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# Solution of a quadratic quaternion equation with mixed coefficients

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## Abstract

A comprehensive analysis of the morphology of the solution space for a special type of quadratic quaternion equation is presented. This equation, which arises in a surface construction problem, incorporates linear terms in a quaternion variable and its conjugate with right and left quaternion coefficients, while the quadratic term has a quaternion coefficient placed between the variable and its conjugate. It is proved that, for generic coefficients, the equation has two, one, or no solutions, but in certain special instances the solution set may comprise a circle or a 3-sphere in the quaternion space  $\mathbb{H}$ . The analysis yields solutions for each case, and intuitive interpretations of them in terms of the four-dimensional geometry of the quaternion space  $\mathbb{H}$ .

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# 1 Introduction

The real quaternions, discovered by Hamilton in 1843, form the first known algebra that involves a non-commutative product, denoted by  $\mathbb{H}$ . This property makes the solution of equations featuring quaternion coefficients and unknowns a much subtler and richer problem than in the case of real or complex numbers. The present study is devoted to developing a comprehensive solution for a novel type of quadratic equation, of the form

$$\mathcal{X}\mathcal{P}\mathcal{X}^* + \mathcal{X}\mathcal{Q} + \mathcal{R}\mathcal{X}^* = \mathcal{S}, \quad (1)$$

in which both the coefficients  $\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S}$  and the variable  $\mathcal{X}$  are quaternions ( $\mathcal{X}^*$  being the conjugate of  $\mathcal{X}$ ). Since we are only interested in the genuinely quadratic case of equation (1), we assume henceforth that  $\mathcal{P} \neq 0$ . We further emphasize that only the algebra  $\mathbb{H}$  of *real* quaternions is considered herein: the results do not apply, for example, to the complexified algebra  $\mathbb{H} \otimes \mathbb{C}$ .

A complete analysis of equation (1) reveals that, in addition to cases with (at most) two distinct *point* solutions, special values of the coefficients yield singly-infinite (*circular*) and triply-infinite (*3-sphere*) families of solutions. A proper identification and treatment of these degenerate cases is therefore an essential feature of any comprehensive solution procedure.

The motivation for studying equation (1) arises [8] from the construction of a surface patch  $\mathbf{x}(u, v)$  defined on  $(u, v) \in [0, 1]^2$  with prescribed boundary curves, such that the  $v = \text{constant}$  isoparametric curves are all polynomial *Pythagorean-hodograph* (PH) *curves* [5]. A brief review of the surface construction problem can be found in Section 3 below. At present, we highlight some unusual properties of equation (1) that distinguish it from prior studies of quadratic (and higher-order) quaternion equations.

The earliest investigations of quaternion equations are found in the papers of Niven [17, 18], Eilenberg and Niven [3], and Gordon and Motzkin [12]. A special case is that of equations of the form  $f(\mathcal{X}) = 0$ , where  $f(\mathcal{X})$  is an element of the algebra of quaternion polynomials  $\mathbb{H}[\mathcal{X}]$ . In this algebra, a convention is fixed for the relative position of the quaternion coefficients and powers of the quaternion variable in each monomial. Thus, one speaks of a left/right quaternion polynomial if the coefficients all appear to the left/right of the corresponding powers of the quaternion unknown.

Several studies [6, 13, 15] have been specifically concerned with (monic) quadratic quaternion equations specified by unilateral coefficients. In this case an essentially closed-form solution, requiring a determination of the

positive real root of a real cubic equation by Cardano's method, is possible (including a complete enumeration of special cases). Although no closed-form solution for the roots of higher-order polynomials is possible, there has been considerable progress in elucidating their fundamental nature, and in developing numerical methods to compute them [1, 2, 9, 10, 11, 14, 16, 19, 20, 21, 22, 23]. In particular, it has been shown that the set of roots of any polynomial  $f(\mathcal{X}) \in \mathbb{H}[\mathcal{X}]$  is a finite union of singletons and 2-spheres.

On the other hand, one may consider the class  $\mathbf{E}$  of functions  $f : \mathbb{H} \rightarrow \mathbb{H}$  defined as finite sums of monomials of type  $\mathcal{A}_0 \mathcal{X} \mathcal{A}_1 \cdots \mathcal{X} \mathcal{A}_n$  with  $\mathcal{A}_0, \dots, \mathcal{A}_n \in \mathbb{H}$  and  $n \in \mathbb{N}$ . This class is extremely large. In fact, it includes each of the functions that map any quaternion  $\mathcal{X} = x_0 + x_1 \mathbf{i} + x_2 \mathbf{j} + x_3 \mathbf{k}$  to one of its four real components  $x_0, \dots, x_3$  since, by direct computation,

$$\begin{aligned} x_0 &= \frac{1}{4}(\mathcal{X} - \mathbf{i} \mathcal{X} \mathbf{i} - \mathbf{j} \mathcal{X} \mathbf{j} - \mathbf{k} \mathcal{X} \mathbf{k}), \\ x_1 &= \frac{\mathbf{i}}{4}(-\mathcal{X} + \mathbf{i} \mathcal{X} \mathbf{i} - \mathbf{j} \mathcal{X} \mathbf{j} - \mathbf{k} \mathcal{X} \mathbf{k}), \\ x_2 &= \frac{\mathbf{j}}{4}(-\mathcal{X} - \mathbf{i} \mathcal{X} \mathbf{i} + \mathbf{j} \mathcal{X} \mathbf{j} - \mathbf{k} \mathcal{X} \mathbf{k}), \\ x_3 &= \frac{\mathbf{k}}{4}(-\mathcal{X} - \mathbf{i} \mathcal{X} \mathbf{i} - \mathbf{j} \mathcal{X} \mathbf{j} + \mathbf{k} \mathcal{X} \mathbf{k}). \end{aligned}$$

Consequently, the class of equations  $f(\mathcal{X}) = 0$  with  $f \in \mathbf{E}$  coincides with the class of all systems of four real polynomial equations in the four real variables  $x_0, \dots, x_3$ . Moreover,  $\mathbf{E}$  includes the map  $\mathcal{X} \mapsto \mathcal{X}^*$ , so equation (1) also has the form  $f(\mathcal{X}) = 0$  with  $f \in \mathbf{E}$ .

In the present paper, we show that the subclass of equations of the form (1) is special enough to allow a comprehensive solution. We show that generically there are two, one, or no solutions, which may be determined geometrically. The remaining special cases, in which an entire circle or 3-sphere of solutions may occur, are also studied in detail. The method adopted herein is to reduce all instances of equation (1) to two special cases, which may be treated as systems of real quadratic equations — as one may expect from the preceding discussion concerning the class of functions  $\mathbf{E}$ .

The plan for this paper is as follows. First, some preparatory notations and results are presented in Section 2, and the surface construction problem resulting in equation (1) is briefly reviewed in Section 3. A comprehensive solution procedure, including treatment of all special cases, is then developed in Section 4, and examples are presented to illustrate the different solution

morphologies that may arise, depending on the nature of the coefficients. Finally, Section 5 assesses the principal results of the present study.

