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## ON BOTT-CHERN COHOMOLOGY OF COMPACT COMPLEX SURFACES

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ABSTRACT. We study Bott-Chern cohomology on compact complex non-Kähler surfaces. In particular, we compute such a cohomology for compact complex surfaces in class VII and for compact complex surfaces diffeomorphic to solvmanifolds.

### INTRODUCTION

For a given complex manifold  $X$ , many cohomological invariants can be defined, and many are known for compact complex surfaces.

Among these, one can consider *Bott-Chern and Aeppli cohomologies*. They are defined as follows:

$$H_{BC}^{\bullet,\bullet}(X) := \frac{\ker \partial \cap \ker \bar{\partial}}{\text{im } \partial \bar{\partial}} \quad \text{and} \quad H_A^{\bullet,\bullet}(X) := \frac{\ker \partial \bar{\partial}}{\text{im } \partial + \text{im } \bar{\partial}}.$$

Note that the identity induces natural maps

$$\begin{array}{ccccc} & & H_{BC}^{\bullet,\bullet}(X) & & \\ & \swarrow & \downarrow & \searrow & \\ H_{\bar{\partial}}^{\bullet,\bullet}(X) & & H_{dR}^{\bullet,\bullet}(X; \mathbb{C}) & & H_{\partial}^{\bullet,\bullet}(X) \\ & \swarrow & \downarrow & \searrow & \\ & & H_A^{\bullet,\bullet}(X) & & \end{array}$$

where  $H_{\bar{\partial}}^{\bullet,\bullet}(X)$  denotes the Dolbeault cohomology and  $H_{\partial}^{\bullet,\bullet}(X)$  its conjugate, and the maps are morphisms of (graded or bi-graded) vector spaces. For compact Kähler manifolds, the natural map  $\bigoplus_{p+q=\bullet} H_{BC}^{p,q}(X) \rightarrow H_{dR}^{\bullet,\bullet}(X; \mathbb{C})$  is an isomorphism.

Assume that  $X$  is compact. The Bott-Chern and Aeppli cohomologies are isomorphic to the kernel of suitable 4th-order differential elliptic operators, see [19, §2.b, §2.c]. In particular, they are finite-dimensional vector spaces. In fact, fixed a Hermitian metric  $g$ , its associated  $\mathbb{C}$ -linear Hodge- $*$ -operator induces the isomorphism

$$H_{BC}^{p,q}(X) \xrightarrow{\cong} H_A^{n-q,n-p}(X),$$

for any  $p, q \in \{0, \dots, n\}$ , where  $n$  denotes the complex dimension of  $X$ . In particular, for any  $p, q \in \{0, \dots, n\}$ , one has

$$\dim_{\mathbb{C}} H_{BC}^{p,q}(X) = \dim_{\mathbb{C}} H_{BC}^{q,p}(X) = \dim_{\mathbb{C}} H_A^{n-p,n-q}(X) = \dim_{\mathbb{C}} H_A^{n-q,n-p}(X).$$

For the Dolbeault cohomology, the Frölicher inequality relates the Hodge numbers and the Betti numbers: for any  $k \in \{0, \dots, 2n\}$ ,

$$\sum_{p+q=k} \dim_{\mathbb{C}} H_{\bar{\partial}}^{p,q}(X) \geq \dim_{\mathbb{C}} H_{dR}^k(X; \mathbb{C}).$$

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Similarly, for Bott-Chern cohomology, the following inequality *à la* Frölicher has been proven in [3, Theorem A]: for any  $k \in \{0, \dots, n\}$ ,

$$\sum_{p+q=k} (\dim_{\mathbb{C}} H_{BC}^{p,q}(X) + \dim_{\mathbb{C}} H_A^{p,q}(X)) \geq 2 \dim_{\mathbb{C}} H_{dR}^k(X; \mathbb{C}).$$

The equality in the Frölicher inequality characterizes the degeneration of the Frölicher spectral sequence at the first level. This always happens for compact complex surfaces. On the other side, in [3, Theorem B], it is proven that the equality in the inequality *à la* Frölicher for the Bott-Chern cohomology characterizes the validity of the  $\partial\bar{\partial}$ -Lemma, namely, the property that every  $\partial$ -closed  $\bar{\partial}$ -closed d-exact form is  $\partial\bar{\partial}$ -exact too, [8]. The validity of the  $\partial\bar{\partial}$ -Lemma implies that the first Betti number is even, which is equivalent to Kählerness for compact complex surfaces. Therefore the positive integer numbers

$$\Delta^k := \sum_{p+q=k} (\dim_{\mathbb{C}} H_{BC}^{p,q}(X) + \dim_{\mathbb{C}} H_A^{p,q}(X)) - 2b_k \in \mathbb{N},$$

varying  $k \in \{1, 2\}$ , measure the non-Kählerness of compact complex surfaces  $X$ .

Compact complex surfaces are divided in seven classes, according to the Kodaira and Enriques classification, see, e.g., [4]. In this note, we compute Bott-Chern cohomology for some classes of compact complex (non-Kähler) surfaces. In particular, we are interested in studying the relations between Bott-Chern cohomology and de Rham cohomology, looking at the injectivity of the natural map  $H_{BC}^{2,1}(X) \rightarrow H_{dR}^3(X; \mathbb{C})$ . This can be intended as a weak version of the  $\partial\bar{\partial}$ -Lemma, compare also [10].

More precisely, we start by proving that the non-Kählerness for compact complex surfaces is encoded only in  $\Delta^2$ , namely,  $\Delta^1$  is always zero. This gives a partial answer to a question by T. C. Dinh to the third author.

**Theorem 1.1.** *Let  $X$  be a compact complex surface. Then:*

- (i) *the natural map  $H_{BC}^{2,1}(X) \rightarrow H_{\partial}^{2,1}(X)$  induced by the identity is injective;*
- (ii)  $\Delta^1 = 0$ .

*In particular, the non-Kählerness of  $X$  is measured by just  $\Delta^2 \in \mathbb{N}$ .*

For compact complex surfaces in class VII, we show the following result, where we denote  $h_{BC}^{p,q} := \dim_{\mathbb{C}} H_{BC}^{p,q}(X)$  for  $p, q \in \{0, 1, 2\}$ .

**Theorem 2.2.** *The Bott-Chern numbers of compact complex surfaces in class VII are:*

$$\begin{array}{ccccccc} & & & h_{BC}^{0,0} = 1 & & & \\ & & & & & h_{BC}^{0,1} = 0 & \\ h_{BC}^{2,0} = 0 & h_{BC}^{1,0} = 0 & & h_{BC}^{1,1} = b_2 + 1 & & h_{BC}^{0,2} = 0 & \\ & h_{BC}^{2,1} = 1 & & & & h_{BC}^{1,2} = 1 & \\ & & & h_{BC}^{2,2} = 1. & & & \end{array}$$

According to Theorem 1.1, the natural map  $H_{BC}^{2,1}(X) \rightarrow H_{\partial}^{2,1}(X)$  is injective for any compact complex surface. One is then interested in studying the injectivity of the natural map  $H_{BC}^{2,1}(X) \rightarrow H_{dR}^3(X; \mathbb{C})$  induced by the identity, at least for compact complex surfaces diffeomorphic to solvmanifolds. In fact, by definition, the property of satisfying the  $\partial\bar{\partial}$ -Lemma, [8], is equivalent to the natural map  $\bigoplus_{p+q=\bullet} H_{BC}^{p,q}(X) \rightarrow H_{dR}^{\bullet}(X; \mathbb{C})$  being injective. Note that, for a compact complex manifold of complex dimension  $n$ , the injectivity of the map  $H_{BC}^{n,n-1}(X) \rightarrow H_{dR}^{2n-1}(X; \mathbb{C})$  implies the  $(n-1, n)$ -th weak  $\partial\bar{\partial}$ -Lemma in the sense of J. Fu and S.-T. Yau, [10, Definition 5].

