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Generalized geometry of Norden manifolds

Antonella Nannicini

Università di Firenze, Italy

ABSTRACT. Let (M, J, g, D) be a Norden manifold with the natural canonical connection D and let \hat{J} be the generalized complex structure on M defined by g and J . We prove that \hat{J} is D -integrable and we find conditions on the curvature of D under which the $\pm i$ -eigenbundles of \hat{J} , $E_{\hat{J}}^{1,0}$, $E_{\hat{J}}^{0,1}$, are complex Lie algebroids. Moreover we prove that $E_{\hat{J}}^{1,0}$ and $(E_{\hat{J}}^{1,0})^*$ are canonically isomorphic and this allow us to define the concept of generalized $\bar{\partial}_{\hat{J}}$ -operator of (M, J, g, D) . Also we describe some generalized holomorphic sections. The class of Kähler-Norden manifolds plays an important role in this paper because for these manifolds $E_{\hat{J}}^{1,0}$ and $E_{\hat{J}}^{0,1}$ are complex Lie algebroids. ^{1 2 3}

1 Introduction

Generalized complex structures were introduced by N. Hitchin in [6], and further investigated by M. Gualtieri in [8], in order to unify symplectic and complex geometry. In this paper we consider a more general concept of generalized complex structure introduced in [15], [16] and also studied in [17], [18], [3]. Let (M, g) be a smooth pseudo-Riemannian manifold, let $T(M)$ be the tangent bundle, let $T^*(M)$ be the cotangent bundle and let $E = T(M) \oplus T^*(M)$ be the generalized tangent bundle of M . In the previous papers [15], [16], we defined a generalized complex structure of M as a complex structure on E and we studied some classes of such structures, in particular calibrated complex structures with respect to the canonical symplectic structure, (\cdot, \cdot) , of E . Using a linear connection, ∇ , on M we introduced a bracket, $[\cdot, \cdot]_{\nabla}$, on sections of E , the corresponding concept of ∇ -integrability for generalized complex structures and we studied integrability conditions. In [18] we concentrated on the canonical generalized complex structure defined by g , $J^g = \begin{pmatrix} O & -g^{-1} \\ g & O \end{pmatrix}$. We proved that in the case J^g is ∇ -integrable the $\pm i$ -eigenbundles of J^g , $E_{J^g}^{1,0}$, $E_{J^g}^{0,1}$, are complex Lie

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algebroids and, by using the canonical isomorphism between $E_{Jg}^{0,1}$ and $(E_{Jg}^{1,0})^*$ induced by the natural symplectic structure of $T(M) \oplus T^*(M)$, we defined the *generalized $\bar{\partial}_{Jg}$ -operator* on M . We remark that this case is strictly related to the field of statistical manifolds introduced in [1]. In this paper we observe that Norden manifolds fit naturally in the context of our concept of generalized complex structures and we extend the results of [18] to the case of Norden manifolds. Precisely we prove that on a Norden manifold, (M, J, g) , with the natural canonical connection D , the generalized complex structure defined by $\hat{J} = \begin{pmatrix} J & O \\ g & -J^* \end{pmatrix}$ is D -integrable. Then we describe the $\pm i$ -eigenbundles of \hat{J} , $E_{\hat{J}}^{1,0}$, $E_{\hat{J}}^{0,1}$, we find conditions under which they are complex Lie algebroids and we prove that for Kähler-Norden manifolds these conditions are automatically satisfied, that is, for this class of manifolds, $E_{\hat{J}}^{1,0}$ and $E_{\hat{J}}^{0,1}$ are complex Lie algebroids. Then we define the *generalized $\bar{\partial}_{\hat{J}}$ -operator* on M , from the Jacobi identity on $E_{\hat{J}}^{1,0}$ it follows that $(\bar{\partial}_{\hat{J}})^2 = 0$ and, as $\bar{\partial}_{\hat{J}}$ is the exterior derivative of the Lie algebroid $E_{\hat{J}}^{1,0}$, we get that $(C^\infty(\wedge^\bullet(E_{\hat{J}}^{1,0})), \wedge, \bar{\partial}_{\hat{J}}, [\cdot, \cdot]_D)$ is a differential Gerstenhaber algebra, where \wedge denotes the Schouten bracket, [12], [24]. The paper is organized as in the following. In section 2 we introduce preliminary material: first we describe the main geometrical properties of the generalized tangent bundle and of generalized complex structures, then we recall the basic definitions in the setting of Norden manifolds, Kähler-Norden manifolds and complex Lie algebroids. Original results are concentrated in section 3: the geometrical description of the generalized complex structure \hat{J} associated naturally to a Norden manifold, the definition of the generalized $\bar{\partial}_{\hat{J}}$ -operator and the description of some generalized holomorphic sections.

2 Preliminaries

2.1 Generalized geometry

Let M be a smooth manifold of real dimension n and let $E = T(M) \oplus T^*(M)$ be the *generalized tangent bundle* of M . Smooth sections of E are elements $X + \xi \in C^\infty(E)$ where $X \in C^\infty(T(M))$ is a vector field and $\xi \in C^\infty(T^*(M))$ is a 1-form.

E is equipped with a natural *symplectic structure* defined by:

$$(X + \xi, Y + \eta) = -\frac{1}{2}(\xi(Y) - \eta(X)) \quad (1)$$

and a natural *indefinite metric* defined by:

$$\langle X + \xi, Y + \eta \rangle = -\frac{1}{2}(\xi(Y) + \eta(X)). \quad (2)$$

$\langle \cdot, \cdot \rangle$ is non degenerate and of signature (n, n) .

