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On a class of pseudo calibrated generalized complex structures: from Norden to para Norden through statistical manifolds

Antonella Nannicini

Abstract. We consider pseudo calibrated generalized complex structures, \widehat{J} , defined by a pseudo Riemannian metric g and a g -symmetric operator H such that $H^2 = \mu I$, $\mu \in \mathbb{R}$, on a smooth manifold M . These structures include the case of complex Norden manifolds for $\mu = -1$, studied in [21], the case of almost tangent structures for $\mu = 0$, $ImH = \ker H$, and the case of para Norden manifolds for $\mu = 1$. The special case $H = O$ is described in [20]. We study first the integrability conditions of \widehat{J} , with respect to a linear connection ∇ , and we describe examples of geometric structures that naturally give rise to integrable pseudo calibrated generalized complex structures. Then we prove that for $\mu \neq -1$ integrability implies that the $\pm i$ -eigenbundles of \widehat{J} , $E_{\widehat{J}}^{1,0}$, $E_{\widehat{J}}^{0,1}$, are complex Lie algebroids. Moreover we define the concept of generalized $\bar{\partial}_{\widehat{J}}$ -operator of (M, H, g, ∇) and we study holomorphic sections.

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Key words: general geometric structures on manifolds; generalized geometry; Norden manifolds; para Norden manifolds; Lie algebroids.

1 Introduction

Generalized complex structures were introduced by Hitchin in [10], and further investigated by Gualtieri in [11], in order to unify symplectic and complex geometry. In this paper we consider the concept of generalized complex structure introduced in [17], [18] and also studied in [19], [20], [21], [5].

Let (M, g) be a smooth pseudo Riemannian manifold, let $T(M)$ be the tangent bundle, let $T^*(M)$ be the cotangent bundle and let $E = T(M) \oplus T^*(M)$ be the generalized tangent bundle of M . A pseudo calibrated generalized complex structure of M is a complex structure on E which is pseudo calibrated with respect to the canonical symplectic structure of E . A linear connection on

M, ∇ , defines a bracket on sections of E , $[\cdot, \cdot]_{\nabla}$, then we can define the corresponding concept of ∇ -integrability for generalized complex structures. In this paper we consider pseudo calibrated generalized complex structures $\widehat{\mathcal{J}}$ defined by a pseudo Riemannian metric g and a g -symmetric operator H such that $H^2 = \mu I$, $\mu \in \mathbb{R}$, on a smooth manifold M . These structures include the case of complex Norden manifolds for $\mu = -1$, studied in [21], the case of almost tangent structures for $\mu = 0$, $ImH = \ker H$, and the case of para Norden manifolds for $\mu = 1$. The special case $H = O$ is described in [20].

In this paper we study first the integrability conditions of $\widehat{\mathcal{J}}$, with respect to a linear connection ∇ with torsion T^{∇} , and we describe examples of geometric structures that naturally give rise to integrable pseudo calibrated generalized complex structures. Then we prove that for $\mu \neq -1$ integrability implies that the $\pm i$ -eigenbundles of $\widehat{\mathcal{J}}$, $E_{\widehat{\mathcal{J}}}^{1,0}$, $E_{\widehat{\mathcal{J}}}^{0,1}$, are complex Lie algebroids. Moreover we define the concept of generalized $\bar{\partial}_{\widehat{\mathcal{J}}}$ -operator of (M, H, g, ∇) . From the Jacobi identity on $E_{\widehat{\mathcal{J}}}^{1,0}$ it follows that $(\bar{\partial}_{\widehat{\mathcal{J}}})^2 = 0$ and, as $\bar{\partial}_{\widehat{\mathcal{J}}}$ is the exterior derivative of the Lie algebroid $E_{\widehat{\mathcal{J}}}^{1,0}$, we get that $(C^{\infty}(\wedge^{\bullet}(E_{\widehat{\mathcal{J}}}^{1,0})), \wedge, \bar{\partial}_{\widehat{\mathcal{J}}}, [\cdot, \cdot]_{\nabla})$ is a differential Gerstenhaber algebra, where \wedge denotes the Schouten bracket, [14], [29]. Finally we study certain holomorphic sections. The paper is organized as in the following. In section 2 we introduce preliminary material of the generalized tangent bundle and of generalized complex structures; in section 3 we compute integrability conditions and in section 4 we give examples of integrable structures; section 5 is devoted to the study of complex Lie algebroids naturally associated to integrable pseudo calibrated generalized complex structures; in section 6 we define the concept of generalized $\bar{\partial}_{\widehat{\mathcal{J}}}$ -operator on M and in section 7 we study some generalized holomorphic sections, in particular, in this context, Hessian manifolds occur as interesting examples.

This paper is a generalization of our previous papers [20], [21] and allows us to unify complex Norden and para Norden manifolds through almost tangent structures and statistical manifolds. The theory reveals that the case of complex Norden manifolds is special.

2 Preliminaries

Let M be a smooth manifold of real dimension n and let $E = T(M) \oplus T^*(M)$ be the generalized tangent bundle of M . Smooth sections of E are elements $X + \xi \in C^{\infty}(E)$ where $X \in C^{\infty}(T(M))$ is a vector field and $\xi \in C^{\infty}(T^*(M))$ is a 1-form.