We then compute the Bott-Chern cohomology for compact complex surfaces diffeomorphic to solvmanifolds, according to the list given by K. Hasegawa in [11], see Theorem 4.1. More precisely, we prove that the cohomologies can be computed by using just left-invariant forms. Furthermore, for such complex surfaces, we note that the natural map  $H_{BC}^{2,1}(X) \rightarrow H_{dR}^3(X; \mathbb{C})$  is injective, see Theorem 4.2.

We note that the above classes do not exhaust the set of compact complex non-Kähler surfaces, the cohomologies of elliptic surfaces being still unknown.

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## 1. NON-KÄHLERNESS OF COMPACT COMPLEX SURFACES AND BOTT-CHERN COHOMOLOGY

We recall that, for a compact complex manifold of complex dimension  $n$ , for  $k \in \{0, \dots, 2n\}$ , we define the “non-Kählerness” degrees, [3, Theorem A],

$$\Delta^k := \sum_{p+q=k} (h_{BC}^{p,q} + h_{BC}^{n-q,n-p}) - 2b_k \in \mathbb{N},$$

where we use the duality in [19, §2.c] giving  $h_{BC}^{p,q} := \dim_{\mathbb{C}} H_{BC}^{p,q}(X) = \dim_{\mathbb{C}} H_{A\bar{A}}^{n-q,n-p}(X)$ . According to [3, Theorem B],  $\Delta^k = 0$  for any  $k \in \{0, \dots, 2n\}$  if and only if  $X$  satisfies the  $\partial\bar{\partial}$ -Lemma, namely, every  $\partial$ -closed  $\bar{\partial}$ -closed d-exact form is  $\partial\bar{\partial}$ -exact too. In particular, for a compact complex surface  $X$ , the condition  $\Delta^1 = \Delta^2 = 0$  is equivalent to  $X$  being Kähler, the first Betti number being even, [14, 17, 20], see also [15, Corollaire 5.7], and [5, Theorem 11].

We prove that  $\Delta^1$  is always zero for any compact complex surface. In particular, a sufficient and necessary condition for compact complex surfaces to be Kähler is  $\Delta^2 = 0$ .

**Theorem 1.1.** *Let  $X$  be a compact complex surface. Then:*

- (i) *the natural map  $H_{BC}^{2,1}(X) \rightarrow H_{\bar{\partial}}^{2,1}(X)$  induced by the identity is injective;*
- (ii)  $\Delta^1 = 0$ .

*In particular, the non-Kählerness of  $X$  is measured by just  $\Delta^2 \in \mathbb{N}$ .*

*Proof.* (i) Let  $\alpha \in \wedge^{2,1} X$  be such that  $[\alpha] = 0 \in H_{\bar{\partial}}^{2,1}(X)$ . Let  $\beta \in \wedge^{2,0} X$  be such that  $\alpha = \bar{\partial}\beta$ . Fix a Hermitian metric  $g$  on  $X$ , and consider the Hodge decomposition of  $\beta$  with respect to the Dolbeault Laplacian  $\bar{\square}$ : let  $\beta = \beta_h + \bar{\partial}^* \lambda$  where  $\beta_h \in \wedge^{2,0} X \cap \ker \bar{\square}$ , and  $\lambda \in \wedge^{2,1} X$ . Therefore we have

$$\alpha = \bar{\partial}\beta = \bar{\partial}\bar{\partial}^* \lambda = -\bar{\partial} * \underbrace{(\partial * \lambda)}_{\in \wedge^{2,0} X} = -\bar{\partial}(\partial * \lambda) = \partial\bar{\partial}(*\lambda),$$

where we have used that any  $(2,0)$ -form is primitive and hence, by the Weil identity, is self-dual. In particular,  $\alpha$  is  $\partial\bar{\partial}$ -exact, so it induces a zero class in  $H_{BC}^{2,1}(X)$ .

(ii) On the one hand, note that

$$\begin{aligned} H_{BC}^{1,0}(X) &= \frac{\ker \partial \cap \ker \bar{\partial} \cap \wedge^{1,0} X}{\text{im } \partial\bar{\partial}} = \ker \partial \cap \ker \bar{\partial} \cap \wedge^{1,0} X \\ &\subseteq \ker \bar{\partial} \cap \wedge^{1,0} X = \frac{\ker \bar{\partial} \cap \wedge^{1,0} X}{\text{im } \bar{\partial}} = H_{\bar{\partial}}^{1,0}(X). \end{aligned}$$

It follows that

$$\dim_{\mathbb{C}} H_{BC}^{0,1}(X) = \dim_{\mathbb{C}} H_{BC}^{1,0}(X) \leq \dim_{\mathbb{C}} H_{\bar{\partial}}^{1,0}(X) = b_1 - \dim_{\mathbb{C}} H_{\bar{\partial}}^{0,1}(X),$$

where we use that the Frölicher spectral sequence degenerates, hence in particular  $b_1 = \dim_{\mathbb{C}} H_{\bar{\partial}}^{1,0}(X) + \dim_{\mathbb{C}} H_{\bar{\partial}}^{0,1}(X)$ .

On the other hand, by part (i), we have

$$\dim_{\mathbb{C}} H_{BC}^{1,2}(X) = \dim_{\mathbb{C}} H_{BC}^{2,1}(X) \leq \dim_{\mathbb{C}} H_{\bar{\partial}}^{2,1}(X) = \dim_{\mathbb{C}} H_{\bar{\partial}}^{0,1}(X),$$

where we use the Kodaira and Serre duality  $H_{\bar{\partial}}^{2,1}(X) \simeq H^1(X; \Omega_X^2) \simeq H^1(X; \mathcal{O}_X) \simeq H_{\bar{\partial}}^{0,1}(X)$ .

By summing up, we get

$$\begin{aligned} \Delta^1 &= \dim_{\mathbb{C}} H_{BC}^{0,1}(X) + \dim_{\mathbb{C}} H_{BC}^{1,0}(X) + \dim_{\mathbb{C}} H_{BC}^{1,2}(X) + \dim_{\mathbb{C}} H_{BC}^{2,1}(X) - 2b_1 \\ &\leq 2 \left( b_1 - \dim_{\mathbb{C}} H_{\bar{\partial}}^{0,1}(X) + \dim_{\mathbb{C}} H_{\bar{\partial}}^{0,1}(X) - b_1 \right) = 0, \end{aligned}$$

concluding the proof. □

## 2. CLASS VII SURFACES

In this section, we compute Bott-Chern cohomology for compact complex surfaces in class VII.