A linear connection on M , ∇ , defines, in a canonical way, a bracket on $C^\infty(E)$, $[\cdot, \cdot]_\nabla$, as follows:

$$[X + \xi, Y + \eta]_\nabla = [X, Y] + \nabla_X \eta - \nabla_Y \xi. \quad (3)$$

The following holds:

Lemma 1 ([15]) *For all $X, Y \in C^\infty(T(M))$, for all $\xi, \eta \in C^\infty(T^*(M))$ and for all $f \in C^\infty(M)$ we have:*

1. $[X + \xi, Y + \eta]_\nabla = -[Y + \eta, X + \xi]_\nabla$,
2. $[f(X + \xi), Y + \eta]_\nabla = f[X + \xi, Y + \eta]_\nabla - Y(f)(X + \xi)$,
3. *Jacobi's identity holds for $[\cdot, \cdot]_\nabla$ if and only if ∇ has zero curvature.*

We consider the following concept of generalized complex structure, introduced in [15], [16] and further investigated in [17], [18], [3] :

Definition 2 *A generalized complex structure on M is an endomorphism \widehat{J} , $\widehat{J} : E \rightarrow E$ such that $\widehat{J}^2 = -I$.*

A pseudo-Riemannian metric on M , g , defines, in a natural way, a complex structure J^g on E by:

$$J^g(X + \xi) = -g^{-1}(\xi) + g(X) \quad (4)$$

where $g : T(M) \rightarrow T^*(M)$ is identified to the bemolle musical isomorphism defined by:

$$g(X)(Y) = g(X, Y), \quad (5)$$

in block matrix form, is:

$$J^g = \begin{pmatrix} O & -g^{-1} \\ g & O \end{pmatrix}. \quad (6)$$

Definition 3 *A generalized complex structure \widehat{J} is called pseudo calibrated if it is (\cdot, \cdot) -invariant and if the bilinear symmetric form on $T(M)$ defined by $(\cdot, J^g \cdot)$ is non degenerate, moreover \widehat{J} is called calibrated if $(\cdot, \widehat{J} \cdot)$ is positive definite, [15].*

A direct computation shows that J^g is pseudo calibrated.

Let ∇ be a linear connection on M and let $[\cdot, \cdot]_\nabla$ be the bracket on $C^\infty(E)$ defined by ∇ , the following holds:

♣

Lemma 4 ([16]) *Let $\hat{J} : E \rightarrow E$ be a generalized complex structure on M and let*

$$N^\nabla(\hat{J}) : C^\infty(E) \times C^\infty(E) \rightarrow C^\infty(E) \quad (7)$$

defined by:

$$N^\nabla(\hat{J})(\sigma, \tau) = \left[\hat{J}\sigma, \hat{J}\tau \right]_\nabla - \hat{J} \left[\hat{J}\sigma, \tau \right]_\nabla - \hat{J} [\sigma, J\tau]_\nabla - [\sigma, \tau]_\nabla \quad (8)$$

for all $\sigma, \tau \in C^\infty(E)$; $N^\nabla(\hat{J})$ is a skew symmetric tensor.

Definition 5 $N^\nabla(\hat{J})$ *is called the Nijenhuis tensor of \hat{J} with respect to ∇ .*

Definition 6 *Let $\hat{J} : E \rightarrow E$ be a generalized complex structure on M , \hat{J} is called ∇ -integrable if $N^\nabla(\hat{J}) = 0$.*

Proposition 7 ([16]) *Let ∇ be a torsion free connection on M and let*

$$J^g = \begin{pmatrix} O & -g^{-1} \\ g & O \end{pmatrix} \quad (9)$$

be the generalized complex structure on M defined by a pseudo-Riemannian metric g , J^g is ∇ -integrable if and only if g is a Codazzi tensor, that is for all $X, Y \in C^\infty(T(M))$ we have:

$$(\nabla_X g)Y = (\nabla_Y g)X. \quad (10)$$

Definition 8 ([1]), ([4]), ([19]) *Let (M, g, ∇) be a pseudo-Riemannian manifold with a torsion free linear connection, if ∇g is symmetric then (M, g, ∇) is called a statistical manifold.*

Corollary 9 *Let ∇ be a torsion free connection on M and let J^g be the generalized complex structure on M defined by a pseudo-Riemannian metric g , J^g is ∇ -integrable if and only if (M, g, ∇) is a statistical manifold.*

2.2 Norden manifolds

Norden manifolds were introduced by A. P. Norden in [20] and then studied also under the names of almost complex manifolds with B-metric and anti-Kählerian manifolds, [2], [9]. They have applications in mathematics and in theoretical physics.

♣

Definition 10 Let (M, J) be an almost complex manifold of real dimension $2n$ and let g be a pseudo-Riemannian metric on M , if J is a g -symmetric operator then g is called Norden metric and (M, J, g) is called Norden manifold.

Remark 11 We can easily prove that a Norden metric g on a $2n$ -dimensional almost complex manifold is of (n, n) -signature, that is g is a neutral metric.

Let (M, J, g) be a complex Norden manifold, that is a Norden manifold with J integrable, then there exists a natural canonical connection on M , precisely the following holds:

Theorem 12 ([9]) On a complex manifold with Norden metric (M, J, g) there exists a unique linear connection D with torsion T such that:

$$(D_X g)(Y, Z) = 0 \quad (11)$$

$$T(JX, Y) = -T(X, JY) \quad (12)$$

$$g(T(X, Y), Z) + g(T(Y, Z), X) + g(T(Z, X), Y) = 0 \quad (13)$$

for all vector fields X, Y, Z on M . D is called the natural canonical connection of the Norden manifold or B -connection and it is defined by:

$$D_X Y = \nabla_X Y - \frac{1}{2} J(\nabla_X J)Y \quad (14)$$

where ∇ is the Levi-Civita connection of g .

