitable complex structure such that the metric h_o turns out to be balanced. The resulting equation has been already treated in [AP] and has a solution if and only if $\mathfrak{g}_o = \mathfrak{su}(p, p+1) \cong \mathfrak{su}(p+1, p)$ for $p \geq 1$.

4. NON-EXISTENCE OF PLURICLOSED METRICS

In this section we prove the following non-existence result

Proposition 4.1. *The compact complex manifolds* (M, J)*, wheer* $\mathrm{M} = \mathrm{G}_o/\Gamma$*, do not admit any pluriclosed metric.*

Note that in the above statement J is the complex structure we have exhibited in section 3.

Now, if h is any such metric, we can obtain a pluriclosed invariant metric h on G_o which is also invariant under right T-translations. It follows that on $\mathfrak{g}$ we have

$$h(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0 \quad \text{if} \quad \beta \neq -\alpha.$$

In order to write down the condition $dd^c \omega = 0$, where ω is the fundamental form of h, we recall the Koszul's formula for the differential of invariant forms. If ϕ is any invariant k-form on G_o or equivalently on $\mathfrak{g}_o$, then for every $v_o, \ldots, v_k$ in $\mathfrak{g}_o$

$$d\phi(v_o, \ldots, v_k) = \sum_{i < j} (-1)^{i+j} \phi([x_i, x_j], v_1, \ldots, \widehat{v_i}, \ldots, \widehat{v_j} \ldots, v_k).$$

We set $\phi := d^c \omega$ and compute $d\phi(E_\alpha, E_{-\alpha}, E_\beta, E_{-\beta})$ for $\alpha, \beta \in R^+$. We have

$$d\phi(E_\alpha, E_{-\alpha}, E_\beta, E_{-\beta}) = -\phi(H_\alpha, E_\beta, E_{-\beta}) + \phi(N_{\alpha\beta} E_{\alpha+\beta}, E_{-\alpha}, E_{-\beta})$$

$$-\phi(N_{\alpha,-\beta}E_{\alpha-\beta},E_{-\alpha},E_{\beta})-\phi(N_{-\alpha,\beta}E_{\beta-\alpha},E_{\alpha},E_{-\beta})+\phi(N_{-\alpha,-\beta}E_{-\alpha-\beta}E_{\alpha},E_{\beta})$$
$$-\phi(H_{\beta},E_{\alpha},E_{-\alpha}),$$

where we use the standard notation $[E_{\gamma},E_{\epsilon}]=N_{\gamma,\epsilon}E_{\gamma+\epsilon}$ for every $\gamma,\epsilon\in R$. Using the known identities for the Weyl basis (see [He], p. 172,176), we can write that

$$d\phi(E_{\alpha},E_{-\alpha},E_{\beta},E_{-\beta})=-\phi(H_{\alpha},E_{\beta},E_{-\beta})-\phi(H_{\beta},E_{\alpha},E_{-\alpha})$$

$$+2\phi(N_{\alpha\beta}E_{\alpha+\beta},E_{-\alpha},E_{-\beta})-2\phi(N_{\alpha,-\beta}E_{\alpha-\beta},E_{-\alpha},E_{\beta}).$$

We also introduce the notation $JE_{\gamma}=i\epsilon_{\gamma}E_{\gamma}$ for every $\gamma\in R$, where $\epsilon_{\gamma}=\pm1$ according to $\gamma\in R^{\pm}$. Then

$$dd^{c}\omega(E_{\alpha},E_{-\alpha},E_{\beta},E_{-\beta})=-d\omega(JH_{\alpha},E_{\beta},E_{-\beta})-d\omega(JH_{\beta},E_{\alpha},E_{-\alpha})$$

$$-2iN_{\alpha,\beta}d\omega(E_{\alpha+\beta},E_{-\alpha},E_{-\beta})-2iN_{\alpha,-\beta}\epsilon_{\alpha-\beta}d\omega(E_{\alpha-\beta},E_{-\alpha},E_{\beta}).$$