Let  $X$  be a compact complex surface. By Theorem 1.1, the natural map  $H_{BC}^{2,1}(X) \rightarrow H_{\bar{\partial}}^{2,1}(X)$  is always injective. Consider now the case when  $X$  is in class VII. If  $X$  is minimal, we prove that the same holds for cohomology with values in a line bundle. We will also prove that the natural map  $H_{BC}^{1,2}(X) \rightarrow H_{\bar{\partial}}^{1,2}(X)$  is not injective.

**Proposition 2.1.** *Let  $X$  be a compact complex surface in class VII<sub>0</sub>. Let  $L \in H^1(X; \mathbb{C}^*) = \text{Pic}^0(X)$ . The natural map  $H_{BC}^{2,1}(X; L) \rightarrow H_{\bar{\partial}}^{2,1}(X; L)$  induced by the identity is injective.*

*Proof.* Let  $\alpha \in \wedge^{2,1} X \otimes L$  be a  $\bar{\partial}_L$ -exact  $(2, 1)$ -form. We need to prove that  $\alpha$  is  $\partial_L \bar{\partial}_L$ -exact too. Consider  $\alpha = \bar{\partial}_L \vartheta$ , where  $\vartheta \in \wedge^{2,0} X \otimes L$ . In particular,  $\partial_L \vartheta = 0$ , hence  $\vartheta$  defines a class in  $H_{\bar{\partial}}^{2,0}(X; L)$ . Note that  $H_{\bar{\partial}}^{2,0}(X; L) \simeq H^2(X; \mathcal{O}_X(L)) \simeq H^0(X; K_X \otimes L^{-1}) = \{0\}$  for surfaces of class VII<sub>0</sub>, [9, Remark 2.21]. It follows that  $\vartheta = -\bar{\partial}_L \bar{\eta}$  for some  $\eta \in \wedge^{1,0} X \otimes L$ . Hence  $\alpha = \partial_L \bar{\partial}_L \eta$ , that is,  $\alpha$  is  $\partial_L \bar{\partial}_L$ -exact.  $\square$

We now compute the Bott-Chern cohomology of class VII surfaces.

**Theorem 2.2.** *The Bott-Chern numbers of compact complex surfaces in class VII are:*

$$\begin{array}{ccccccc} & & & h_{BC}^{0,0} = 1 & & & \\ & & & & h_{BC}^{0,1} = 0 & & \\ h_{BC}^{2,0} = 0 & h_{BC}^{1,0} = 0 & & h_{BC}^{1,1} = b_2 + 1 & & h_{BC}^{0,2} = 0 & \\ & h_{BC}^{2,1} = 1 & & & h_{BC}^{1,2} = 1 & & \\ & & & h_{BC}^{2,2} = 1 & & & \end{array}$$

*Proof.* It holds  $H_{BC}^{1,0}(X) = \frac{\ker \partial \cap \ker \bar{\partial} \cap \wedge^{1,0} X}{\text{im } \partial \bar{\partial}} = \ker \partial \cap \ker \bar{\partial} \cap \wedge^{1,0} X \subseteq \ker \bar{\partial} \cap \wedge^{1,0} X = \frac{\ker \bar{\partial} \cap \wedge^{1,0} X}{\text{im } \bar{\partial}} = H_{\bar{\partial}}^{1,0}(X) = \{0\}$  hence  $h_{BC}^{1,0} = h_{BC}^{0,1} = 0$ .

On the other side, by Theorem 1.1,  $0 = \Delta^1 = 2 \left( h_{BC}^{1,0} + h_{BC}^{2,1} - b_1 \right) = 2 \left( h_{BC}^{2,1} - 1 \right)$  hence  $h_{BC}^{2,1} = h_{BC}^{1,2} = 1$ .

Similarly, it holds  $H_{BC}^{2,0}(X) = \frac{\ker \partial \cap \ker \bar{\partial} \cap \wedge^{2,0} X}{\text{im } \partial \bar{\partial}} = \ker \partial \cap \ker \bar{\partial} \cap \wedge^{2,0} X \subseteq \ker \bar{\partial} \cap \wedge^{2,0} X = \frac{\ker \bar{\partial} \cap \wedge^{2,0} X}{\text{im } \bar{\partial}} = H_{\bar{\partial}}^{2,0}(X) = \{0\}$  hence  $h_{BC}^{2,0} = h_{BC}^{0,2} = 0$ .

Note that, from [3, Theorem A], we have  $0 \leq \Delta^2 = 2 \left( h_{BC}^{2,0} + h_{BC}^{1,1} + h_{BC}^{0,2} - b_2 \right) = 2 \left( h_{BC}^{1,1} - b_2 \right)$  hence  $h_{BC}^{1,1} \geq b_2$ . More precisely, from [3, Theorem B] and Theorem 1.1, we have that  $h_{BC}^{1,1} = b_2$  if and only if  $\Delta^2 = 0$  if and only if  $X$  satisfies the  $\partial \bar{\partial}$ -Lemma, in fact  $X$  is Kähler, which is not the case.

Finally, we prove that  $h_{BC}^{1,1} = b_2 + 1$ . Consider the following exact sequences from [21, Lemma 2.3]. More precisely, the sequence

$$0 \rightarrow \frac{\text{im } d \cap \wedge^{1,1} X}{\text{im } \partial \bar{\partial}} \rightarrow H_{BC}^{1,1}(X) \rightarrow \text{im} \left( H_{BC}^{1,1}(X) \rightarrow H_{dR}^2(X; \mathbb{C}) \right) \rightarrow 0$$

is clearly exact. Furthermore, fix a Gauduchon metric  $g$ . Denote by  $\omega := g(J \cdot, \cdot)$  the  $(1, 1)$ -form associated to  $g$ , where  $J$  denotes the integrable almost-complex structure. By definition of  $g$  being Gauduchon, we have  $\partial \bar{\partial} \omega = 0$ . The sequence

$$0 \rightarrow \frac{\text{im } d \cap \wedge^{1,1} X}{\text{im } \partial \bar{\partial}} \xrightarrow{\langle \cdot | \omega \rangle} \mathbb{C}$$

is exact. Indeed, firstly note that for  $\eta = \partial \bar{\partial} f \in \text{im } \partial \bar{\partial} \cap \wedge^{1,1} X$ , we have

$$\langle \eta | \omega \rangle = \int_X \partial \bar{\partial} f \wedge \bar{*} \omega = \int_X \partial \bar{\partial} f \wedge \omega = \int_X f \partial \bar{\partial} \omega = 0$$

by applying twice the Stokes theorem. Then, we recall the argument in [21, Lemma 2.3(ii)] for proving that the map

$$\langle \cdot | \omega \rangle : \frac{\text{im } d \cap \wedge^{1,1} X}{\text{im } \partial \bar{\partial}} \rightarrow \mathbb{C}$$