We remark that (14) is equivalent to:

$$D_X Y = \frac{1}{2} (\nabla_X Y - J\nabla_X JY) \quad (15)$$

then, by direct computation we get the following Proposition.

Proposition 13 If D is the natural canonical connection of the complex Norden manifold (M, J, g) then

$$DJ = 0. \quad (16)$$

Definition 14 Let (M, J, g) be a Norden manifold and let

$$\tilde{g}(X, Y) = g(JX, Y). \quad (17)$$

for all X and Y vector fields on M . \tilde{g} is a pseudo-Riemannian metric on M with (n, n) -signature and (M, J, \tilde{g}) is a Norden manifold. \tilde{g} is called the associated metric to g . \tilde{g} is also called the twin or the dual metric of g .

2.3 Kähler-Norden manifolds

Kähler-Norden manifolds are strictly related with complex analysis and they will be the main object of our theory. We recall here the definition and the main properties of Kähler-Norden manifolds, for details see [2],[11], [23].

Definition 15 *Let (M, J, g) be a Norden manifold and let ∇ be the Levi-Civita connection of g , if $\nabla J = 0$ then (M, J, g) is called Kähler-Norden manifold.*

We remark that for a Kähler-Norden manifold (M, J, g) the structure J is integrable and the natural canonical connection is the Levi-Civita connection.

Moreover the following holds:

Theorem 16 ([22]) *Let (M, J, g) be a Kähler-Norden manifold, the Levi-Civita connection of g coincides with the Levi-Civita connection of the associated metric \tilde{g} , in particular the Riemann curvature tensors of g and \tilde{g} coincide.*

A large class of Kähler-Norden manifolds is given by complex parallelisable manifolds, ([2]).

An interesting property of Kähler-Norden manifolds is the following:

Proposition 17 ([2]) *Let (M, J, g) be a Kähler-Norden manifold then, extending g by \mathbb{C} -linearity to the complexified tangent bundle $T(M) \otimes \mathbb{C}$, the components of the complex extended metric, \hat{g} , are holomorphic functions.*

We recall that on a complex manifold (M, J) an element $X \in C^\infty(TM)$ is an *infinitesimal automorphism* of the complex structure J on M if and only if X satisfies the following condition:

$$[X, JY] = J[X, Y] \quad (18)$$

for all $Y \in C^\infty(TM)$.

On Kähler-Norden manifolds, from the condition $\nabla J = 0$, (18) can be written as:

$$\nabla_{JY} X = \nabla_Y JX. \quad (19)$$

The Riemannian curvature tensor of a Kähler-Norden manifold has interesting properties, precisely we have the following:

Theorem 18 ([11]), ([22]) *In a Kähler-Norden manifold the Riemannian curvature tensor, R^∇ , of the Norden metric g is pure in all arguments, that is, for all $X, Y, Z, W \in C^\infty(T(M))$:*

$$\begin{aligned} g(R^\nabla(JX, Y)Z, W) &= g(R^\nabla(X, JY)Z, W) \\ &= g(R^\nabla(X, Y)JZ, W) \\ &= g(R^\nabla(X, Y)Z, JW). \end{aligned} \quad (20)$$

2.4 Complex Lie algebroids

Lie algebroids were introduced by J. Pradines in [21]; we recall here the definition and the main properties.

Definition 19 A complex Lie algebroid is a complex vector bundle L over a smooth real manifold M such that: a Lie bracket $[,]$ is defined on $C^\infty(L)$, a smooth bundle map $\rho : L \rightarrow T(M)$, called anchor, is defined and, for all $\sigma, \tau \in C^\infty(L)$, for all $f \in C^\infty(M)$ the following conditions hold:

1. $\rho([\sigma, \tau]) = [\rho(\sigma), \rho(\tau)]$
2. $[f\sigma, \tau] = f([\sigma, \tau]) - (\rho(\tau)(f))\sigma$.

Let L and its dual vector bundle L^* be Lie algebroids; on sections of $\wedge L$, respectively $\wedge L^*$, the Schouten bracket is defined by:

$$[,]_L : C^\infty(\wedge^p L) \times C^\infty(\wedge^q L) \longrightarrow C^\infty(\wedge^{p+q-1} L) \quad (21)$$

$$\begin{aligned} & [X_1 \wedge \dots \wedge X_p, Y_1 \wedge \dots \wedge Y_q]_L = \\ \dagger & = \sum_{i=1}^p \sum_{j=1}^q (-1)^{i+j} [X_i, Y_j]_L \wedge X_1 \wedge \dots \wedge \widehat{X_i} \wedge \dots \wedge X_p \wedge Y_1 \wedge \dots \wedge \widehat{Y_j} \wedge \dots \wedge Y_q \end{aligned} \quad (22)$$

and, for $f \in C^\infty(M)$, $X \in C^\infty(L)$

$$[X, f]_L = -[f, X]_L = \rho(X)(f); \quad (23)$$

respectively, by:

$$[,]_{L^*} : C^\infty(\wedge^p L^*) \times C^\infty(\wedge^q L^*) \longrightarrow C^\infty(\wedge^{p+q-1} L^*) \quad (24)$$

$$\begin{aligned} & [X_1^* \wedge \dots \wedge X_p^*, Y_1^* \wedge \dots \wedge Y_q^*]_{L^*} = \\ & = \sum_{i=1}^p \sum_{j=1}^q (-1)^{i+j} [X_i^*, Y_j^*]_{L^*} \wedge X_1^* \wedge \dots \wedge \widehat{X_i^*} \wedge \dots \wedge X_p^* \wedge Y_1^* \wedge \dots \wedge \widehat{Y_j^*} \wedge \dots \wedge Y_q^* \end{aligned} \quad (25)$$