## 2 Notations and preliminary results

Before proceeding to the study of equation (1), we fix some notations and briefly recall the definition and basic properties of the algebra of quaternions  $\mathbb{H}$ . It is the vector space  $\mathbb{R}^4$  endowed with a multiplicative operation through the following steps:

- denote the standard basis of  $\mathbb{R}^4$  as  $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ ;
- let the (left or right) multiplication by 1 have no effect on  $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ ;
- set  $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$  and  $\mathbf{i}\mathbf{j} = -\mathbf{j}\mathbf{i} = \mathbf{k}$ ,  $\mathbf{j}\mathbf{k} = -\mathbf{k}\mathbf{j} = \mathbf{i}$ ,  $\mathbf{k}\mathbf{i} = -\mathbf{i}\mathbf{k} = \mathbf{j}$ ;
- extend the multiplication to all quaternions  $\mathcal{X} = x_0\mathbf{1} + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$  (with  $x_0, \dots, x_3 \in \mathbb{R}$ ) in a bilinear fashion.

The resulting algebra is associative and non-commutative. It is also unitary, with identity element 1. In view of this, we henceforth write  $x_0$  instead of  $x_0\mathbf{1}$ , and we identify the axis  $\mathbb{R}\mathbf{1}$  with the real field  $\mathbb{R}$ .

Moreover,  $\mathbb{H}$  is a skew field — namely, every non-zero element admits a multiplicative inverse. In order to give a formula for the inverse, we introduce some further notations. For each quaternion  $\mathcal{X} = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \in \mathbb{H}$ , we denote by

$$\mathbf{x} = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$$

its *vector part*, and by  $x_0$  its *scalar part*. The notations  $\text{scal}(\mathcal{X}) = x_0$  and  $\text{vect}(\mathcal{X}) = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$  are also used. A quaternion whose scalar part vanishes is called a *pure vector* quaternion. To each quaternion  $\mathcal{X} = x_0 + \mathbf{x} = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ , one associates the *conjugate*

$$\mathcal{X}^* = x_0 - \mathbf{x} = x_0 - x_1\mathbf{i} - x_2\mathbf{j} - x_3\mathbf{k},$$

such that  $\mathcal{X}\mathcal{X}^* = x_0^2 + |\mathbf{x}|^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2$  coincides with the square of the Euclidean norm  $|\mathcal{X}|$  of  $\mathcal{X}$ , i.e.,

$$\mathcal{X}\mathcal{X}^* = \mathcal{X}^*\mathcal{X} = |\mathcal{X}|^2.$$

By analogy with the complex case, the (non-negative) real number  $|\mathcal{X}|$  is also called the *modulus* of  $\mathcal{X}$ . Clearly  $|\mathcal{X}|^2$  is a positive real number for all non-zero  $\mathcal{X} \in \mathbb{H}$ , and the preceding equation implies that  $\mathcal{X}^*|\mathcal{X}|^{-2} = |\mathcal{X}|^{-2}\mathcal{X}^*$  is the (left and right) inverse of  $\mathcal{X}$ , i.e.,

$$\mathcal{X}^{-1} = \mathcal{X}^*|\mathcal{X}|^{-2} = |\mathcal{X}|^{-2}\mathcal{X}^*.$$

The following properties will also prove useful in our computations.

- If we denote by  $\langle \cdot, \cdot \rangle$  and  $\times$ , respectively, the Euclidean scalar product and the vector product in  $\mathbb{R}^3$ , which we may identify with  $\text{vect}(\mathbb{H}) = \mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$ , then for all  $\mathbf{x}, \mathbf{y} \in \text{vect}(\mathbb{H})$ , we have

$$\mathbf{x}\mathbf{y} = -\langle \mathbf{x}, \mathbf{y} \rangle + \mathbf{x} \times \mathbf{y}.$$

- If we denote by  $\langle \cdot, \cdot \rangle$  the Euclidean scalar product in  $\mathbb{R}^4 = \mathbb{H}$ , then for all  $\mathcal{X}, \mathcal{Y} \in \mathbb{H}$  we have

$$\langle \mathcal{X}, \mathcal{Y} \rangle = \text{scal}(\mathcal{X}\mathcal{Y}^*).$$

Finally, we denote by  $\mathbb{S} = \{\mathcal{X} : \mathcal{X}^2 = -1\} = \{\mathbf{x} \in \text{vect}(\mathbb{H}) : |\mathbf{x}| = 1\}$  the 2-sphere of *pure vector quaternion units*. Then the following properties hold.

- For any fixed  $\mathbf{u} \in \mathbb{S}$  the subalgebra of  $\mathbb{H}$  generated by 1 and  $\mathbf{u}$ , namely  $\mathbb{R} + \mathbf{u}\mathbb{R}$ , is isomorphic to the complex plane and is denoted by  $\mathbb{C}_{\mathbf{u}}$ .
- $\mathbb{C}_{\mathbf{u}} = \mathbb{C}_{\mathbf{v}}$  if and only if  $\mathbf{u} = \pm\mathbf{v}$ ; otherwise  $\mathbb{C}_{\mathbf{u}} \cap \mathbb{C}_{\mathbf{v}} = \mathbb{R}$ .
- Every  $\mathcal{X} \in \mathbb{H} \setminus \mathbb{R}$  (i.e.,  $\mathcal{X} = x_0 + \mathbf{x}$  with  $\mathbf{x} \neq \mathbf{0}$ ) belongs to  $\mathbb{C}_{\hat{\mathbf{x}}}$ , where

$$\hat{\mathbf{x}} := \frac{\mathbf{x}}{|\mathbf{x}|}.$$

Every  $\mathcal{X} \in \mathbb{R}$  belongs to  $\mathbb{C}_{\mathbf{u}}$  for all  $\mathbf{u} \in \mathbb{S}$ . Therefore:

$$\mathbb{H} = \bigcup_{\mathbf{u} \in \mathbb{S}} \mathbb{C}_{\mathbf{u}}.$$

These properties will be used extensively in the proofs of our main results.

### 3 Surface construction problem

The motivation for the study of equation (1) arises [8] in the construction of a surface patch  $\mathbf{x}(u, v)$  for  $(u, v) \in [0, 1]^2$  with prescribed boundary curves, such that the  $v = \text{constant}$  isoparametric curves are Pythagorean–hodograph (PH) curves.<sup>2</sup> Such a surface is obtained by integrating the expression

$$\mathbf{x}_u(u, v) = \mathcal{A}(u, v) \mathbf{i} \mathcal{A}^*(u, v), \quad (2)$$

where  $\mathcal{A}(u, v)$  is a bivariate tensor–product quaternion polynomial

$$\mathcal{A}(u, v) = \sum_{i=0}^m \sum_{j=0}^n \mathcal{A}_{ij} b_i^m(u) b_j^n(v). \quad (3)$$

expressed in terms of the Bernstein basis

$$b_k^d(t) = \binom{d}{k} (1-t)^{d-k} t^k, \quad k = 0, \dots, d.$$

The simplest non–trivial solutions correspond to  $m = n = 2$ . Integrating (2) allows the surface to be expressed in terms of *Bézier control points*  $\mathbf{p}_{ij}$  [4] as

$$\mathbf{x}(u, v) = \sum_{i=0}^5 \sum_{j=0}^4 \mathbf{p}_{ij} b_i^5(u) b_j^4(v). \quad (4)$$

The points  $\mathbf{p}_{ij}$  for  $i > 0$  can be expressed in terms of the coefficients  $\mathcal{A}_{ij}$  for  $0 \leq i, j \leq 2$ , while  $\mathbf{p}_{0j}$  for  $j = 0, \dots, 4$  amount to free integration constants that specify  $\mathbf{x}(0, v)$ . Note, in particular, that  $\mathbf{p}_{i0}$  and  $\mathbf{p}_{i4}$  for  $i = 0, \dots, 5$  depend only on the coefficients  $\mathcal{A}_{i0}$  and  $\mathcal{A}_{i2}$  for  $i = 0, \dots, 2$ , respectively.