is injective. Take  $\alpha = d\beta \in \text{im } d \cap \wedge^{1,1} X \cap \ker \langle \cdot | \omega \rangle$ . Then

$$\langle \Lambda \alpha | 1 \rangle = \langle \alpha | \omega \rangle = 0,$$

where  $\Lambda$  is the adjoint operator of  $\omega \wedge \cdot$  with respect to  $\langle \cdot | \cdot \rangle$ . Then  $\Lambda \alpha \in \ker \langle \cdot | 1 \rangle = \text{im } \Lambda \partial \bar{\partial}$ , by extending [16, Corollary 7.2.9] by  $\mathbb{C}$ -linearity. Take  $u \in C^\infty(X; \mathbb{C})$  such that  $\Lambda \alpha = \Lambda \partial \bar{\partial} u$ . Then, by defining  $\alpha' := \alpha - \partial \bar{\partial} u$ , we have  $[\alpha'] = [\alpha] \in \frac{\text{im } d \cap \wedge^{1,1} X}{\text{im } \partial \bar{\partial}}$ , and  $\Lambda \alpha' = 0$ , and  $\alpha' = d\beta'$  where  $\beta' := \beta - \bar{\partial} u$ . In particular,  $\alpha'$  is primitive. Since  $\alpha'$  is primitive and of type  $(1, 1)$ , then it is anti-self-dual by the Weil identity. Then

$$\|\alpha'\|^2 = \langle \alpha' | \alpha' \rangle = \int_X \alpha' \wedge \bar{*} \alpha' = - \int_X \alpha' \wedge \bar{\alpha}' = - \int_X d\beta' \wedge d\bar{\beta}' = - \int_X d(\beta' \wedge d\bar{\beta}') = 0$$

and hence  $\alpha' = 0$ , and therefore  $[\alpha] = 0$ .

Since the space  $\frac{\text{im } d \cap \wedge^{1,1} X}{\text{im } \partial \bar{\partial}}$  is finite-dimensional, being a sub-space of  $H_{BC}^{1,1}(X)$ , and since the space  $\text{im} \left( H_{BC}^{1,1}(X) \rightarrow H_{dR}^2(X; \mathbb{C}) \right)$  is finite-dimensional, being a sub-space of  $H_{dR}^2(X; \mathbb{C})$ , we get that

$$\dim_{\mathbb{C}} \frac{\text{im } d \cap \wedge^{1,1} X}{\text{im } \partial \bar{\partial}} \leq \dim_{\mathbb{C}} \mathbb{C} = 1,$$

and hence

$$b_2 < \dim_{\mathbb{C}} H_{BC}^{1,1}(X) = \dim_{\mathbb{C}} \text{im} \left( H_{BC}^{1,1}(X) \rightarrow H_{dR}^2(X; \mathbb{C}) \right) + \dim_{\mathbb{C}} \frac{\text{im } d \cap \wedge^{1,1} X}{\text{im } \partial \bar{\partial}} \leq b_2 + 1.$$

We get that  $\dim_{\mathbb{C}} H_{BC}^{1,1}(X) = b_2 + 1$ .  $\square$

Finally, we prove that the natural map  $H_{BC}^{1,2}(X) \rightarrow H_{\bar{\partial}}^{1,2}(X)$  is not injective.

**Proposition 2.3.** *Let  $X$  be a compact complex surface in class VII. Then the natural map  $H_{BC}^{1,2}(X) \rightarrow H_{\bar{\partial}}^{1,2}(X)$  induced by the identity is the zero map and not an isomorphism.*

*Proof.* Note that, for class VII surfaces, the pluri-genera are zero. In particular,  $H_{\bar{\partial}}^{1,2}(X) \simeq H_{\bar{\partial}}^{1,0}(X) = \{0\}$ , by Kodaira and Serre duality. By Theorem 2.2, one has  $H_{BC}^{1,2}(X) \neq \{0\}$ .  $\square$

**2.1. Cohomologies of Calabi-Eckmann surface.** In this section, as an explicit example, we list the representatives of the cohomologies of a compact complex surface in class VII: namely, we consider the Calabi-Eckmann structure on the differentiable manifolds underlying the Hopf surfaces.

Consider the differentiable manifold  $X := \mathbb{S}^1 \times \mathbb{S}^3$ . As a Lie group,  $\mathbb{S}^3 = SU(2)$  has a global left-invariant co-frame  $\{e^1, e^2, e^3\}$  such that  $de^1 = -2e^2 \wedge e^3$  and  $de^2 = 2e^1 \wedge e^3$  and  $de^3 = -2e^1 \wedge e^2$ . Hence, we consider a global left-invariant co-frame  $\{f, e^1, e^2, e^3\}$  on  $X$  with structure equations

$$\begin{cases} df &= 0 \\ de^1 &= -2e^2 \wedge e^3 \\ de^2 &= 2e^1 \wedge e^3 \\ de^3 &= -2e^1 \wedge e^2 \end{cases}.$$

Consider the left-invariant almost-complex structure defined by the  $(1,0)$ -forms

$$\begin{cases} \varphi^1 &:= e^1 + i e^2 \\ \varphi^2 &:= e^3 + i f \end{cases}.$$

By computing the complex structure equations, we get

$$\begin{cases} \partial \varphi^1 &= i \varphi^1 \wedge \varphi^2 \\ \partial \varphi^2 &= 0 \end{cases} \quad \text{and} \quad \begin{cases} \bar{\partial} \varphi^1 &= i \varphi^1 \wedge \bar{\varphi}^2 \\ \bar{\partial} \varphi^2 &= -i \varphi^1 \wedge \bar{\varphi}^1 \end{cases}.$$

We note that the almost-complex structure is in fact integrable.

The manifold  $X$  is a compact complex manifold not admitting Kähler metrics. It is bi-holomorphic to the complex manifold  $M_{0,1}$  considered by Calabi and Eckmann, [6], see [18, Theorem 4.1].

Consider the Hermitian metric  $g$  whose associated  $(1,1)$ -form is

$$\omega := \frac{i}{2} \sum_{j=1}^2 \varphi^j \wedge \bar{\varphi}^j.$$

As for the de Rham cohomology, from the Künneth formula we get

$$H_{dR}^{\bullet}(X; \mathbb{C}) = \mathbb{C} \langle 1 \rangle \oplus \mathbb{C} \langle \varphi^2 - \bar{\varphi}^2 \rangle \oplus \mathbb{C} \langle \varphi^{12\bar{1}} - \varphi^{1\bar{1}2} \rangle \oplus \mathbb{C} \langle \varphi^{12\bar{1}\bar{2}} \rangle,$$

(where, here and hereafter, we shorten, e.g.,  $\varphi^{12\bar{1}} := \varphi^1 \wedge \varphi^2 \wedge \bar{\varphi}^1$ ).