and, for $f \in C^\infty(M)$, $X \in C^\infty(L^*)$

$$[X, f]_{L^*} = -[f, X]_{L^*} = \rho(X)(f). \quad (26)$$

Moreover the exterior derivatives d and d_* associated with the Lie algebroid structure of L and L^* are defined respectively by:

$$d : C^\infty(\wedge^p L^*) \longrightarrow C^\infty(\wedge^{p+1} L^*) \quad (27)$$

$$\begin{aligned} & (d\alpha)(\sigma_0, \dots, \sigma_p) = \\ & = \sum_{i=0}^p (-1)^i \rho(\sigma_i) \alpha(\sigma_0, \dots, \widehat{\sigma_i}, \dots, \sigma_p) + \sum_{i < j} (-1)^{i+j} \alpha([\sigma_i, \sigma_j]_L, \sigma_0, \dots, \widehat{\sigma_i}, \widehat{\sigma_j}, \dots, \sigma_p) \end{aligned} \quad (28)$$

for $\alpha \in C^\infty(\wedge^p L^*)$, $\sigma_0, \dots, \sigma_p \in C^\infty(L)$,

and:

$$\begin{aligned} d_* : C^\infty(\wedge^p L) &\longrightarrow C^\infty(\wedge^{p+1} L) & (29) \\ (d_* \alpha)(\sigma_0, \dots, \sigma_p) &= \\ &= \sum_{i=0}^p (-1)^i \rho(\sigma_i) \alpha(\sigma_0, \dots, \widehat{\sigma}_i, \dots, \sigma_p) + \sum_{i < j} (-1)^{i+j} \alpha([\sigma_i, \sigma_j]_{L^*}, \sigma_0, \dots, \widehat{\sigma}_i, \dots, \widehat{\sigma}_j, \dots, \sigma_p) \end{aligned} \quad (30)$$

for $\alpha \in C^\infty(\wedge^p L)$, $\sigma_0, \dots, \sigma_p \in C^\infty(L^*)$.

3 Generalized geometry of Norden manifolds

3.1 Generalized complex structures

Let (M, J, g) be a Norden manifold, the almost complex structure J and the pseudo Riemannian metric g define, in a natural way, a complex structure \widehat{J} on E by:

$$\widehat{J}(X + \xi) = J(X) + g(X) - J^*(\xi) \quad (31)$$

where $J^* : T^*(M) \rightarrow T^*(M)$ is the dual operator of J defined by:

$$J^*(\xi)(X) = \xi(J(X)). \quad (32)$$

In block matrix form, is:

$$\widehat{J} = \begin{pmatrix} J & O \\ g & -J^* \end{pmatrix}. \quad (33)$$

Remark 20 From the g -symmetry of J it follows immediately that \widehat{J} is a pseudo calibrated generalized complex structure on M , see also [16].

A direct computation gives the following:

Proposition 21 Let (M, J, g) be a Norden manifold and let ∇ be a linear connection on M with torsion T , let \widehat{J} be the generalized complex structure defined by J and g , we have:

$$\begin{aligned} N^\nabla(\widehat{J})(X, Y) &= (\nabla_{JX} J)Y - J(\nabla_X J)Y - (\nabla_{JY} J)X + J(\nabla_Y J)X + \\ &\quad -T(JX, JY) + JT(X, JY) + JT(JX, Y) + T(X, Y) + \\ &\quad + g((\nabla_Y J)X - (\nabla_X J)Y) + g(T(X, JY) + T(JX, Y)) + \\ &\quad + (\nabla_{JX} g)Y - (\nabla_{JY} g)X + (\nabla_X g)JY - (\nabla_Y g)JX \end{aligned} \quad (34)$$

$$N^\nabla(\widehat{J})(X, \xi) = -J^*(\nabla_X J^*)\xi - (\nabla_{JX} J^*)\xi \quad (35)$$

$$N^\nabla(\widehat{J})(\xi, \eta) = 0 \quad (36)$$

for all $X, Y \in C^\infty(T(M))$ and for all $\xi, \eta \in C^\infty(T^*(M))$.

Corollary 22 \hat{J} is ∇ -integrable if and only if the following conditions hold:

$$041, \dagger (\nabla_{JX}J) = J(\nabla_X J) \quad (37)$$

$$T(JX, JY) - JT(X, JY) - JT(JX, Y) - T(X, Y) = 0 \quad (38)$$

$$g((\nabla_Y J)X - (\nabla_X J)Y) + g(T(X, JY) + T(JX, Y)) + \quad (39)$$

$$+ (\nabla_{JX}g)Y - (\nabla_{JY}g)X + (\nabla_X g)JY - (\nabla_Y g)JX = 0$$

for all $X, Y \in C^\infty(T(M))$.

Corollary 23 If \hat{J} is ∇ -integrable then J is integrable.

Proof. Let $N(J)$ be the Nijenhuis tensor of the almost complex structure J , we have:

$$N(J)(X, Y) = (\nabla_{JX}J)Y - J(\nabla_X J)Y - (\nabla_{JY}J)X + J(\nabla_Y J)X + \quad (40)$$

$$-T(JX, JY) + JT(X, JY) + JT(JX, Y) + T(X, Y)$$

for all $X, Y \in C^\infty(T(M))$, then the statement follows from Corollary 22. ■

As we are interested in integrable generalized complex structures in the following we will assume that (M, J, g) is a complex Norden manifold. In particular we get:

Proposition 24 Let (M, J, g) be a complex Norden manifold and let D be the natural canonical connection on M , let \hat{J} be the generalized complex structure defined by J and g , then \hat{J} is D -integrable.