Now  $\mathcal{A}_{00}, \mathcal{A}_{10}, \mathcal{A}_{20}$  and  $\mathcal{A}_{02}, \mathcal{A}_{12}, \mathcal{A}_{22}$  may be used to fix the boundary PH curves  $\mathbf{x}(u, 0)$  and  $\mathbf{x}(u, 1)$  — i.e., to determine  $\mathbf{p}_{i0}$  and  $\mathbf{p}_{i4}$  for  $i = 0, \dots, 5$  — as Hermite interpolants [7]. The remaining coefficients  $\mathcal{A}_{01}, \mathcal{A}_{11}, \mathcal{A}_{21}$  must then be used to achieve desired positions for the three interior control points  $\mathbf{p}_{51}, \mathbf{p}_{52}, \mathbf{p}_{53}$  that specify  $\mathbf{x}(1, v)$ . This problem is under–determined, with three free parameters, but we assume that they are chosen *a priori*. One can formulate  $\mathbf{p}_{51}, \mathbf{p}_{52}, \mathbf{p}_{53}$  as quadratic expressions in  $\mathcal{A}_{01}, \mathcal{A}_{11}, \mathcal{A}_{21}$  and known quantities. Two of the resulting equations are linear in  $\mathcal{A}_{01}, \mathcal{A}_{21}$  and can be used to express these unknowns in terms of  $\mathcal{A}_{11}$ . Finally, substituting these expressions into the third equation yields the form (1), where  $\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S}$  are known, and we set  $\mathcal{X} = \mathcal{A}_{11}$ . Complete details may be found in [8].

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<sup>2</sup>The PH curves have rational unit tangents, polynomial arc length functions, and many other attractive properties — complete details may be found in [5].

## 4 The solution procedure

Different approaches to equation (1) are appropriate, according to whether or not the coefficient  $\mathcal{P}$  lies on the real axis  $\mathbb{R}$ . These cases will be analyzed in detail in Sections 4.1 and 4.2 below. The principal results of this analysis are sketched in the following theorem.

**Theorem 1.** *For generic  $\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S} \in \mathbb{H}$  with  $\mathcal{P} \neq 0$ , equation (1) has two, one, or no solutions. However, there are two non-generic instances of  $\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S}$  in which equation (1) admits infinitely-many solutions — namely, a 3-sphere of solutions or a circle of solutions.*

We now proceed to investigate individually the two cases  $\mathcal{P} \in \mathbb{R}$  and  $\mathcal{P} \notin \mathbb{R}$  of equation (1), resulting in Theorems 2 and 3 below — which immediately imply Theorem 1.

### 4.1 The case $\mathcal{P} \in \mathbb{R}$

We begin with the special case of equation (1) in which the coefficient  $\mathcal{P}$  lies on the real axis  $\mathbb{R}$ . Since we will have occasion to refer individually to the scalar and vector parts of the quaternion variable and coefficients, we recall from Section 2 their splitting denoted by  $\mathcal{X} = x_0 + \mathbf{x}$ ,  $\mathcal{P} = p_0 + \mathbf{p}$ ,  $\mathcal{Q} = q_0 + \mathbf{q}$ ,  $\mathcal{R} = r_0 + \mathbf{r}$ ,  $\mathcal{S} = s_0 + \mathbf{s}$ .

**Theorem 2.** *If  $\mathcal{P} = p_0 \in \mathbb{R}$  with  $p_0 \neq 0$ , equation (1) becomes*

$$p_0 |\mathcal{X}|^2 + \mathcal{X}\mathcal{Q} + \mathcal{R}\mathcal{X}^* = \mathcal{S}, \quad (5)$$

*whose solutions in  $\mathbb{H}$  are the points  $\mathcal{X} = \mathcal{Y} - (\mathcal{Q}^* + \mathcal{R})/2p_0$ , where  $\mathcal{Y}$  satisfies*

$$|\mathcal{Y}|^2 = \rho, \quad \rho := \frac{\text{scal}(\mathcal{S})}{p_0} + \frac{|\mathcal{Q} + \mathcal{R}^*|^2}{4p_0^2} \quad (6)$$

*and*

$$\text{vect}(\mathcal{Y}(\mathcal{Q} - \mathcal{R}^*)) = \text{vect}\left(\mathcal{S} + \frac{\mathcal{R}\mathcal{Q}}{p_0}\right). \quad (7)$$

*In particular, we may distinguish the following cases.*

1. *When  $\rho < 0$ , there is no solution.*

2. When  $\rho = 0$ , either  $\mathcal{X} = -(\mathcal{Q}^* + \mathcal{R})/2p_0$  is the unique solution or there is no solution, depending on whether or not

$$\mathcal{S} + \frac{\mathcal{R}\mathcal{Q}}{p_0} \in \mathbb{R}.$$

3. When  $\rho > 0$ , then

- (a) if  $\mathcal{Q} = \mathcal{R}^*$  the set of solutions is the 3-sphere (6) or the empty set, depending on whether or not  $\mathcal{S} \in \mathbb{R}$ ;
- (b) if  $\mathcal{Q} \neq \mathcal{R}^*$  there are two, one, or no solutions, namely, the points  $\mathcal{X} = \mathcal{Y} - (\mathcal{Q}^* + \mathcal{R})/2p_0$  where  $\mathcal{Y}$  is a point of intersection of the 3-sphere (6) with the affine line (7). Specifically,

$$\mathcal{Y} = \left[ \text{vect} \left( \mathcal{S} + \frac{\mathcal{R}\mathcal{Q}}{p_0} \right) \pm \sqrt{\Delta} \right] (\mathcal{Q} - \mathcal{R}^*)^{-1}$$

where  $\Delta$  is given by

$$\Delta = \rho |\mathcal{Q} - \mathcal{R}^*|^2 - \left| \text{vect} \left( \mathcal{S} + \frac{\mathcal{R}\mathcal{Q}}{p_0} \right) \right|^2.$$

A positive, zero, or negative  $\Delta$  identifies cases with two, one, or no solutions.