By [12, Appendix II, Theorem 9.5], one has that a model for the Dolbeault cohomology is given by

$$H_{\bar{\partial}}^{\bullet, \bullet}(X) \simeq \bigwedge_5 \langle x_{2,1}, x_{0,1} \rangle,$$



(A) differentiable structure underlying a *complex torus*:

$$\begin{cases} d e^1 = 0 \\ d e^2 = 0 \\ d e^3 = 0 \\ d e^4 = 0 \end{cases} ;$$

(B) differentiable structure underlying a *hyperelliptic surface*:

$$\begin{cases} d e^1 = e^2 \wedge e^4 \\ d e^2 = -e^1 \wedge e^4 \\ d e^3 = 0 \\ d e^4 = 0 \end{cases} ;$$

(C) differentiable structure underlying a *Inoue surface of type  $\mathcal{S}_M$* :

$$\begin{cases} d e^1 = \alpha e^1 \wedge e^4 + \beta e^2 \wedge e^4 \\ d e^2 = -\beta e^1 \wedge e^4 + \alpha e^2 \wedge e^4 \\ d e^3 = -2\alpha e^3 \wedge e^4 \\ d e^4 = 0 \end{cases} ;$$

(D) differentiable structure underlying a *primary Kodaira surface*:

$$\begin{cases} d e^1 = 0 \\ d e^2 = 0 \\ d e^3 = e^1 \wedge e^2 \\ d e^4 = 0 \end{cases} ;$$

(E) differentiable structure underlying a *secondary Kodaira surface*:

$$\begin{cases} d e^1 = e^2 \wedge e^4 \\ d e^2 = -e^1 \wedge e^4 \\ d e^3 = e^1 \wedge e^2 \\ d e^4 = 0 \end{cases} ;$$

(F) differentiable structure underlying a *Inoue surface of type  $\mathcal{S}^\pm$* :

$$\begin{cases} d e^1 = e^2 \wedge e^3 \\ d e^2 = e^2 \wedge e^4 \\ d e^3 = -e^3 \wedge e^4 \\ d e^4 = 0 \end{cases} .$$

In cases (A), (B), (C), (D), (E), consider the  $G$ -left-invariant almost-complex structure  $J$  on  $X$  defined by

$$J e_1 := e_2 \quad \text{and} \quad J e_2 := -e_1 \quad \text{and} \quad J e_3 := e_4 \quad \text{and} \quad J e_4 := -e_3 .$$

Consider the  $G$ -left-invariant  $(1,0)$ -forms

$$\begin{cases} \varphi^1 := e^1 + i e^2 \\ \varphi^2 := e^3 + i e^4 \end{cases} .$$

In case (F), consider the  $G$ -left-invariant almost-complex structure  $J$  on  $X$  defined by

$$J e_1 := e_2 \quad \text{and} \quad J e_2 := -e_1 \quad \text{and} \quad J e_3 := e_4 - q e_2 \quad \text{and} \quad J e_4 := -e_3 - q e_1 ,$$

where  $q \in \mathbb{R}$ . Consider the  $G$ -left-invariant  $(1,0)$ -forms

$$\begin{cases} \varphi^1 := e^1 + i e^2 + i q e^4 \\ \varphi^2 := e^3 + i e^4 \end{cases} .$$

With respect to the  $G$ -left-invariant coframe  $\{\varphi^1, \varphi^2\}$  for the holomorphic tangent bundle  $T^{1,0} \Gamma \backslash G$ , we have the following structure equations. (As for notation, we shorten, e.g.,  $\varphi^{1\bar{2}} := \varphi^1 \wedge \bar{\varphi}^2$ .)

(A) *torus*:

$$\begin{cases} d\varphi^1 &= 0 \\ d\varphi^2 &= 0 \end{cases}$$

(B) *hyperelliptic surface*:

$$\begin{cases} d\varphi^1 &= -\frac{1}{2}\varphi^{12} + \frac{1}{2}\varphi^{1\bar{2}} \\ d\varphi^2 &= 0 \end{cases}$$

(C) *Inoue surface  $S_M$* :

$$\begin{cases} d\varphi^1 &= \frac{\alpha-i\beta}{2i}\varphi^{12} - \frac{\alpha-i\beta}{2i}\varphi^{1\bar{2}} \\ d\varphi^2 &= -i\alpha\varphi^{2\bar{2}} \end{cases}$$

(where  $\alpha \in \mathbb{R} \setminus \{0\}$  and  $\beta \in \mathbb{R}$ );

(D) *primary Kodaira surface*:

$$\begin{cases} d\varphi^1 &= 0 \\ d\varphi^2 &= \frac{i}{2}\varphi^{1\bar{1}} \end{cases}$$

(E) *secondary Kodaira surface*:

$$\begin{cases} d\varphi^1 &= -\frac{1}{2}\varphi^{12} + \frac{1}{2}\varphi^{1\bar{2}} \\ d\varphi^2 &= \frac{i}{2}\varphi^{1\bar{1}} \end{cases}$$

(F) *Inoue surface  $S^\pm$* :

$$\begin{cases} d\varphi^1 &= \frac{1}{2i}\varphi^{12} + \frac{1}{2i}\varphi^{2\bar{1}} + \frac{q}{2}i\varphi^{2\bar{2}} \\ d\varphi^2 &= \frac{1}{2i}\varphi^{2\bar{2}} \end{cases}$$

#### 4. COHOMOLOGIES OF COMPLEX SURFACES DIFFEOMORPHIC TO SOLVMANIFOLDS

In this section, we compute the Dolbeault and Bott-Chern cohomologies of the compact complex surfaces diffeomorphic to a solvmanifold.

We prove the following theorem.

**Theorem 4.1.** *Let  $X$  be a compact complex surface diffeomorphic to a solvmanifold  $\Gamma \backslash G$ ; denote the Lie algebra of  $G$  by  $\mathfrak{g}$ . Then the inclusion  $(\wedge^\bullet \bullet \mathfrak{g}^*, \partial, \bar{\partial}) \hookrightarrow (\wedge^\bullet \bullet X, \partial, \bar{\partial})$  induces an isomorphism both in Dolbeault and in Bott-Chern cohomologies. In particular, the dimensions of the de Rham, Dolbeault, and Bott-Chern cohomologies and the degrees of non-Kählerness are summarized in Table 5.*

*Proof.* Firstly, we compute the cohomologies of the sub-complex of  $G$ -left-invariant forms. The computations are straightforward from the structure equations.