Proof. It follows from the properties of D described in Theorem 12 and in Proposition 13. ■

Analogous statement can be given for the associated metric, precisely the following holds:

Proposition 25 Let (M, J, g) be a complex Norden manifold and let \tilde{D} be the natural canonical connection of the associated metric \tilde{g} , let \tilde{J} be the generalized complex structure defined by J and \tilde{g} , then \tilde{J} is \tilde{D} -integrable.

3.2 Generalized $\bar{\partial}_{\hat{J}}$ -operator

Let (M, J, g) be a complex Norden manifold and let \hat{J} be the generalized complex structure on M defined by J and g , let

$$E^{\mathbb{C}} = (T(M) \oplus T^*(M)) \otimes \mathbb{C} \quad (41)$$

be the complexified generalized tangent bundle. The splitting in $\pm i$ eigenspaces of \widehat{J} is denoted by:

$$E^{\mathbb{C}} = E_{\widehat{J}}^{1,0} \oplus E_{\widehat{J}}^{0,1} \quad (42)$$

with

$$E_{\widehat{J}}^{0,1} = \overline{E_{\widehat{J}}^{1,0}}. \quad (43)$$

A direct computation gives:

$$E_{\widehat{J}}^{1,0} = \{Z - iJZ + g(W + iJW - iZ) \mid Z, W \in T(M) \otimes \mathbb{C}\}, \quad (44)$$

equivalently $E_{\widehat{J}}^{1,0}$ is generated by elements of the following type:

$$X - iJX - ig(X) \text{ with } X \in C^\infty(TM), \quad (45)$$

$$g(Y + iJY) \text{ with } Y \in C^\infty(TM). \quad (46)$$

Analogously we have:

$$E_{\widehat{J}}^{0,1} = \{Z + iJZ + g(W - iJW + iZ) \mid Z, W \in T(M) \otimes \mathbb{C}\} \quad (47)$$

and $E_{\widehat{J}}^{0,1}$ is generated by elements of the following type:

$$X + iJX + ig(X) \text{ with } X \in C^\infty(TM), \quad (48)$$

$$g(Y - iJY) \text{ with } Y \in C^\infty(TM). \quad (49)$$

Moreover, for any linear connection ∇ , the following holds:

Lemma 26 $E_{\widehat{J}}^{1,0}$ and $E_{\widehat{J}}^{0,1}$ are $[\cdot, \cdot]_{\nabla}$ -involutive if and only if $N^\nabla(\widehat{J}) = 0$.

Proof. Let $P_+ : E^{\mathbb{C}} \rightarrow E_{\widehat{J}}^{1,0}$ and $P_- : E^{\mathbb{C}} \rightarrow E_{\widehat{J}}^{0,1}$ be the projection operators:

$$P_{\pm} = \frac{1}{2}(I \mp i\widehat{J}), \quad (50)$$

for all $\sigma, \tau \in C^\infty(E^{\mathbb{C}})$ we have:

$$\begin{aligned} P_{\mp} [P_{\pm}(\sigma), P_{\pm}(\tau)]_{\nabla} &= P_{\mp} \left[\frac{1}{2}(\sigma \mp i\widehat{J}\sigma), \frac{1}{2}(\tau \mp i\widehat{J}\tau) \right]_{\nabla} \\ &= -\frac{1}{8}(N^\nabla(\widehat{J})(\sigma, \tau) \pm i\widehat{J}N^\nabla(\widehat{J})(\sigma, \tau)) = -\frac{1}{4}P_{\mp} (N^\nabla(\widehat{J})(\sigma, \tau)). \end{aligned} \quad (51)$$

■

From now on we suppose that (M, J, g, D) is a complex Norden manifold with the natural canonical connection. A direct computation of the bracket associated to D on $E_{\widehat{J}}^{1,0}$ and $E_{\widehat{J}}^{0,1}$ gives the following:

or

$$\begin{aligned}\sigma &= X - iJX - ig(X) \\ \tau &= Y - iJY - ig(Y) \\ \nu &= Z - iJZ - ig(Z).\end{aligned}\tag{66}$$

Let us compute

$$\sharp \quad Jac [[g(X + iJX), Y - iJY - ig(Y)]_D, Z - iJZ - ig(Z)]_D.\tag{67}$$

We have:

$$[[g(X + iJX), Y - iJY - ig(Y)]_D, Z - iJZ - ig(Z)]_D = g(K + iJK)\tag{68}$$

$$[[Y - iJY - ig(Y), Z - iJZ - ig(Z)]_D, g(X + iJX)]_D = g(L + iJL)\tag{69}$$

$$g [[Z - iJZ - ig(Z), g(X + iJX)]_D Y - iJY - ig(Y)]_D = g(H + iJH)\tag{70}$$

where

$$K = D_Z D_Y X + D_Z J D_{JY} X + J D_{JZ} D_Y X + J D_{JZ} J D_{JY} X\tag{71}$$

$$L = D_{[Y, Z]} X + J D_{J[Y, Z]} X - D_{[JY, JZ]} X - J D_{J[JY, JZ]} X\tag{72}$$

$$H = -D_Y D_Z X - J D_Y D_{JZ} X - J D_{JY} D_Z X + D_{JY} D_{JZ} X.\tag{73}$$

Then we get

$$Jac [[\sigma, \tau]_D, \nu]_D = O\tag{74}$$

if and only if

$$\sharp \quad K + L + H = O\tag{75}$$

or, by direct computation, if and only if:

$$R^D(JY, JZ) - JR^D(JY, Z) - JR^D(Y, JZ) - R^D(Y, Z) - J D_{JN(J)(Y, Z)} = O\tag{76}$$

where $N(J)$ is the Nijenhuis tensor of J . By using the integrability of J , we have the first condition.