*Proof.* Setting  $\mathcal{Y} := \mathcal{X} + (\mathcal{Q}^* + \mathcal{R})/2p_0$  and substituting  $\mathcal{X} = \mathcal{Y} - (\mathcal{Q}^* + \mathcal{R})/2p_0$  into equation (5), we obtain

$$\begin{aligned} \mathcal{S} &= p_0 \left| \mathcal{Y} - \frac{\mathcal{Q}^* + \mathcal{R}}{2p_0} \right|^2 + \left( \mathcal{Y} - \frac{\mathcal{Q}^* + \mathcal{R}}{2p_0} \right) \mathcal{Q} + \mathcal{R} \left( \mathcal{Y} - \frac{\mathcal{Q}^* + \mathcal{R}}{2p_0} \right)^* \\ &= p_0 |\mathcal{Y}|^2 - \mathcal{Y} \frac{\mathcal{Q} + \mathcal{R}^*}{2} + \mathcal{Y}\mathcal{Q} - \frac{\mathcal{Q}^* + \mathcal{R}}{2} \mathcal{Y}^* + \mathcal{R}\mathcal{Y}^* \\ &\quad + \frac{|\mathcal{Q}^* + \mathcal{R}|^2}{4p_0} - \frac{\mathcal{Q}^* + \mathcal{R}}{2p_0} \mathcal{Q} - \mathcal{R} \frac{\mathcal{Q} + \mathcal{R}^*}{2p_0} \\ &= p_0 |\mathcal{Y}|^2 + \mathcal{Y} \frac{\mathcal{Q} - \mathcal{R}^*}{2} + \frac{-\mathcal{Q}^* + \mathcal{R}}{2} \mathcal{Y}^* \\ &\quad + \frac{|\mathcal{Q}^* + \mathcal{R}|^2}{4p_0} - \frac{|\mathcal{Q}^2| + 2\mathcal{R}\mathcal{Q} + |\mathcal{R}|^2}{2p_0} \\ &= p_0 |\mathcal{Y}|^2 + \mathcal{Y}\tilde{\mathcal{Q}} - \tilde{\mathcal{Q}}^* \mathcal{Y}^* - \frac{|\mathcal{Q}^* + \mathcal{R}|^2}{4p_0} + \frac{-\mathcal{R}\mathcal{Q} + \mathcal{Q}^* \mathcal{R}^*}{2p_0} \end{aligned}$$

where  $\tilde{\mathcal{Q}} = \frac{1}{2}(\mathcal{Q} - \mathcal{R}^*)$ . Hence, setting

$$\tilde{\mathcal{S}} = \mathcal{S} + \frac{|\mathcal{Q}^* + \mathcal{R}|^2}{4p_0} + \frac{\mathcal{R}\mathcal{Q} - \mathcal{Q}^*\mathcal{R}^*}{2p_0},$$

we have

$$p_0|\mathcal{Y}|^2 + \mathcal{Y}\tilde{\mathcal{Q}} - \tilde{\mathcal{Q}}^*\mathcal{Y}^* = \tilde{\mathcal{S}}. \quad (8)$$

Writing  $\tilde{\mathcal{S}} = \tilde{s}_0 + \tilde{\mathfrak{s}}$ , equation (8) is equivalent to the system

$$p_0|\mathcal{Y}|^2 = \tilde{s}_0, \quad \mathcal{Y}\tilde{\mathcal{Q}} - \tilde{\mathcal{Q}}^*\mathcal{Y}^* = \tilde{\mathfrak{s}}. \quad (9)$$

The solutions to this system are those points that simultaneously satisfy  $|\mathcal{Y}|^2 = \rho$ , with

$$\rho = \frac{\tilde{s}_0}{p_0} = \frac{s_0}{p_0} + \frac{|\mathcal{Q}^* + \mathcal{R}|^2}{4p_0^2},$$

and

$$\text{vect}(\mathcal{Y}(\mathcal{Q} - \mathcal{R}^*)) = 2 \text{vect}(\mathcal{Y}\tilde{\mathcal{Q}}) = \tilde{\mathfrak{s}} = \text{vect}\left(\mathcal{S} + \frac{\mathcal{R}\mathcal{Q}}{p_0}\right). \quad (10)$$

If  $\rho < 0$ , then there is no solution in  $\mathbb{H}$ , whence case 1. If  $\rho = 0$ , then 0 is the only solution to  $|\mathcal{Y}|^2 = \rho$ . It is also a solution of equation (10) if, and only if,  $\tilde{\mathfrak{s}} = \mathbf{0}$ . This verifies case 2.

Consider now case 3, in which  $\rho > 0$ . If  $\mathcal{Q} = \mathcal{R}^*$ , the solution to (10) is either the entire space  $\mathbb{H}$  or the empty set, according to whether or not  $\mathfrak{s} = 0$ . On the other hand, the solutions to (10) comprise an affine line if  $\mathcal{Q} \neq \mathcal{R}^*$ . In this case, the solutions to (9) are the points

$$\mathcal{Y} = \left(\xi + \mathcal{S} + \frac{\mathcal{R}\mathcal{Q}}{p_0}\right) (\mathcal{Q} - \mathcal{R}^*)^{-1},$$

where  $\xi \in \mathbb{R}$  is a root of the real quadratic equation

$$\left|\xi + \mathcal{S} + \frac{\mathcal{R}\mathcal{Q}}{p_0}\right|^2 = \rho|\mathcal{Q} - \mathcal{R}^*|^2.$$

A positive, zero, or negative discriminant leads to two, one, or no solutions.

Finally, the translation  $\mathcal{X} = \mathcal{Y} - (\mathcal{Q}^* + \mathcal{R})/2p_0$  yields the solutions of equation (1) in terms of those of (9).  $\square$

We now describe some examples that serve to illustrate all the possible types of solution sets to equation (1) covered by Theorem 2. We begin with two examples that illustrate cases 1 and 2 of Theorem 2.

**Example 1.** If  $\mathcal{P} = 1 = -\mathcal{S}$  and  $\mathcal{Q} = \mathcal{R} = 0$ , equation (1) becomes

$$|\mathcal{X}|^2 + 1 = 0,$$

which clearly has no solution in  $\mathbb{H}$ .

**Example 2.** If  $\mathcal{P} = \mathcal{Q} = \mathcal{R} = 1 = -\mathcal{S}$ , equation (1) becomes

$$|\mathcal{X}|^2 + \mathcal{X} + \mathcal{X}^* + 1 = 0,$$

which is equivalent to  $|\mathcal{X} + 1|^2 = 0$ . Thus,  $\mathcal{X} = -1$  is the only solution in  $\mathbb{H}$ .

The following two examples are instances of case 3(a) in Theorem 2.

**Example 3.** If  $\mathcal{P} = \mathcal{Q} = \mathcal{R} = 1$  and  $\mathcal{S} = 0$ , equation (1) becomes

$$|\mathcal{X}|^2 + \mathcal{X} + \mathcal{X}^* = 0,$$

which reduces to  $|\mathcal{X} + 1|^2 = 1$ . The set of solutions in  $\mathbb{H}$  is therefore the 3-sphere of radius 1 centered at  $-1$ .