(p, q)	(A) torus				(B) hyperelliptic				(C) Inoue $S_M$			
	$H_{\mathbb{R}}^{p,q}$	$\dim_{\mathbb{C}} H_{BC}^{p,q}$	$H_{BC}^{p,q}$	$\dim_{\mathbb{C}} H_{BC}^{p,q}$	$H_{\mathbb{R}}^{p,q}$	$\dim_{\mathbb{C}} H_{BC}^{p,q}$	$H_{BC}^{p,q}$	$\dim_{\mathbb{C}} H_{BC}^{p,q}$	$H_{\mathbb{R}}^{p,q}$	$\dim_{\mathbb{C}} H_{BC}^{p,q}$	$H_{BC}^{p,q}$	$\dim_{\mathbb{C}} H_{BC}^{p,q}$
(0, 0)	(1)	1	(1)	1	(1)	1	(1)	1	(1)	1	(1)	1
(1, 0)	$\langle \varphi^1, \varphi^2 \rangle$	2	$\langle \varphi^1, \varphi^2 \rangle$	2	$\langle \varphi^2 \rangle$	1	$\langle \varphi^2 \rangle$	1	(0)	0	(0)	0
(0, 1)	$\langle \varphi^{\bar{1}}, \varphi^{\bar{2}} \rangle$	2	$\langle \varphi^{\bar{1}}, \varphi^{\bar{2}} \rangle$	2	$\langle \varphi^{\bar{3}} \rangle$	1	$\langle \varphi^{\bar{3}} \rangle$	1	$\langle \varphi^{\bar{3}} \rangle$	1	(0)	0
(2, 0)	$\langle \varphi^{12} \rangle$	1	$\langle \varphi^{12} \rangle$	1	(0)	0	(0)	0	(0)	0	(0)	0
(1, 1)	$\langle \varphi^{1\bar{1}}, \varphi^{1\bar{2}}, \varphi^{2\bar{1}}, \varphi^{2\bar{2}} \rangle$	4	$\langle \varphi^{1\bar{1}}, \varphi^{1\bar{2}}, \varphi^{2\bar{1}}, \varphi^{2\bar{2}} \rangle$	4	$\langle \varphi^{1\bar{1}}, \varphi^{2\bar{2}} \rangle$	2	$\langle \varphi^{1\bar{1}}, \varphi^{2\bar{2}} \rangle$	2	(0)	0	$\langle \varphi^{2\bar{2}} \rangle$	1
(0, 2)	$\langle \varphi^{\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{\bar{1}\bar{2}} \rangle$	1	(0)	0	(0)	0	(0)	0	(0)	0
(2, 1)	$\langle \varphi^{12\bar{1}}, \varphi^{12\bar{2}} \rangle$	2	$\langle \varphi^{12\bar{1}}, \varphi^{12\bar{2}} \rangle$	2	$\langle \varphi^{12\bar{1}} \rangle$	1	$\langle \varphi^{12\bar{1}} \rangle$	1	$\langle \varphi^{12\bar{1}} \rangle$	1	$\langle \varphi^{12\bar{1}} \rangle$	1
(1, 2)	$\langle \varphi^{1\bar{1}\bar{2}}, \varphi^{2\bar{1}\bar{2}} \rangle$	2	$\langle \varphi^{1\bar{1}\bar{2}}, \varphi^{2\bar{1}\bar{2}} \rangle$	2	$\langle \varphi^{1\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{1\bar{1}\bar{2}} \rangle$	1	(0)	0	$\langle \varphi^{1\bar{1}\bar{2}} \rangle$	1
(2, 2)	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1

TABLE 1. Dolbeault and Bott-Chern cohomologies of compact complex surfaces diffeomorphic to solvmanifolds, part 1.

(p, q)	(D) primary Kodaira				(E) secondary Kodaira				(F) Inoue $S_{\pm}$			
	$H_{\bar{\partial}}^{p,q}$	$\dim_{\mathbb{C}} H_{\bar{\partial}}^{p,q}$	$H_{BC}^{p,q}$	$\dim_{\mathbb{C}} H_{BC}^{p,q}$	$H_{\bar{\partial}}^{p,q}$	$\dim_{\mathbb{C}} H_{\bar{\partial}}^{p,q}$	$H_{BC}^{p,q}$	$\dim_{\mathbb{C}} H_{BC}^{p,q}$	$H_{\bar{\partial}}^{p,q}$	$\dim_{\mathbb{C}} H_{\bar{\partial}}^{p,q}$	$H_{BC}^{p,q}$	$\dim_{\mathbb{C}} H_{BC}^{p,q}$
(0, 0)	(1)	1	(1)	1	(1)	1	(1)	1	(1)	1	(1)	1
(1, 0)	$\langle \varphi^1 \rangle$	1	$\langle \varphi^1 \rangle$	1	(0)	0	(0)	0	(0)	0	(0)	0
(0, 1)	$\langle \varphi^1, \varphi^{\bar{3}} \rangle$	2	$\langle \varphi^{\bar{1}} \rangle$	1	$\langle \varphi^{\bar{3}} \rangle$	1	(0)	0	$\langle \varphi^{\bar{3}} \rangle$	1	(0)	0
(2, 0)	$\langle \varphi^{12} \rangle$	1	$\langle \varphi^{12} \rangle$	1	(0)	0	(0)	0	(0)	0	(0)	0
(1, 1)	$\langle \varphi^{12}, \varphi^{2\bar{1}} \rangle$	2	$\langle \varphi^{1\bar{1}}, \varphi^{12}, \varphi^{2\bar{1}} \rangle$	3	(0)	0	$\langle \varphi^{1\bar{1}} \rangle$	1	(0)	0	$\langle \varphi^{2\bar{2}} \rangle$	1
(0, 2)	$\langle \varphi^{\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{\bar{1}\bar{2}} \rangle$	1	(0)	0	(0)	0	(0)	0	(0)	0
(2, 1)	$\langle \varphi^{12\bar{1}}, \varphi^{12\bar{2}} \rangle$	2	$\langle \varphi^{12\bar{1}}, \varphi^{12\bar{2}} \rangle$	2	$\langle \varphi^{12\bar{1}} \rangle$	1	$\langle \varphi^{12\bar{1}} \rangle$	1	$\langle \varphi^{12\bar{1}} \rangle$	1	$\langle \varphi^{12\bar{1}} \rangle$	1
(1, 2)	$\langle \varphi^{2\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{1\bar{1}\bar{2}}, \varphi^{2\bar{1}\bar{2}} \rangle$	2	(0)	0	$\langle \varphi^{1\bar{1}\bar{2}} \rangle$	1	(0)	0	$\langle \varphi^{1\bar{1}\bar{2}} \rangle$	1
(2, 2)	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1

TABLE 2. Dolbeault and Bott-Chern cohomologies of compact complex surfaces diffeomorphic to solvmanifolds, part 2.

k	(A) torus		(B) hyperelliptic		(C) Inoue $S_M$	
	$H_{dR}^k$	$\dim_{\mathbb{C}} H_{dR}^k$	$H_{dR}^k$	$\dim_{\mathbb{C}} H_{dR}^k$	$H_{dR}^k$	$\dim_{\mathbb{C}} H_{dR}^k$
0	$\langle 1 \rangle$	1	$\langle 1 \rangle$	1	$\langle 1 \rangle$	1
1	$\langle \varphi^1, \varphi^2, \varphi^{\bar{1}}, \varphi^{\bar{2}} \rangle$	4	$\langle \varphi^2, \varphi^{\bar{2}} \rangle$	2	$\langle \varphi^2 - \varphi^{\bar{2}} \rangle$	1
2	$\langle \varphi^{12}, \varphi^{1\bar{1}}, \varphi^{1\bar{2}}, \varphi^{2\bar{1}}, \varphi^{2\bar{2}}, \varphi^{\bar{1}\bar{2}} \rangle$	6	$\langle \varphi^{1\bar{1}}, \varphi^{2\bar{2}} \rangle$	2	$\langle 0 \rangle$	0
3	$\langle \varphi^{12\bar{1}}, \varphi^{12\bar{2}}, \varphi^{1\bar{1}\bar{2}}, \varphi^{2\bar{1}\bar{2}} \rangle$	4	$\langle \varphi^{12\bar{1}}, \varphi^{1\bar{1}\bar{2}} \rangle$	2	$\langle \varphi^{12\bar{1}} - \varphi^{1\bar{1}\bar{2}} \rangle$	1
4	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1