Let us compute

$$Jac [[X - iJX - ig(X), Y - iJY - ig(Y)]_D, Z - iJZ - ig(Z)]_D.\tag{77}$$

We have:

$$\begin{aligned}[[X - iJX - ig(X), Y - iJY - ig(Y)]_D, Z - iJZ - ig(Z)]_D &= \\ &= A - iJA - ig(A) + g(B + iJB)\end{aligned}\tag{78}$$

where

$$A = [[X, Y] - [JX, JY], Z] - [J[X, Y] - J[JX, JY], JZ] \quad (79)$$

and

$$B = D_{JZ}[X, Y] + D_{JZ}T^D(JX, JY) - D_{J[X, Y]}Z + \quad (80)$$

$$+ D_{J[JX, JY]}Z - D_Z D_{JY}X + D_Z D_{JX}Y$$

where T^D denotes the torsion tensor of the connection D .
 From the Jacobi identity of $[,]$ we have that $Jac(A) = O$, then it is enough to compute $Jac(B)$.

From the properties of the torsion tensor T^D we get:

$$Jac(B) = (R^D(JX, Y) + R^D(X, JY))Z + \quad (81)$$

$$+ (R^D(JZ, X) + R^D(Z, JX))Y + (R^D(Y, JZ) + R^D(JY, Z))X.$$

Analogous computations for $E_J^{0,1}$ gives exactly the same conditions, then the Proof is complete. ■

Remark 29 We observe that (61) is equivalent to:

$$(R^D)^{(0,2)} = O \quad (82)$$

where $(R^D)^{(0,2)}$ denotes the $(0, 2)$ -part of the curvature with respect to the complex structure J on M . Moreover, if the torsion is zero, from the first Bianchi identity with zero torsion, we get that (62) is automatically satisfied; instead, from the first Bianchi identity with torsion:

$$R^D(X, Y)Z + R^D(Y, Z)X + R^D(Z, X)Y + \quad (83)$$

$$- T^D(X, [Y, Z]) - T^D(Y, [Z, X]) - T^D(Z, [X, Y]) +$$

$$- D_X T(Y, Z) - D_Y T(Z, X) - D_Z T(X, Y) = O,$$

we obtain that (62) is equivalent to the following:

$$(R^D(JX, JY) - R^D(X, Y))Z + (R^D(JZ, JX) - R^D(Z, X))Y + \quad (84)$$

$$+ (R^D(JY, JZ) - R^D(Y, Z))X = O.$$

From Proposition 26 we get in particular the following:

Proposition 30 If $R^D = O$ then $E_J^{1,0}$ and $E_J^{0,1}$ are complex Lie algebroids.

In this sense the following result provides a class of examples, ([10]), ([13]).

Theorem 31 ([10]), ([13]) Each hyper-Kähler NH-manifold is a flat pseudo-Riemannian manifold of signature $(2n, 2n)$.

More generally we have the following:

Theorem 32 *Let (M, J, g) be a Kähler-Norden manifold then $E_{\hat{J}}^{1,0}$ and $E_{\hat{J}}^{0,1}$ are complex Lie algebroids.*

Proof. In this case the natural canonical connection D is the Levi-Civita connection ∇ and, as its torsion is zero, (62) is automatically satisfied. Moreover from (20) we get that (61) is equivalent to:

$$R^\nabla(Y, Z) + R^\nabla(JY, Z)J = O \quad (85)$$

and, by using again the fact that R^∇ is a pure tensor, we have that, for all $Y, Z, W \in C^\infty(T(M))$, (85) becomes:

$$R^\nabla(Y, Z)W + R^\nabla(Y, Z)JJW = O \quad (86)$$

which is automatically satisfied. Thus the proof is complete. ■

Remark 33 *Analogous statement can be given for $E_{\hat{J}}^{1,0}$ and $E_{\hat{J}}^{0,1}$. In the following we will consider only \hat{J} .*

The following holds:

Proposition 34 *The natural symplectic structure on E defines a canonical isomorphism between $E_{\hat{J}}^{0,1}$ and the dual bundle of $E_{\hat{J}}^{1,0}$, $(E_{\hat{J}}^{1,0})^*$.*

Proof. We define

$$\varphi : E_{\hat{J}}^{0,1} \rightarrow (E_{\hat{J}}^{1,0})^* \quad (87)$$

by:

$$\begin{aligned} (\varphi(Z + iJZ + g(W - iJW + iZ)))(X - iJX + g(Y + iJY - iX)) = \\ = (Z + iJZ + g(W - iJW + iZ), X - iJX + g(Y + iJY - iX)) \end{aligned} \quad (88)$$

for all $X, Y, Z, W \in T(M) \otimes \mathbb{C}$.

We get:

$$\begin{aligned} (\varphi(Z + iJZ + g(W - iJW + iZ)))(X - iJX + g(Y + iJY - iX)) = \\ = g(Y, Z) - g(W, X) + i(g(W, JX) + g(Y, JZ) - g(X, Z)) \end{aligned} \quad (89)$$

and we extend by linearity. We have immediately that φ is injective and furthermore φ is an isomorphism. ■