**Example 4.** If  $\mathcal{P} = \mathcal{Q} = \mathcal{R} = 1$  and  $\mathcal{S} = \mathbf{i}$ , equation (1) becomes

$$|\mathcal{X}|^2 + \mathcal{X} + \mathcal{X}^* = \mathbf{i}.$$

Since this cannot be satisfied by any  $\mathcal{X}$ , there are no solutions in  $\mathbb{H}$ .

Next is a family of examples corresponding to case 3(b) in Theorem 2.

**Example 5.** If  $\mathcal{P} = \mathcal{Q} = 1 = -\mathcal{R}$  and  $\mathcal{S} = 1 + \mathbf{s}$ , equation (1) becomes

$$|\mathcal{X}|^2 + \mathcal{X} - \mathcal{X}^* = 1 + \mathbf{s}.$$

The solutions correspond to the intersections of the 3-sphere  $|\mathcal{X}|^2 = 1$  with the line  $\mathcal{X} - \mathcal{X}^* = \mathbf{s}$ . If  $\mathbf{s} = 2 \sin \theta \hat{\mathbf{s}}$ , there are two or one intersections, namely  $\pm \cos(\theta) + \sin(\theta) \hat{\mathbf{s}}$ . If  $|\mathbf{s}| > 2$ , on the other hand, there is no intersection.

## 4.2 The case $\mathcal{P} \notin \mathbb{R}$

Consider now equation (1) when  $\mathcal{P}$  is not a real number. In this case, we shall make use of a known result (see Section 2.1 of [7]) concerning the instance of (1) in which  $\mathcal{P}, \mathcal{S}$  are pure imaginary quaternions  $\mathbf{p}, \mathbf{s}$  and  $\mathcal{Q} = \mathcal{R} = 0$ . Recall from Section 2 that  $\hat{\mathbf{p}} := \mathbf{p}/|\mathbf{p}|$  and  $\hat{\mathbf{s}} := \mathbf{s}/|\mathbf{s}|$  are unit vectors in the direction of  $\mathbf{p}$  and  $\mathbf{s}$ , and  $\mathbb{C}_{\hat{\mathbf{p}}}$  is the 2-plane spanned by 1 and  $\hat{\mathbf{p}}$ .

**Definition 1.** Let  $\mathbf{p}, \mathbf{s}$  be non-zero pure vector quaternions. If  $\hat{\mathbf{s}} \neq -\hat{\mathbf{p}}$ , the 2-plane  $\Pi_{\mathbf{p},\mathbf{s}}$  is defined by

$$\Pi_{\mathbf{p},\mathbf{s}} := \mathbf{u}_{\mathbf{p},\mathbf{s}} \mathbb{C}_{\hat{\mathbf{p}}}, \quad \text{with } \mathbf{u}_{\mathbf{p},\mathbf{s}} := \frac{\hat{\mathbf{p}} + \hat{\mathbf{s}}}{|\hat{\mathbf{p}} + \hat{\mathbf{s}}|}.$$

On the other hand, if  $\hat{\mathbf{s}} = -\hat{\mathbf{p}}$ , then  $\Pi_{\mathbf{p},\mathbf{s}}$  is defined to be the 2-plane through the origin that is orthogonal to  $\hat{\mathbf{p}}$  in the 3-space  $\text{vect}(\mathbb{H})$ .

Note that the 2-plane  $\Pi_{\mathbf{p},\mathbf{s}}$  bisects the angle between  $\hat{\mathbf{p}}$  and  $\hat{\mathbf{s}}$ .

**Lemma 1.** Let  $\mathbf{p}, \mathbf{s}$  be pure vector quaternions with  $\mathbf{p} \neq \mathbf{0}$ . Then if  $\mathbf{s} \neq \mathbf{0}$ , the 2-plane  $\Pi_{\mathbf{p},\mathbf{s}}$  is the space of solutions  $\mathcal{V}$  to equation

$$\mathcal{V} \hat{\mathbf{p}} - \hat{\mathbf{s}} \mathcal{V} = 0. \tag{11}$$

Moreover, the set of solutions to the quaternion equation

$$\mathcal{X} \mathbf{p} \mathcal{X}^* = \mathbf{s} \tag{12}$$

is the circle  $\mathcal{C}_{\mathbf{p},\mathbf{s}}$  in the 2-plane  $\Pi_{\mathbf{p},\mathbf{s}}$  with center 0 and radius  $\tau = \sqrt{|\mathbf{s}|/|\mathbf{p}|}$ . On the other hand, when  $\mathbf{s} = 0$ , it is simply  $\mathcal{C}_{\mathbf{p},0} := \{0\}$ .

Although this result is already known [7], a proof is included below to make the presentation self-contained.

*Proof.* Clearly, if  $\mathbf{s} = 0$ , the unique solution is  $\mathcal{X} = 0$ . We therefore consider only the case  $\mathbf{s} \neq 0$ . Equation (12) then implies that

$$|\mathcal{X}|^2 = \frac{|\mathbf{s}|}{|\mathbf{p}|}.$$

Therefore, any solution must have the form  $\mathcal{X} = \tau \mathcal{U}$  with  $\tau^2 = |\mathbf{s}|/|\mathbf{p}|$  and  $|\mathcal{U}| = 1$ . Substituting into (12) then gives  $\mathcal{U} \hat{\mathbf{p}} \mathcal{U}^* = \hat{\mathbf{s}}$ , whose solution set is

the intersection of the 3–sphere  $|\mathcal{U}| = 1$  with the set of solutions  $\mathcal{V}$  to (11). To verify Lemma 1, we show that this set of solutions is precisely  $\Pi_{\mathbf{p},\mathbf{s}}$ .

If  $\hat{\mathbf{s}} \neq -\hat{\mathbf{p}}$  it is evident by inspection that  $\mathcal{V} = \mathbf{u}_{\mathbf{p},\mathbf{s}}$  is a solution of (11). Moreover, if  $\mathcal{V}$  satisfies (11), the product of  $\mathcal{V}$  with any element of  $\mathbb{C}_{\hat{\mathbf{p}}}$  also satisfies it. Thus,  $\Pi_{\mathbf{p},\mathbf{s}} = \mathbf{u}_{\mathbf{p},\mathbf{s}} \mathbb{C}_{\hat{\mathbf{p}}}$  belongs to the space of solutions of (11). We will now prove that the dimension of this space cannot exceed two. Indeed, an orthonormal transformation gives  $\hat{\mathbf{p}} = \mathbf{i}$  and  $\hat{\mathbf{s}} = \mathbf{i} \cos \varphi + \mathbf{j} \sin \varphi$  for some  $\varphi \in \mathbb{R}$  and the rank of the linear map  $\mathcal{V} \mapsto \mathcal{V} \mathbf{i} - (\mathbf{i} \cos \varphi + \mathbf{j} \sin \varphi) \mathcal{V}$  cannot be less than 2 since, by inspection, its values at  $\mathcal{V} = \mathbf{i}$  and  $\mathcal{V} = \mathbf{k}$  are linearly independent. Hence, the solution space of (11) coincides precisely with  $\Pi_{\mathbf{p},\mathbf{s}}$ .