TABLE 3. de Rham cohomology of compact complex surfaces diffeomorphic to solvmanifolds, part 1.

k	(D) primary Kodaira		(E) secondary Kodaira		(F) Inoue $S^{\pm}$	
	$H_{dR}^k$	$\dim_{\mathbb{C}} H_{dR}^k$	$H_{dR}^k$	$\dim_{\mathbb{C}} H_{dR}^k$	$H_{dR}^k$	$\dim_{\mathbb{C}} H_{dR}^k$
0	$\langle 1 \rangle$	1	$\langle 1 \rangle$	1	$\langle 1 \rangle$	1
1	$\langle \varphi^1, \varphi^{\bar{1}}, \varphi^2 - \varphi^{\bar{2}} \rangle$	3	$\langle \varphi^2 - \varphi^{\bar{2}} \rangle$	1	$\langle \varphi^2 - \varphi^{\bar{2}} \rangle$	1
2	$\langle \varphi^{12}, \varphi^{1\bar{2}}, \varphi^{2\bar{1}}, \varphi^{\bar{1}\bar{2}} \rangle$	4	$\langle 0 \rangle$	0	$\langle 0 \rangle$	0
3	$\langle \varphi^{12\bar{2}}, \varphi^{2\bar{1}\bar{2}}, \varphi^{12\bar{1}} - \varphi^{1\bar{1}\bar{2}} \rangle$	3	$\langle \varphi^{12\bar{1}} - \varphi^{1\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}} - q\varphi^{12\bar{2}} - \varphi^{1\bar{1}\bar{2}} + q\varphi^{2\bar{1}\bar{2}} \rangle$	1
4	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1

TABLE 4. de Rham cohomology of compact complex surfaces diffeomorphic to solvmanifolds, part 2.

In Tables 1 and 2 and in Tables 3 and 4, we summarize the results of the computations. The subcomplexes of left-invariant forms are depicted in Figure 1 (each dot represents a generator, vertical arrows depict the  $\bar{\partial}$ -operator, horizontal arrows depict the  $\partial$ -operator, and trivial arrows are not shown.) The dimensions are listed in Table 5.

On the one side, recall that the inclusion of left-invariant forms into the space of forms induces an injective map in Dolbeault and Bott-Chern cohomologies, see, e.g., [7, Lemma 9], [1, Lemma 3.6]. On the other side, recall that the Frölicher spectral sequence of a compact complex surface  $X$  degenerates at the first level, equivalently, the equalities

$$\dim_{\mathbb{C}} H_{\bar{\partial}}^{1,0}(X) + \dim_{\mathbb{C}} H_{\bar{\partial}}^{0,1}(X) = \dim_{\mathbb{C}} H_{dR}^1(X; \mathbb{C})$$

and

$$\dim_{\mathbb{C}} H_{\bar{\partial}}^{2,0}(X) + \dim_{\mathbb{C}} H_{\bar{\partial}}^{1,1}(X) + \dim_{\mathbb{C}} H_{\bar{\partial}}^{0,2}(X) = \dim_{\mathbb{C}} H_{dR}^2(X; \mathbb{C})$$

hold. By comparing the dimensions in Table 5 with the Betti numbers case by case, we find that the left-invariant forms suffice in computing the Dolbeault cohomology for each case. Then, by [1, Theorem 3.7], see also [2, Theorem 1.3, Theorem 1.6], it follows that also the Bott-Chern cohomology is computed using just left-invariant forms.  $\square$

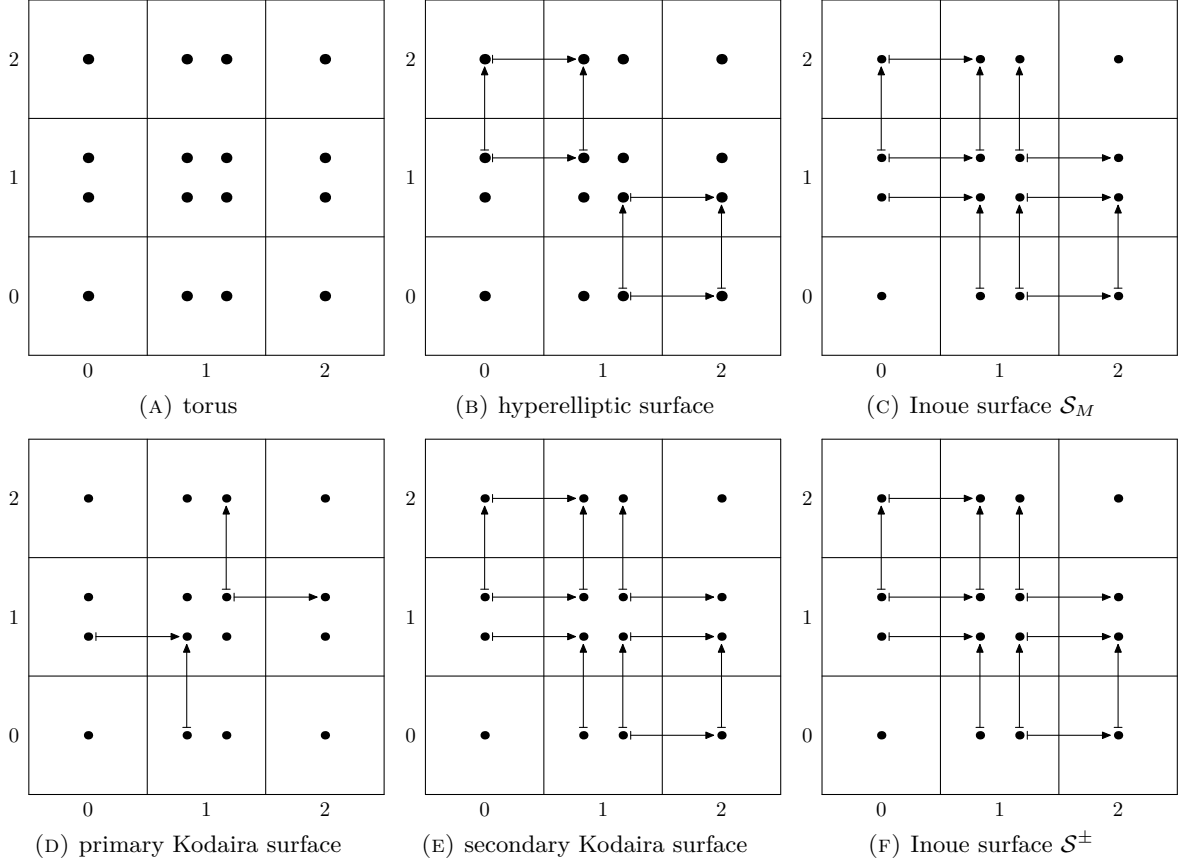


FIGURE 1. The double-complexes of left-invariant forms over 4-dimensional solvmanifolds.