The canonical isomorphism φ between $E_{\hat{J}}^{0,1}$ and the dual bundle $(E_{\hat{J}}^{1,0})^*$ allows us to define the $\bar{\partial}_{\hat{J}}$ -operator associated to the complex structure \hat{J} as in the following:

let $f \in C^\infty(M)$ and let $df \in C^\infty(T^*(M)) \hookrightarrow C^\infty(T(M) \oplus T^*(M))$, we pose

$$\bar{\partial}_{\hat{J}} f = 2(df)^{0,1} = df + i\hat{J}df \quad (90)$$

or:

$$\begin{aligned} \bar{\partial}_{\hat{J}} f &= df - iJ^*(df) \\ &= df - i(df)J; \end{aligned} \quad (91)$$

moreover we define:

$$\bar{\partial}_{\hat{J}} : C^\infty(E_{\hat{J}}^{0,1}) \rightarrow C^\infty(\wedge^2(E_{\hat{J}}^{0,1})) \quad (92)$$

via the natural isomorphism

$$E_{\hat{J}}^{0,1} \xrightarrow{\cong} (E_{\hat{J}}^{1,0})^* \quad (93)$$

as:

$$\bar{\partial}_{\hat{J}} : C^\infty((E_{\hat{J}}^{1,0})^*) \rightarrow C^\infty(\wedge^2(E_{\hat{J}}^{1,0})^*) \quad (94)$$

$$(\bar{\partial}_{\hat{J}}\alpha)(\sigma, \tau) = \rho(\sigma)\alpha(\tau) - \rho(\tau)\alpha(\sigma) - \alpha([\sigma, \tau]_D) \quad (95)$$

for $\alpha \in C^\infty((E_{\hat{J}}^{1,0})^*)$, $\sigma, \tau \in C^\infty(E_{\hat{J}}^{1,0})$.

In general:

$$\bar{\partial}_{\hat{J}} : C^\infty(\wedge^p(E_{\hat{J}}^{1,0})^*) \rightarrow C^\infty(\wedge^{p+1}(E_{\hat{J}}^{1,0})^*) \quad (96)$$

is defined by:

$$\begin{aligned} &(\bar{\partial}_{\hat{J}}\alpha)(\sigma_0, \dots, \sigma_p) = \\ &= \sum_{i=0}^p (-1)^i \rho(\sigma_i) \alpha(\sigma_0, \dots, \hat{\sigma}_i, \dots, \sigma_p) + \sum_{i < j} (-1)^{i+j} \alpha([\sigma_i, \sigma_j]_D, \sigma_0, \dots, \hat{\sigma}_i, \dots, \hat{\sigma}_j, \dots, \sigma_p) \end{aligned} \quad (97)$$

for $\alpha \in C^\infty(\wedge^p(E_{\hat{J}}^{1,0})^*)$, $\sigma_0, \dots, \sigma_p \in C^\infty(E_{\hat{J}}^{1,0})$.

Definition 35 $\bar{\partial}_{\hat{J}}$ is called generalized $\bar{\partial}$ -operator of (M, J, g, D) or generalized $\bar{\partial}_{\hat{J}}$ -operator.

We get the following:

Proposition 36 If (61) and (62) hold then $(\bar{\partial}_{\hat{J}})^2 = 0$ and $(\partial_{\hat{J}})^2 = 0$.

Proof. It follows from the fact that Jacobi identity holds on $E_{\hat{f}}^{1,0}$ and $(E_{\hat{f}}^{1,0})^*$. ■

From now on we suppose that (61) and (62) hold. We have immediately that $\bar{\partial}_{\hat{f}}$ is the exterior derivative, d_L , of the Lie algebroid $L = E_{\hat{f}}^{1,0}$. Moreover the exterior derivative d_{L^*} of $L^* = (E_{\hat{f}}^{1,0})^*$ is given by the operator $\partial_{\hat{f}}$ defined by:

$$\begin{aligned} \partial_{\hat{f}} : C^\infty \left(\wedge^p \left(E_{\hat{f}}^{1,0} \right) \right) &\rightarrow C^\infty \left(\wedge^{p+1} \left(E_{\hat{f}}^{1,0} \right) \right) & (98) \\ (\partial_{\hat{f}} \sigma) (\alpha_0^*, \dots, \alpha_p^*) &= \\ = \sum_{i=0}^{\hat{p}} (-1)^i \rho(\alpha_i^*) \sigma (\alpha_0^*, \dots, \hat{\alpha}_i^*, \dots, \alpha_p^*) &+ \sum_{i < j} (-1)^{i+j} \sigma \left([\alpha_i^*, \alpha_j^*]_D, \alpha_0^*, \dots, \hat{\alpha}_i^*, \dots, \hat{\alpha}_j^*, \dots, \alpha_p^* \right) & (99) \end{aligned}$$

for $\sigma \in C^\infty \left(\wedge^p \left(E_{\hat{f}}^{1,0} \right) \right)$, $\alpha_0^*, \dots, \alpha_p^* \in C^\infty \left(\left(E_{\hat{f}}^{1,0} \right)^* \right)$.

3.3 Generalized holomorphic sections

Definition 37 Let $\alpha \in C^\infty \left(\wedge^p \left(E_{\hat{f}}^{1,0} \right)^* \right)$, α is called generalized holomorphic section if

$$\bar{\partial}_{\hat{f}} \alpha = 0. \quad (100)$$

We remark that for all $f \in C^\infty(M)$ we have $\bar{\partial}_{\hat{f}} f = 0$ if and only if $df = 0$, so the generalized holomorphic condition for functions gives only constant functions on connected components of M .