Otherwise, if  $\hat{\mathbf{s}} = -\hat{\mathbf{p}}$ , the solutions  $\mathcal{V} = v_0 + \mathbf{v}$  must satisfy

$$0 = \mathcal{V} \hat{\mathbf{p}} + \hat{\mathbf{p}} \mathcal{V} = 2v_0 \hat{\mathbf{p}} + \mathbf{v} \hat{\mathbf{p}} + \hat{\mathbf{p}} \mathbf{v} = 2v_0 \hat{\mathbf{p}} - 2 \langle \mathbf{v}, \hat{\mathbf{p}} \rangle$$

i.e., we must have  $v_0 = 0$  and  $\mathbf{v} \perp \hat{\mathbf{p}}$ . This is exactly the definition of  $\Pi_{\mathbf{p},\mathbf{s}}$  in the case  $\hat{\mathbf{s}} = -\hat{\mathbf{p}}$  under consideration.  $\square$

**Remark 1.** When  $\hat{\mathbf{s}} \neq -\hat{\mathbf{p}}$ , the circle  $\mathcal{C}_{\mathbf{p},\mathbf{s}}$  can be explicitly described in terms of  $\tau = \sqrt{|\mathbf{s}|/|\mathbf{p}|}$ ,  $\mathbf{u}_{\mathbf{p},\mathbf{s}}$ , and a parameter  $\phi$  as

$$\mathcal{C}_{\mathbf{p},\mathbf{s}} = \{ \tau \mathbf{u}_{\mathbf{p},\mathbf{s}} (\cos \phi + \sin \phi \hat{\mathbf{p}}) \mid \phi \in [0, 2\pi) \}. \quad (13)$$

Using again the notations  $\mathcal{X} = x_0 + \mathbf{x}$ ,  $\mathcal{P} = p_0 + \mathbf{p}$ ,  $\mathcal{Q} = q_0 + \mathbf{q}$ ,  $\mathcal{R} = r_0 + \mathbf{r}$ ,  $\mathcal{S} = s_0 + \mathbf{s}$  for the splittings of the variable and coefficients into scalar and vector parts, we are now ready to complete the study of the general equation.

**Theorem 3.** Defining  $\tilde{\mathcal{R}}$  and  $\tilde{\mathcal{S}} = (\tilde{s}_0, \tilde{\mathbf{s}})$  by

$$\begin{aligned} \tilde{\mathcal{R}} &:= (\mathcal{Q}^* \mathcal{P} - \mathcal{R} \mathcal{P}^*) (\mathcal{P} - \mathcal{P}^*)^{-1}, \\ \tilde{\mathcal{S}} &:= \mathcal{S} + \frac{\mathcal{Q}^* \mathcal{P} \mathcal{R}^* + \mathcal{R} \mathcal{P}^* \mathcal{Q}}{|\mathcal{P} - \mathcal{P}^*|^2} + \frac{\mathcal{R} (\mathcal{P}^* - \mathcal{P}) \mathcal{Q}}{|\mathcal{P} - \mathcal{P}^*|^2} - \frac{\mathcal{Q}^* \mathcal{P}^* \mathcal{Q}}{|\mathcal{P} - \mathcal{P}^*|^2} - \frac{\mathcal{R} \mathcal{P}^* \mathcal{R}^*}{|\mathcal{P} - \mathcal{P}^*|^2}, \end{aligned}$$

the solutions in  $\mathbb{H}$  to equation (1) with  $\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S} \in \mathbb{H}$  and  $\mathcal{P} \notin \mathbb{R}$  are given by

$$\mathcal{X} = \mathcal{Z} - (\mathcal{R} - \mathcal{Q}^*) (\mathcal{P} - \mathcal{P}^*)^{-1},$$

where  $\mathcal{Z} \in \mathcal{C}_{\mathbf{p},\tilde{\mathbf{s}}}$  satisfies

$$\langle \mathcal{Z}, \tilde{\mathcal{R}} \rangle = \frac{\tilde{s}_0 |\mathbf{p}| - p_0 |\tilde{\mathbf{s}}|}{2|\mathbf{p}|}. \quad (14)$$

Specifically, the following cases may be distinguished.

1. If  $\tilde{\mathcal{S}} = 0$ , then  $\mathcal{X} = -(\mathcal{R} - \mathcal{Q}^*)(\mathcal{P} - \mathcal{P}^*)^{-1}$  is the unique solution.
2. If  $\tilde{\mathcal{S}}$  is a non-zero real number, there is no solution.
3. If  $\tilde{\mathcal{S}}$  is not a real number, the following cases arise.

(a) If  $\tilde{\mathcal{R}} \perp \Pi_{\mathbf{p}, \tilde{\mathbf{s}}}$  the set of solutions  $\mathcal{X}$  is either the circle

$$\mathcal{C}_{\mathbf{p}, \tilde{\mathbf{s}}} - (\mathcal{R} - \mathcal{Q}^*)(\mathcal{P} - \mathcal{P}^*)^{-1}$$

or the empty set, depending on whether or not  $\tilde{s}_0 |\mathbf{p}| = p_0 |\tilde{\mathbf{s}}|$ .

(b) If  $\tilde{\mathcal{R}} \not\perp \Pi_{\mathbf{p}, \tilde{\mathbf{s}}}$  there are two, one or no solutions, namely, the points  $\mathcal{X} = \mathcal{Z} - (\mathcal{R} - \mathcal{Q}^*)(\mathcal{P} - \mathcal{P}^*)^{-1}$  where  $\mathcal{Z}$  is an intersection point of the circle  $\mathcal{C}_{\mathbf{p}, \tilde{\mathbf{s}}}$  with the coplanar affine line specified by intersecting  $\Pi_{\mathbf{p}, \tilde{\mathbf{s}}}$  with the hyperplane (14).

*Proof.* Upon setting  $\mathcal{Z} := \mathcal{X} + (\mathcal{R} - \mathcal{Q}^*)(\mathcal{P} - \mathcal{P}^*)^{-1}$  and substituting  $\mathcal{X} = \mathcal{Z} - (\mathcal{R} - \mathcal{Q}^*)(\mathcal{P} - \mathcal{P}^*)^{-1}$  into equation (1), we obtain