$(p, q)$	(A) torus				(B) hyperell				(C) Inoue $S_M$				(D) prim Kod				(E) sec Kod				(F) Inoue $S^\pm$							
	$h_{\bar{\partial}}^{p,q}$	$h_{BC}^{p,q}$	$b_k$	$\Delta^k$	$h_{\bar{\partial}}^{p,q}$	$h_{BC}^{p,q}$	$b_k$	$\Delta^k$	$h_{\bar{\partial}}^{p,q}$	$h_{BC}^{p,q}$	$b_k$	$\Delta^k$	$h_{\bar{\partial}}^{p,q}$	$h_{BC}^{p,q}$	$b_k$	$\Delta^k$	$h_{\bar{\partial}}^{p,q}$	$h_{BC}^{p,q}$	$b_k$	$\Delta^k$	$h_{\bar{\partial}}^{p,q}$	$h_{BC}^{p,q}$	$b_k$	$\Delta^k$				
$(0, 0)$	1	1	1	0	1	1	1	0	1	1	1	0	1	1	1	0	1	1	1	0	1	1	1	0	1	1	1	0
$(1, 0)$	2	2	4	0	1	1	2	0	0	0	1	0	1	1	3	0	0	0	1	0	0	0	1	0	0	0	1	0
$(0, 1)$	2	2	4	0	1	1	2	0	1	0	1	0	2	1	3	0	1	0	1	0	1	0	1	0	1	0	1	0
$(2, 0)$	1	1	6	0	0	0	2	0	0	0	0	2	1	1	4	2	0	0	0	2	0	0	0	2	0	0	0	2
$(1, 1)$	4	4			2	2			0	1			2	3			0	1			0	1			0	1		
$(0, 2)$	1	1			0	0			0	0			1	1			0	0			0	0			0	0		
$(2, 1)$	2	2	4	0	1	1	2	0	1	1	1	0	2	2	3	0	1	1	1	0	1	1	1	0	1	1	1	0
$(1, 2)$	2	2	4	0	1	1	2	0	0	1	1	0	1	2	3	0	0	1	1	0	0	1	1	0	0	1	1	0
$(2, 2)$	1	1	1	0	1	1	1	0	1	1	1	0	1	1	1	0	1	1	1	0	1	1	1	0	1	1	1	0

TABLE 5. Summary of the dimensions of de Rham, Dolbeault, and Bott-Chern cohomologies and of the degree of non-Kählerness for compact complex surfaces diffeomorphic to solvmanifolds.

We prove the following result.

**Theorem 4.2.** *Let  $X$  be a compact complex surface diffeomorphic to a solvmanifold. Then the natural map  $H_{BC}^{2,1}(X) \rightarrow H_{\bar{\partial}}^{2,1}(X)$  induced by the identity is an isomorphism, and the natural map  $H_{BC}^{2,1}(X) \rightarrow H_{dR}^3(X; \mathbb{C})$  induced by the identity is injective.*

*Proof.* By the general result in Theorem 1.1, the natural map  $H_{BC}^{2,1}(X) \rightarrow H_{\bar{\partial}}^{2,1}(X)$  is injective. In fact, it is an isomorphism as follows from the computations summarized in Tables 1 and 2. As for the injectivity of the natural map  $H_{BC}^{2,1}(X) \rightarrow H_{dR}^3(X; \mathbb{C})$ , it is a straightforward computation from Tables 1 and 2 and Tables 3 and 4.

As an example, we offer an explicit calculation of the injectivity of the map  $H_{BC}^{2,1}(X) \rightarrow H_{dR}^3(X; \mathbb{C})$  for the Inoue surfaces of type 0, see [13], see also [22]. We will change a little bit the notation. Recall the construction of Inoue surfaces: let  $M \in \mathrm{SL}(3; \mathbb{Z})$  be a unimodular matrix having a real eigenvalue  $\lambda > 1$  and two complex eigenvalues  $\mu \neq \bar{\mu}$ . Take a real eigenvector  $(\alpha_1, \alpha_2, \alpha_3)$  and an eigenvector  $(\beta_1, \beta_2, \beta_3)$  of  $M$ . Let  $\mathbb{H} = \{z \in \mathbb{C} \mid \Im z > 0\}$ ; on the product  $\mathbb{H} \times \mathbb{C}$  consider the following transformations defined as

$$\begin{aligned} f_0(z, w) &:= (\lambda z, \mu w) \\ f_j(z, w) &:= (z + \alpha_j, w + \beta_j) \quad \text{for } j \in \{1, 2, 3\}. \end{aligned}$$

Denote by  $\Gamma_M$  the group generated by  $f_0, \dots, f_3$ ; then  $\Gamma_M$  acts in a properly discontinuous way and without fixed points on  $\mathbb{H} \times \mathbb{C}$ , and  $\mathcal{S}_M := \mathbb{H} \times \mathbb{C} / \Gamma_M$  is an Inoue surface of type 0, as in case (C) in [11]. Denoting by  $z = x + iy$  and  $w = u + iv$ , consider the following differential forms on  $\mathbb{H} \times \mathbb{C}$ :

$$e^1 := \frac{1}{y} dx, \quad e^2 := \frac{1}{y} dy, \quad e^3 := \sqrt{y} du, \quad e^4 := \sqrt{y} dv.$$

(Note that  $e^1$  and  $e^2$ , and  $e^3 \wedge e^4$  are  $\Gamma_M$ -invariant, and consequently they induce global differential forms on  $\mathcal{S}_M$ .) We obtain

$$de^1 = e^1 \wedge e^2, \quad de^2 = 0, \quad de^3 = \frac{1}{2} e^2 \wedge e^3, \quad de^4 = \frac{1}{2} e^2 \wedge e^4.$$

Consider the natural complex structure on  $\mathcal{S}_M$  induced by  $\mathbb{H} \times \mathbb{C}$ . Locally, we have

$$Je^1 = -e^2 \quad \text{and} \quad Je^2 = e^1 \quad \text{and} \quad Je^3 = -e^4 \quad \text{and} \quad Je^4 = e^3.$$

Considering the  $\Gamma_M$ -invariant (2, 1)-Bott-Chern cohomology of  $\mathcal{S}_M$ , we obtain that

$$H_{BC}^{2,1}(\mathcal{S}_M) = \mathbb{C} \langle [e^1 \wedge e^3 \wedge e^4 + ie^2 \wedge e^3 \wedge e^4] \rangle.$$

Clearly  $\bar{\partial}(e^1 \wedge e^3 \wedge e^4 + ie^2 \wedge e^3 \wedge e^4) = 0$  and  $e^1 \wedge e^3 \wedge e^4 + ie^2 \wedge e^3 \wedge e^4 = e^1 \wedge e^3 \wedge e^4 + i d(e^3 \wedge e^4)$ , therefore the de Rham cohomology class  $[e^1 \wedge e^3 \wedge e^4 + ie^2 \wedge e^3 \wedge e^4] = [e^1 \wedge e^3 \wedge e^4] \in H_{dR}^3(\mathcal{S}_M)$  is non-zero.  $\square$

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