Proposition 38 Let $W \in C^\infty(T(M))$ and let $\sigma = g(W - iJW) \in E_{\hat{f}}^{0,1}$ then $\bar{\partial}_{\hat{f}} \sigma = 0$ if and only if for all $X, Y \in C^\infty(T(M))$ holds:

$$g(D_X W - D_{JX} JW, Y) = g(D_Y W - D_{JY} JW, X). \quad (101)$$

Proof. Let $X, Y \in C^\infty(T(M))$, from (95), direct computations give:

$$\bar{\partial}_{\hat{f}} \sigma (g(X + iJX), g(Y + iJY)) = 0 \quad (102)$$

$$\bar{\partial}_{\hat{f}} \sigma (g(X + iJX), Y - iJY - ig(Y)) = 0 \quad (103)$$

$$\begin{aligned} \bar{\partial}_{\hat{f}} \sigma (X - iJX - ig(X), Y - iJY - ig(Y)) &= \\ = g(-D_X W + D_{JX} JW + i(D_{JX} W + iJ D_X W, Y) + & (104) \\ + g(D_Y W - D_{JY} JW - i(D_{JY} W + J D_Y W), X). \end{aligned}$$

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In particular we have $(\bar{\partial}_{\mathcal{J}}\sigma) = 0$ if and only if:

$$\begin{aligned} &g(-D_X W + D_{JX} JW + i(D_{JX} W + iJD_X W, Y) + \\ &+ g(D_Y W - D_{JY} JW - i(D_{JY} W + JD_Y W), X) = 0 \end{aligned} \quad (105)$$

and then, by separating real and imaginary parts, we get the statement. ■

Equivalently we can state Proposition 36 as follows:

Proposition 39 *Let $W \in C^\infty(T(M))$ and let $\sigma = g(W - iJW) \in E_{\mathcal{J}}^{0,1}$ then $\bar{\partial}_{\mathcal{J}}\sigma = 0$ if and only if for all $X, Y \in C^\infty(T(M))$ holds:*

$$(d(g(W)))(X, Y) = (d(g(W)))(JX, JY). \quad (106)$$

Proof. We have:

$$\begin{aligned} (d(g(W)))(X, Y) &= Xg(W, Y) - Yg(W, X) - g(W, [X, Y]) \\ &= g(D_X W, Y) - g(D_Y W, X) - g(W, T^D(X, Y)). \end{aligned} \quad (107)$$

On the other hand:

$$\begin{aligned} (d(g(W)))(JX, JY) &= JXg(W, JY) - JYg(W, JX) - g(W, [JX, JY]) \\ &= g(D_{JX} W, JY) - g(D_{JY} W, JX) - g(W, T^D(JX, JY)) \\ &= g(D_{JX} JW, Y) - g(D_{JY} JW, X) - g(W, T^D(JX, JY)). \end{aligned} \quad (108)$$

From the property (12) of the torsion T^D of the natural canonical connection we get the conclusion. ■

Moreover:

Proposition 40 *Let $Z \in C^\infty(T(M))$ and let $\sigma = Z + iJZ + ig(Z) \in E_{\mathcal{J}}^{0,1}$ then $\bar{\partial}_{\mathcal{J}}\sigma = 0$ if and only if for all $X, Y \in C^\infty(T(M))$ the following conditions hold:*

$$D_{JY} JZ = -D_Y Z \quad (109)$$

$$g(D_X Z, Y) = g(D_Y Z, X). \quad (110)$$

Proof. Let $X, Y \in C^\infty(T(M))$, direct computations give:

$$(\bar{\partial}_{\mathcal{J}}\sigma)(g(X + iJX), g(Y + iJY)) = 0 \quad (111)$$

$$\begin{aligned} &(\bar{\partial}_{\mathcal{J}}\sigma)(g(X + iJX), Y - iJY - ig(Y)) = \\ &= -g(D_Y Z + D_{JY} JZ, X) + ig(D_{JY} Z - D_Y JZ, X) \end{aligned} \quad (112)$$

$$\begin{aligned} &(\bar{\partial}_{\mathcal{J}}\sigma)(X - iJX - ig(X), Y - iJY - ig(Y)) = \\ &= -g(iD_X Z + D_{JX} Z, Y) + g(iD_Y Z + D_{JY} Z, X) \\ &= g(D_{JY} Z, X) - g(D_{JX} Z, Y) + i(g(D_Y Z, X) - g(D_X Z, Y)). \end{aligned} \quad (113)$$

and, by separating real and imaginary parts, we get the following conditions:

$$D_{JY}JZ + D_Y Z = O \quad (114)$$

$$g(D_{JY}Z, X) - g(D_{JX}Z, Y) = O; \quad (115)$$

From (114) we get

$$D_{JY}Z = JD_Y Z \quad (116)$$

and, substituting in (115), we have

$$g(D_Y Z, JX) - g(D_{JX}Z, Y) = O \quad (117)$$

for all $X, Y \in C^\infty(T(M))$, then we get the statement. ■

Corollary 41 *Given $Z \in C^\infty(T(M))$, infinitesimal automorphism of J , Z defines the following generalized holomorphic sections of $E_J^{0,1}$:*

$$\sigma = g(Z - iJZ) \quad (118)$$

$$\tau = Z + iJZ + ig(Z) \quad (119)$$

if and only if for all $X, Y \in C^\infty(T(M))$ the following condition hold:

$$g(D_X Z, Y) = g(D_Y Z, X). \quad (120)$$

In particular for Kähler-Norden manifolds, as D is the Levi-Civita connection and then torsion free, condition (120) is equivalent to the d -closure of $g(Z)$, and, by using a classical result in symplectic geometry, [14], we have:

Proposition 42 *Let M be a Kähler-Norden manifold and let $Z \in C^\infty(T(M))$ be an infinitesimal automorphism of J then $g(Z - iJZ)$ and $Z + iJZ + ig(Z)$ are generalized holomorphic sections of $E_J^{0,1}$ if and only if $g(Z)$ is a Lagrangian submanifold of $T^*(M)$ with respect to the standard symplectic structure.*

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Author's address:
Dipartimento di Matematica e Informatica "U. Dini"
Piazza Ghiberti, 27 - 50122 Firenze
email: antonella.nannicini@unifi.it