$$\begin{aligned}
\mathcal{S} &= \mathcal{Z}\mathcal{P}\mathcal{Z}^* - \mathcal{Z}\mathcal{P}(\mathcal{P}^* - \mathcal{P})^{-1}(\mathcal{R}^* - \mathcal{Q}) + \mathcal{Z}\mathcal{Q} \\
&\quad - (\mathcal{R} - \mathcal{Q}^*)(\mathcal{P} - \mathcal{P}^*)^{-1}\mathcal{P}\mathcal{Z}^* + \mathcal{R}\mathcal{Z}^* \\
&\quad + (\mathcal{R} - \mathcal{Q}^*)(\mathcal{P} - \mathcal{P}^*)^{-1}\mathcal{P}(\mathcal{P}^* - \mathcal{P})^{-1}(\mathcal{R}^* - \mathcal{Q}) \\
&\quad - (\mathcal{R} - \mathcal{Q}^*)(\mathcal{P} - \mathcal{P}^*)^{-1}\mathcal{Q} - \mathcal{R}(\mathcal{P}^* - \mathcal{P})^{-1}(\mathcal{R}^* - \mathcal{Q}) \\
&= \mathcal{Z}\mathcal{P}\mathcal{Z}^* + \mathcal{Z}(\mathcal{P}^* - \mathcal{P})^{-1}[-\mathcal{P}\mathcal{R}^* + \mathcal{P}\mathcal{Q} + (\mathcal{P}^* - \mathcal{P})\mathcal{Q}] \\
&\quad + [\mathcal{Q}^*\mathcal{P} + \mathcal{R}(-\mathcal{P} + \mathcal{P} - \mathcal{P}^*)](\mathcal{P} - \mathcal{P}^*)^{-1}\mathcal{Z}^* \\
&\quad + |\mathcal{P} - \mathcal{P}^*|^{-2}(\mathcal{R} - \mathcal{Q}^*)\mathcal{P}(\mathcal{R}^* - \mathcal{Q}) \\
&\quad - |\mathcal{P} - \mathcal{P}^*|^{-2}(\mathcal{R} - \mathcal{Q}^*)(\mathcal{P}^* - \mathcal{P})\mathcal{Q} \\
&\quad - |\mathcal{P} - \mathcal{P}^*|^{-2}\mathcal{R}(\mathcal{P} - \mathcal{P}^*)(\mathcal{R}^* - \mathcal{Q}) \\
&= \mathcal{Z}\mathcal{P}\mathcal{Z}^* + \mathcal{Z}(\mathcal{P}^* - \mathcal{P})^{-1}(-\mathcal{P}\mathcal{R}^* + \mathcal{P}^*\mathcal{Q}) \\
&\quad + (\mathcal{Q}^*\mathcal{P} - \mathcal{R}\mathcal{P}^*)(\mathcal{P} - \mathcal{P}^*)^{-1}\mathcal{Z}^* \\
&\quad + |\mathcal{P} - \mathcal{P}^*|^{-2}(\mathcal{R} - \mathcal{Q}^*)(\mathcal{P}\mathcal{R}^* - \mathcal{P}^*\mathcal{Q}) \\
&\quad + |\mathcal{P} - \mathcal{P}^*|^{-2}\mathcal{R}(-\mathcal{P}\mathcal{R}^* + \mathcal{P}\mathcal{Q} + \mathcal{P}^*\mathcal{R}^* - \mathcal{P}^*\mathcal{Q}) \\
&= \mathcal{Z}\mathcal{P}\mathcal{Z}^* + \mathcal{Z}\tilde{\mathcal{R}}^* + \tilde{\mathcal{R}}\mathcal{Z}^* \\
&\quad + |\mathcal{P} - \mathcal{P}^*|^{-2}[-\mathcal{Q}^*(\mathcal{P}\mathcal{R}^* - \mathcal{P}^*\mathcal{Q}) + \mathcal{R}(-2\mathcal{P}^*\mathcal{Q} + \mathcal{P}\mathcal{Q} + \mathcal{P}^*\mathcal{R}^*)],
\end{aligned}$$

where  $\tilde{\mathcal{R}} = (\mathcal{Q}^*\mathcal{P} - \mathcal{R}\mathcal{P}^*)(\mathcal{P} - \mathcal{P}^*)^{-1}$ . This gives

$$\mathcal{Z}\mathcal{P}\mathcal{Z}^* + \mathcal{Z}\tilde{\mathcal{R}}^* + \tilde{\mathcal{R}}\mathcal{Z}^* = \tilde{\mathcal{S}} \tag{15}$$

where

$$\begin{aligned}\tilde{\mathcal{S}} &= \mathcal{S} + \frac{\mathcal{Q}^*(\mathcal{P}\mathcal{R}^* - \mathcal{P}^*\mathcal{Q}) + \mathcal{R}(2\mathcal{P}^*\mathcal{Q} - \mathcal{P}\mathcal{Q} - \mathcal{P}^*\mathcal{R}^*)}{|\mathcal{P} - \mathcal{P}^*|^2} \\ &= \mathcal{S} + \frac{\mathcal{Q}^*\mathcal{P}\mathcal{R}^* + \mathcal{R}\mathcal{P}^*\mathcal{Q}}{|\mathcal{P} - \mathcal{P}^*|^2} + \frac{\mathcal{R}(\mathcal{P}^* - \mathcal{P})\mathcal{Q}}{|\mathcal{P} - \mathcal{P}^*|^2} - \frac{\mathcal{Q}^*\mathcal{P}^*\mathcal{Q}}{|\mathcal{P} - \mathcal{P}^*|^2} - \frac{\mathcal{R}\mathcal{P}^*\mathcal{R}^*}{|\mathcal{P} - \mathcal{P}^*|^2}.\end{aligned}$$

Equation (15) is equivalent to the system

$$p_0 |\mathcal{Z}|^2 + \mathcal{Z}\tilde{\mathcal{R}}^* + \tilde{\mathcal{R}}\mathcal{Z}^* = \tilde{s}_0, \quad \mathcal{Z}\mathbf{p}\mathcal{Z}^* = \tilde{s}. \quad (16)$$

If  $\tilde{s} = 0$ , the only solution to the second of these equations is  $\mathcal{Z} = 0$ . This also satisfies the first equation if and only if  $\tilde{s}_0 = 0$ . We have thus established cases 1 and 2. For case 3, with  $\tilde{s} \neq 0$ , equations (16) are equivalent to

$$\mathcal{Z}\tilde{\mathcal{R}}^* + \tilde{\mathcal{R}}\mathcal{Z}^* = \frac{\tilde{s}_0|\mathbf{p}| - p_0|\tilde{s}|}{|\mathbf{p}|}, \quad \mathcal{Z}\mathbf{p}\mathcal{Z}^* = \tilde{s}. \quad (17)$$

By Lemma 1, the solutions of the latter equation comprise the circle  $\mathcal{C}_{\mathbf{p},\tilde{s}}$  of radius  $\sqrt{|\tilde{s}|/|\mathbf{p}|}$  and center 0 in the 2-plane  $\Pi_{\mathbf{p},\tilde{s}}$ . The solutions to the system (17) are determined by intersecting this circle with the set  $\mathcal{H}$  of solutions to

$$\langle \mathcal{Z}, \tilde{\mathcal{R}} \rangle = \frac{\tilde{s}_0|\mathbf{p}| - p_0|\tilde{s}|}{2|\mathbf{p}|}.$$

For any  $\mathcal{Z}_0 \in \mathcal{H}$ , this equation is equivalent to

$$\langle \mathcal{Z} - \mathcal{Z}_0, \tilde{\mathcal{R}} \rangle = 0.$$

Hence, the set  $\mathcal{H}$  is an affine space orthogonal to  $\tilde{\mathcal{R}}$ . Case 3(a) is verified by observing that, since  $\mathcal{C}_{\mathbf{p},\tilde{s}}$  is a circle centered at 0 in the plane  $\Pi_{\mathbf{p},\tilde{s}}$ , the following are equivalent:

- $\mathcal{C}_{\mathbf{p},\tilde{s}}$  is the set of solutions to system (17);
- $\Pi_{\mathbf{p},\tilde{s}} \subset \mathcal{H}$ ;
- $0 \in \mathcal{H}$  and the equality  $\langle \mathcal{Z}, \tilde{\mathcal{R}} \rangle = 0$  holds for all  $\mathcal{Z} \in \Pi_{\mathbf{p},\tilde{s}}$ ;
- $\tilde{s}_0|\mathbf{p}| = p_0|\tilde{s}|$  and  $\tilde{\mathcal{R}} \perp \Pi_{\mathbf{p},\tilde{s}}$ .