Now we easily compute

$$d\omega(JH_{\alpha},E_{\beta},E_{-\beta})=-\omega(H_{\beta},JH_{\alpha})$$

and

$$d\omega(E_{\alpha+\beta},E_{-\alpha},E_{-\beta})=N_{\alpha,\beta}(\omega(E_{\alpha},E_{-\alpha})+\omega(E_{\beta},E_{-\beta})-\omega(E_{\alpha+\beta},E_{-\alpha-\beta})),$$

where we have used the fact that $N_{\alpha,\beta}=N_{\alpha+\beta,-\beta}=-N_{\alpha+\beta,-\alpha}$ (see [He], p. 172). Similarly,

$$d\omega(E_{\alpha-\beta},E_{-\alpha},E_{\beta})=N_{\alpha,-\beta}(-\omega(E_{\beta},E_{-\beta})+\omega(E_{\alpha},E_{-\alpha})-\omega(E_{\alpha-\beta},E_{\beta-\alpha})).$$

Summing up we have

$$dd^{c}\omega(E_{\alpha},E_{-\alpha},E_{\beta},E_{-\beta})=-2\omega(JH_{\alpha},H_{\beta})$$

$$-2iN_{\alpha,\beta}^{2}(\omega(E_{\alpha},E_{-\alpha})+\omega(E_{\beta},E_{-\beta})-\omega(E_{\alpha+\beta},E_{-\alpha-\beta}))$$

$$-2iN_{\alpha,-\beta}^{2}\epsilon_{\alpha-\beta}(-\omega(E_{\beta},E_{-\beta})+\omega(E_{\alpha},E_{-\alpha})-\omega(E_{\alpha-\beta},E_{\beta-\alpha})).$$

We now set $a_{\alpha}:h(E_{\alpha},E_{-\alpha})$. The pluriclosed condition implies that

$$0=-h(H_{\alpha},H_{\beta})-iN_{\alpha,\beta}^{2}(-ia_{\alpha}-ia_{\beta}+ia_{\alpha+\beta})$$

$$-iN_{\alpha,-\beta}^{2}\epsilon_{\alpha-\beta}(-i\epsilon_{\beta-\alpha}a_{\alpha-\beta}+ia_{\beta}-ia_{\alpha})$$

hence

(4.8) $$h(H_{\alpha},H_{\beta})=N_{\alpha,\beta}^{2}(a_{\alpha+\beta}-a_{\alpha}-a_{\beta})+N_{\alpha,-\beta}^{2}\epsilon_{\alpha-\beta}(\epsilon_{\alpha-\beta}a_{\alpha-\beta}+a_{\beta}-a_{\alpha})$$

We recall that

$$a_{\alpha}=h(E_{\alpha},E_{-\alpha})=-h(v_{\alpha},v_{\alpha})<0,\qquad\alpha\in R_{\mathfrak{k}}^{+},$$

$$a_{\alpha}=h(E_{\alpha},E_{-\alpha})=h(v_{\alpha},v_{\alpha})>0,\qquad\alpha\in R_{\mathfrak{p}}^{+},$$

$$h(H_{\alpha},H_{\beta})=-h(iH_{\alpha},iH_{\beta})\in\mathbb{R},\qquad h(H_{\alpha},H_{\alpha})<0.$$

Now, we recall that the existence of the complex structure J, which we constructed in section 3, relies on Lemma 3.3. In particular, when $\mathfrak{g}_{o}\neq\mathfrak{so}(1,2n)$, we have the existence of two simple roots $\psi_{1},\psi_{2}\in\Pi_{nc}$ with $\psi_{1}+\psi_{2}=\phi\in R_{\mathfrak{k}}$. The following lemma is elementary.

Lemma 4.2. *Either $\psi_{1}+2\psi_{2}\notin R$ or $\psi_{2}+2\psi_{1}\notin R$.*

Proof. As ψ_{1},ψ_{2} are simple, we have $\pm(\psi_{1}-\psi_{2})\notin R$. Now, $\psi_{i}+n\psi_{j}\in R$ if and only if $0\leq n\leq q_{j}$ with $q_{j}=-2\frac{\langle\psi_{1},\psi_{2}\rangle}{||\psi_{j}||^{2}}\in\mathbb{N}$ for $i\neq j$. It is then clear that $q_{1},q_{2}\geq2$ is impossible, as $\psi_{1}\neq\psi_{2}$ implies $q_{1}\cdot q_{2}<4$. $\qquad\square$