Finally, consider case 3(b) with  $\tilde{\mathcal{R}} \not\perp \Pi_{\mathbf{p},\tilde{\mathbf{s}}}$  (whence  $\tilde{\mathcal{R}} \neq 0$ ). Then  $\mathcal{H}$  is a hyperplane that, by Lemma 1, intersects  $\Pi_{\mathbf{p},\tilde{\mathbf{s}}} \supset \mathcal{C}_{\mathbf{p},\tilde{\mathbf{s}}}$  in the affine line defined by the equations

$$\mathcal{Z}\hat{\mathbf{p}} - \hat{\mathbf{s}}\mathcal{Z} = 0, \quad \mathcal{Z}\tilde{\mathcal{R}}^* + \tilde{\mathcal{R}}\mathcal{Z}^* = \frac{\tilde{s}_0|\mathbf{p}| - p_0|\tilde{\mathbf{s}}|}{|\mathbf{p}|}.$$

Finally, the translation  $\mathcal{X} = \mathcal{Z} - (\mathcal{R} - \mathcal{Q}^*)(\mathcal{P} - \mathcal{P}^*)^{-1}$  gives the solutions of quation (1) in terms of those of (15).  $\square$

Cases 3(a) and 3(b) of Theorem 3 are characterized by whether or not  $\tilde{\mathcal{R}}$  is orthogonal to the plane  $\Pi_{\mathbf{p},\tilde{\mathbf{s}}}$ . In an algorithm, this could be determined by checking to see if the scalar product of  $\tilde{\mathcal{R}}$  with any two linearly independent quaternions in  $\Pi_{\mathbf{p},\tilde{\mathbf{s}}}$  vanishes. Note also that the computation of the points  $\mathcal{Z}$  in case 3(b) can be performed by: (a) writing a parameterization  $\xi \mapsto \mathcal{A}\xi + \mathcal{B}$  of the affine line specified by intersecting  $\Pi_{\mathbf{p},\tilde{\mathbf{s}}}$  with the hyperplane (14); and (b) solving the real quadratic equation  $|\mathcal{A}\xi + \mathcal{B}|^2 = |\tilde{\mathbf{s}}|^2/|\tilde{\mathbf{p}}|^2$ .

We now illustrate cases 1. and 2. of Theorem 3 by the following examples.

**Example 6.** If  $\mathcal{P} = \mathbf{i}$ ,  $\mathcal{Q} = \mathcal{R} = 1$ , and  $\mathcal{S} = 0$ , equation (1) becomes

$$\mathcal{X}\mathbf{i}\mathcal{X}^* + \mathcal{X} + \mathcal{X}^* = 0,$$

which has the unique solution  $\mathcal{X} = 0$  in  $\mathbb{H}$ .

**Example 7.** If  $\mathcal{P} = \mathbf{i}$  and  $\mathcal{Q} = \mathcal{R} = \mathcal{S} = 1$ , equation (1) becomes

$$\mathcal{X}\mathbf{i}\mathcal{X}^* + \mathcal{X} + \mathcal{X}^* = 1,$$

which has no solution in  $\mathbb{H}$ .

The following two instances exemplify case 3(a) in Theorem 3.

**Example 8.** If  $\mathcal{P} = \mathbf{i}$ ,  $\mathcal{Q} = \mathcal{R} = 1$ , and  $\mathcal{S} = -\mathbf{i}$ , equation (1) becomes

$$\mathcal{X}\mathbf{i}\mathcal{X}^* + \mathcal{X} + \mathcal{X}^* = -\mathbf{i}.$$

in this case, the set of solutions in  $\mathbb{H}$  is the circle

$$\mathcal{C}_{\mathbf{i},-\mathbf{i}} = \{x_2\mathbf{j} + x_3\mathbf{k} \mid x_2^2 + x_3^2 = 1\}.$$

**Example 9.** If  $\mathcal{P} = \mathbf{i}$ ,  $\mathcal{Q} = \mathcal{R} = 1$ , and  $\mathcal{S} = 1 - \mathbf{i}$ , equation (1) becomes

$$\mathcal{X} \mathbf{i} \mathcal{X}^* + \mathcal{X} + \mathcal{X}^* = 1 - \mathbf{i},$$

which is impossible to satisfy, since the (purely imaginary) circle  $\mathcal{C}_{\mathbf{i}, -\mathbf{i}}$  does not intersect the hyperplane  $\mathcal{X} + \mathcal{X}^* = 1$ .

Case 3(b) of Theorem 3 is illustrated by the following family of examples.

**Example 10.** If  $\mathcal{P} = \mathbf{i}$ ,  $\mathcal{Q} = \mathcal{R} = 1$ , and  $\mathcal{S} = s_0 + \mathbf{i}$ , equation (1) becomes

$$\mathcal{X} \mathbf{i} \mathcal{X}^* + \mathcal{X} + \mathcal{X}^* = s_0 + \mathbf{i}.$$

The solutions are the intersections of the circle  $\mathcal{C}_{\mathbf{i}, \mathbf{i}} = \{\cos \phi + \sin \phi \mathbf{i} : \phi \in \mathbb{R}\}$  with the hyperplane  $\mathcal{X} + \mathcal{X}^* = s_0$ . If  $s_0 = 2 \cos \phi_0$  there are two or one solutions in  $\mathbb{H}$ , namely  $\cos \phi_0 \pm \sin \phi_0 \mathbf{i}$ . If  $|s_0| > 2$ , on the other hand, there is no solution in  $\mathbb{H}$ .

Further examples of numerical solutions to equation (1), computed in the context of the surface construction problem (Section 3), are presented in [8].

## 5 Closure

Although the solution of equations in the space of quaternions  $\mathbb{H}$  has recently attracted considerable attention, most studies have been restricted to the case of unilateral coefficients. In the present study, we have considered a special quadratic quaternion equation in the quaternion variable *and* its conjugate, with mixed coefficients. This equation, arising from a surface construction problem [8], was shown to admit a complete characterization of its solutions, for all possible instances of the coefficients. In addition to point solutions, *circles* or *3-spheres* of solutions are observed — as distinct from the case of unilateral coefficients, which admits [10, 19, 23] only point solutions and 2-spheres of solutions. We have thus determined a significant class of low-degree quaternion equations (disjoint from that of polynomial equations with unilateral coefficients), for which a comprehensive solution can be achieved.

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