Suppone then that $\phi + \psi_1 = \psi_2 + 2\psi_1 \notin R$. We now apply (4.8) with two possible choices for α, β, namely:

(1) $\alpha = \psi_1, \beta = \psi_2$. Then

$$h(H_{\psi_1}, H_{\psi_2}) = N^2_{\psi_1, \psi_2}(a_\phi - a_{\psi_1} - a_{\psi_2}).$$

(2) $\alpha = \phi, \beta = \psi_1$. Then

$$h(H_\phi, H_{\psi_2}) = N^2_{\phi, -\psi_1}(a_{\psi_2} + a_{\psi_1} - a_\phi).$$

Subtracting (1) from (2) we get

$$h(H_{\psi_2}, H_{\psi_2}) = \left(N^2_{\phi, -\psi_1} + N^2_{\psi_1, \psi_2}\right)(a_{\psi_2} + a_{\psi_1} - a_\phi).$$

This is a contradiction, as $h(H_{\psi_2}, H_{\psi_2}) < 0$, while $a_{\psi_i} > 0$ for $i = 1, 2$ and $a_\phi < 0$.

We are left with the case $\mathfrak{g}_o = \mathfrak{so}(1, 2n)$, that we have dealt with separately in section 3. In this case the complex structure J is defined by the standard system of positive roots, namely $R^+ = \{\epsilon_i \pm \epsilon_j, \epsilon_i, \ 1 \le i \ne j \le n\}$. In particular $R^+_{\mathfrak{k}} = \{\epsilon_i \pm \epsilon_j\}_{i \ne j}$ and $R^+_{\mathfrak{p}} = \{\epsilon_i\}_{i=1,\dots,n}$. We now consider $\psi_i = \epsilon_i$, $i = 1, 2$, $\phi_1 = \psi_1 + \psi_2 \in R^+_{\mathfrak{k}}$ and $\phi_2 = \psi_1 - \psi_2 \in R^+_{\mathfrak{k}}$. We apply (4.8) in two different ways:

(1) $\alpha = \psi_1, \beta = \psi_2$. Then

$$h(H_{\psi_1}, H_{\psi_2}) = N^2_{\psi_1, \psi_2}(a_\phi - a_{\psi_1} - a_{\psi_2}) + N^2_{\psi_1, -\psi_2}(a_{\phi_2} + a_{\psi_2} - a_{\psi_1}).$$

(2) $\alpha = \phi_1, \beta = \psi_2$. Note that $\phi_1 + \psi_1 \notin R$. Then

$$h(H_{\phi_1}, H_{\psi_1}) = N^2_{\phi_1, -\psi_1}(a_{\psi_2} + a_{\psi_1} - a_{\phi_1}).$$

Therefore

$$h(H_{\psi_1}, H_{\psi_1}) = (N^2_{\phi_1, -\psi_1} + N^2_{\psi_1, \psi_2})(a_{\psi_2} + a_{\psi_1} - a_{\phi_1}) + N^2_{\psi_1, -\psi_2}(a_{\psi_1} - a_{\phi_2} - a_{\psi_2})$$

We now recall that, if $\gamma, \delta \in R$, then $N^2_{\gamma, \delta} = \frac{q(1-p)}{2}||\gamma||^2$, where $\delta + n\gamma$, $p \le n \le q$, is the γ-series containing δ (see [He], p.176). We then immediately see that $N^2_{\psi_1, \psi_2} = N^2_{\psi_1, -\psi_2}$ and noting furthermore that $N^2_{\phi_1, -\psi_1} = N^2_{\psi_1, \psi_2}$, we can write

$$h(H_{\psi_1}, H_{\psi_1}) = N^2_{\psi_1, \psi_2}(a_{\psi_2} + 3a_{\psi_1} - 2a_{\phi_1} - a_{\phi_2}),$$

giving the contradiction $h(H_{\psi_1}, H_{\psi_1}) > 0$.

5. GEOMETRIC PROPERTIES

In this section, we prove the following result, which may contribute to shed to some light on the geometry of the complex balanced manifolds we have constructed in the previous sections.

Theorem 5.1. *If* (M, J, h) *is a balanced* n-*dimensional manifold, where* $M = G_o/\Gamma$, J *is a standard invariant complex structure and* h *is a balanced Hermitian metric, then the metric* h *has vanishing Chern scalar curvature.*

Moreover $c_1(M) = 0$ *and the Kodaira dimension* $\kappa(M) = -\infty$.

We consider a standard complex structure J on a manifold $M = G_o/\Gamma$. We denote by D the Chern connection relative to a Hermitian metric h which is induced by an invariant metric on G_o, again denoted by h. We can moreover suppose that h is invariant by the right T-translations.

If $x \in \mathfrak{g}_o$ and if we still denote by x the induced left-invariant vector field on G_o, we consider $D_x \in \text{End}(\mathfrak{g}_o)$ the endomorphism of $\mathfrak{g}_o$ which assigns to every $y \in \mathfrak{g}_o$ the element

$D_x y$ corresponding to the left invariant vector field $D_x y$. Clearly $D_x \in \mathfrak{so}(\mathfrak{g}_o, h)$ and $[D_x, \mathrm{J}] = 0$. Moreover

(5.9) $$D_x y = [x, y]^{10}, \qquad \forall x \in \mathfrak{g}_o^{01}, \; y \in \mathfrak{g}_o^{10},$$

that follows from the fact that $T^{1,1} = 0$, where T is torsion of D.

If R denote the curvature, where $R_{xy} = [D_x, D_y] - D_{[x,y]}$, we are interested in the first Ricci tensor ρ given by

$$\rho(x, y) = -\frac{1}{2} \mathrm{Tr}(J \circ R_{xy}).$$

As the complex structure and the metric are both invariant under the adjoint action of the group $T = \exp(\mathfrak{t})$, we see that

$$\rho(\mathfrak{t}, E_\alpha) = 0, \; \forall \alpha \in R,$$

$$\rho(E_\alpha, E_\beta) \neq 0 \text{ implies } \beta = -\alpha, \; \alpha, \beta \in R.$$

Therefore we can compute

$$\rho(E_\alpha, E_{-\alpha}) = \frac{1}{2} \mathrm{Tr}(J D_{H_\alpha}).$$

Lemma 5.2. *For every $x \in \mathfrak{h}$*

$$D_x = \mathrm{ad}(x).$$

Proof. We use similar arguments as in [Po]. It will suffice to consider the case where $x \in \mathfrak{h}^{10}$; then for every $\alpha \in R^+$ we have

$$D_x E_{-\alpha} = [x, E_{-\alpha}]^{01} = [x, E_{-\alpha}], \quad D_x \mathfrak{h}^{01} = 0$$

by (5.9). Then if $\beta \in R^+$ we have

$$h(D_x E_\alpha, E_{-\beta}) = -h(E_\alpha, D_x E_{-\beta}) = -\beta(x) h(E_\alpha, E_{-\beta}) = 0 \qquad \text{if } \alpha \neq \beta,$$

so that $D_x E_\alpha = \alpha(x) E_\alpha = [x, E_\alpha] \pmod{\mathfrak{h}}$. As $h(D_x E_\alpha, \mathfrak{h}^{01}) = -h(E_\alpha, D_x \mathfrak{h}^{01}) = 0$, we conclude that

$$D_x E_\alpha = [x, E_\alpha].$$

Finally, $h(D_x \mathfrak{h}^{10}, \mathfrak{h}^{01}) = 0$ and $h(D_x \mathfrak{h}^{10}, E_{-\alpha}) = -h(\mathfrak{h}^{10}, [x, E_{-\alpha}]) = 0$, so that $D_x \mathfrak{h} = 0 = [x, \mathfrak{h}]$. $\square$

It follows that

$$\rho(\mathfrak{h}, \mathfrak{h}) = 0$$

and

$$\rho(E_\alpha, E_{-\alpha}) = \frac{1}{2} \left(2 \sum_{\beta \in R^+} i\beta(H_\alpha) \right) = B(H_\alpha, \delta),$$

where

$$\delta = \sum_{\beta \in R^+} iH_\beta \in \mathfrak{t}.$$

This means that for every $\alpha, \beta \in R$ we have

$$\rho(E_\alpha, E_\beta) = -B(E_\alpha, [\delta, E_\beta]) = B([E_\alpha, E_\beta], \delta)$$

and therefore for every $x, y \in \mathfrak{g}_o$

(5.10) $$\rho(x, y) = B([x, y], \delta).$$

This means that $\rho = d\phi$, where ϕ is the left-invariant 1-form that is given by $\phi(v) = B(v, \delta)$. Then clearly $c_1(M) = 0$.

We now show that the tensor powers $K_{\mathrm{M}}^{\otimes k}$ are holomorphically non trivial for every $m \geq 1$. Indeed, the metric h induces a Hermitian metric on the line bundles $K_{\mathrm{M}}^{\otimes m}$ with curvature form $m\rho$. If Ω is a nowhere vanishing holomorphic section of $K_{\mathrm{M}}^{\otimes k}$, then $m\rho = -i\partial\overline{\partial}\ln(||\Omega||^2)$. If we denote by $\widehat{}$ the result of the symmetrization process, which commutes with the operators ∂ and $\overline{\partial}$, we obtain on $\mathrm{G_o}$ that $\widehat{\rho} = -i\partial\overline{\partial}\widehat{\ln(||\Omega||^2)} = 0$. As ρ is invariant, $\widehat{\rho} = \rho = 0$ and we get a contradiction as $\delta \neq 0$.

We now compute the Chern scalar curvature s^{Ch} of the metric h using formula (5.10). We use the orthonormal frame $e_1, \ldots, e_{2n}$. Then

$$s^{Ch} = \sum_i \rho(Je_i, e_i) = -2 \sum_{\alpha \in R^+} \frac{1}{h_\alpha^2} \rho(v_\alpha, w_\alpha) =$$

$$= 2iB\left(\sum_{\alpha \in R_{\mathfrak{p}}^+} \frac{1}{h_\alpha^2} H_\alpha - \sum_{\alpha \in R_{\mathfrak{k}}^+} \frac{1}{h_\alpha^2} H_\alpha, \delta \right) = 0$$

if we consider the system of positive roots satisfying equation (2.3). The claim $\kappa(\mathrm{M}) = -\infty$ now follows from Thm. 1.4 in [Y].

Remark 5.3. Note that also for a compact group K endowed with an invariant complex structure we have $h^{n,0}(\mathrm{K}) = 0$, see [Pi], Prop. 3.7.

We finally remark here that the balanced condition implies that the two scalar curvatures that one can obtain tracing the Chern curvature tensor coincide (see [Ga3], p. 501).

REFERENCES

[AB1] L.Alessandrini, G. Bassanelli, *Modifications of compact balanced manifolds*, C. R. Acad. Sci.Paris Sér. I Math. **320**, 1517–1522 (1995)

[AB2] L. Alessandrini, G. Bassanelli, *Small deformations of a class of compact non-Kähler manifolds*, Proc. Amer. Math. Soc. **109**, 1059–1062 (1990)

[AG] E. Abbena, A. Grassi *Hermitian left invariant metrics on complex Lie groups and cosymplectic Hermitian manifolds*, Boll. Un. Mat. Ital. A **5**, 371–379 (1986)

[AI] B. Alexandrov, S. Ivanov, *Vanishing theorems on Hermitian manifolds*, Differential Geom. Appl.**14**, 251–265 (2001)

[AP] D. Alekseevsky, F. Podestà, *Homogeneous almost-Kähler manifolds and the Chern-Einstein equation*, Math.Z. **296**, 831–846 (2020)

[AU] D. Angella, L. Ugarte, *On small deformations of balanced manifolds*, Differential Geom. Appl. **54**, 464–474 (2017)

[AV] A. Andrada, R. Villacampa, *Abelian balanced Hermitian structures on unimodular Lie algebras*, Transform. Groups **21**, 903–927 (2016).

[Be] F. Belgun, *On the metric structure of non-Kähler complex surfaces*, Math.Ann. **317**, 1–40 (2000)

[Bo] A. Borel, *Compact Clifford-Klein forms of symmetric spaces*, Topology **2** (1963), 111–122

[BV] L. Bedulli, L. Vezzoni, *A parabolic flow of balanced metrics*, J. Reine Angew. Math. **723** , 79–99 (2017)

[Ch] B.Y. Chu, *Symplectic homogeneous spaces*, Trans. Amer. Math. Soc. **197**, 145–159 (1974)

[Fo] U. Fowdar, *Einstein metrics on bundles over hyperKähler manifolds*, arXiv:2105.04254v1 (2021)

[FV] A. Fino, L. Vezzoni, *On the existence of balanced and SKT metrics on nilmanifolds*, Proc. AMS **144**, 2455–2459 (2016)

[FeY] T. Fey, S.-T. Yau, *Invariant solutions to the Strominger system on complex Lie groups and their quotients*, Comm. Math. Phys. **338**, 1183–1195 (2015)

[FuY] J. Fu, S.-T. Yau, *A note on small deformations of balanced manifolds*, C.R. Math. Acad. Sci. Paris **349**, 793–796 (2011)

[FGV] A. Fino, G. Grantcharov, L. Vezzoni, *Astheno-Kähler and balanced structures on fibrations* Int. Math. Res. Not., **22**, 7093–7117 (2019)

[FLY] J. Fu, J. Li, S.-T. Yau, *Balanced metrics on non-Kähler Calabi Yau threefolds*, J.Differential Geom. **90**, 81–129 (2012)

[FPS] A. Fino, M. Parton, S. Salamon, *Families of strong KT structures in six dimensions*, Comment. Math. Helv. **79**, 317–340 (2004)

[Ga1] P. Gauduchon, *Fibré hermitiens à endomorphisme de Ricci non négativ*, Bulletin de la S.M.F. **105**, 113–140 (1977)

[Ga2] P. Gauduchon, *Le théorème de l'excentricité nulle*, C. R. Acad. Sci. Paris Sér. A-B **285**, A387–A390 (1977)

[Ga3] P. Gauduchon, *La 1-form de torsion d'une variété hermitienne compacte*, Math. Ann. **267**, 495–518 (1984)

[GF]   M. Garcia-Fernandez, *Lectures on the Strominger system*, Trav. Math. **24**, 7–61 (2016)

[Gr]   G. Grantcharov, *Geometry of compact complex manifolds with vanishing first Chern class*, Adv. Math. **226**, 3136–3159 (2011)

[Gra] A. Gray, *Some examples of almost hermitian manifolds*, Ill. J. Math. **10**, 353–366 (1966)

[He]   S. Helgason, Differential Geometry, Lie groups, and Symmetric spaces, Academic Press, Inc (1978)

[Mi]   M.L. Michelsohn, *On the existence of special metrics in complex geometry*, Acta Math. **149**, 261–295 (1982)

[Mo]   A. Morimoto, *Structures complexes invariantes sur les groupes de Lie semisimples*, C.R. **242**, 1101–1103 (1956)

[MTU] C. Medori, A. Tomassini, L. Ugarte, *On balanced Hermitian structures on Lie groups*, Geom. Dedicata **166**, 233–250 (2013)

[Pi]   H. Pittie, *The Dolbeault-cohomology ring of a compact, even-dimensional Lie group*, Proc.Indian Acad. Sci. **98**,117–152 (1988)

[Po]   F. Podestà, *Homogeneous Hermitian manifolds and special metrics* , Transform. Groups, **23**, 1129-–1147 (2018)

[Sam] H. Samelson, *A class of compact-analytic manifolds*, Portugaliae Math. **12** 129–132 (1953)

[Sas] T. Sasaki, *Classification of left invariant complex structures on* $SL(3, \mathbb{R})$, Kumamoto J. Sci. **15**, 59–72 (1982)

[Sn]   D. Snow, *Invariant complex structures on reductive Lie groups*, J. für di Reine und Angew. Math. **371**, 191–215 (1986)

[To]   A. Tomberg, *Twistor spaces of hypercomplex manifolds are balanced*, Adv. Math. **280**, 282–300 (2015)

[UV]   L. Ugarte, R. Villacampa, *Balanced Hermitian geometry on 6-dimensional nilmanifolds*, Forum Math. **27**, 1025–1070 (2015)

[Y]   X. Yang, *Scalar curvature on compact complex manifolds*, Trans. AMS **371**, 2073–2087 (2019)

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