E is equipped with a natural symplectic structure, (\cdot, \cdot) , defined by:

$$(X + \xi, Y + \eta) = -\frac{1}{2}(\xi(Y) - \eta(X)) \quad (1)$$

and a natural indefinite metric, $\langle \cdot, \cdot \rangle$, defined by:

$$\langle X + \xi, Y + \eta \rangle = -\frac{1}{2}(\xi(Y) + \eta(X)). \quad (2)$$

$\langle \cdot, \cdot \rangle$ is non degenerate and of signature (n, n) .

A linear connection on M , ∇ , defines, in a canonical way, a bracket on $C^\infty(E)$, $[\cdot, \cdot]_\nabla$, as follows:

$$[X + \xi, Y + \eta]_\nabla = [X, Y] + \nabla_X \eta - \nabla_Y \xi. \quad (3)$$

As in [17], a direct computation gives the following:

Lemma 2.1. *For all $X, Y \in C^\infty(T(M))$, for all $\xi, \eta \in C^\infty(T^*(M))$ and for all $f \in C^\infty(M)$ we have:*

1. $[X + \xi, Y + \eta]_\nabla = -[Y + \eta, X + \xi]_\nabla$,
2. $[f(X + \xi), Y + \eta]_\nabla = f[X + \xi, Y + \eta]_\nabla - Y(f)(X + \xi)$,
3. *Jacobi's identity holds for $[\cdot, \cdot]_\nabla$ if and only if ∇ has zero curvature.*

We consider the following concept of generalized complex structure, introduced in [17], [18] and further investigated in [19], [20], [21], [5] :

Definition 2.2. A *generalized complex structure* on M is an endomorphism \widehat{J} , $\widehat{J} : E \rightarrow E$ such that $\widehat{J}^2 = -I$.

Definition 2.3. A generalized complex structure \widehat{J} is called *pseudo calibrated* if it is (\cdot, \cdot) -invariant and if the bilinear symmetric form defined by $(\cdot, \widehat{J}\cdot)$ on $T(M)$ is non degenerate, moreover \widehat{J} is called *calibrated* if it is pseudo calibrated and $(\cdot, \widehat{J}\cdot)$ is positive definite, [17].

From the definition we get that a pseudo calibrated complex structure \widehat{J} on E can be written in the following block matrix form:

$$\widehat{J} = \begin{pmatrix} H & -(I + H^2)g^{-1} \\ g & -H^* \end{pmatrix} \quad (4)$$

where $g : T(M) \rightarrow T^*(M)$ is identified to the bemolle musical isomorphism of the pseudo Riemannian metric g on M , $H : T(M) \rightarrow T(M)$ is a g -symmetric operator, $H^* : T^*(M) \rightarrow T^*(M)$ is the dual operator of H defined by: $H^*(\xi)(X) = \xi(H(X))$.

We have:

$$(g(X))(Y) = g(X, Y) = 2(X, \widehat{J}Y) \quad (5)$$

for all $X, Y \in T(M)$.

In the following we will consider g -symmetric operators $H : T(M) \rightarrow T(M)$ such that $H^2 = \mu I$ where $\mu \in \mathbb{R}$ and I denotes identity. In that case we have:

$$\widehat{J} = \begin{pmatrix} H & \lambda g^{-1} \\ g & -H^* \end{pmatrix} \quad (6)$$

where $\lambda = -1 - \mu$.

We remark that for $\mu = -1$ (M, H, g) is a Norden manifold, [21], for $\mu = 0$ and $ImH = \ker H$, (M, H) is an almost tangent manifold, [2], and for $\mu = 1$ (M, H, g) is a para Norden manifold. The special case $H = O$ is described in [20].

3 Integrability

Let ∇ be a linear connection on M and let $[,]_{\nabla}$ be the bracket on $C^{\infty}(E)$ defined by ∇ , the following holds:

Lemma 3.1. ([18]) *Let $\hat{J} : E \rightarrow E$ be a generalized complex structure on M and let*

$$N^{\nabla}(\hat{J}) : C^{\infty}(E) \times C^{\infty}(E) \rightarrow C^{\infty}(E) \quad (7)$$

defined by:

$$N^{\nabla}(\hat{J})(\sigma, \tau) = \left[\hat{J}\sigma, \hat{J}\tau \right]_{\nabla} - \hat{J} \left[\hat{J}\sigma, \tau \right]_{\nabla} - \hat{J} [\sigma, \tau]_{\nabla} - [\sigma, \tau]_{\nabla} \quad (8)$$

for all $\sigma, \tau \in C^{\infty}(E)$; $N^{\nabla}(\hat{J})$ is a skew symmetric tensor.

Definition 3.2. $N^{\nabla}(\hat{J})$ is called the *Nijenhuis tensor* of \hat{J} with respect to ∇ .

Definition 3.3. Let $\hat{J} : E \rightarrow E$ be a generalized complex structure on M , \hat{J} is called ∇ -*integrable* if $N^{\nabla}(\hat{J}) = 0$.

Let T^{∇} be the torsion of ∇ :

$$T^{\nabla}(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \quad (9)$$

for all $X, Y \in C^{\infty}(TM)$ and let d^{∇} be the exterior differential associated to ∇ .

In particular:

$$(d^{\nabla}g)(X, Y) = (\nabla_X g)(Y) - (\nabla_Y g)(X) + g(T^{\nabla}(X, Y)). \quad (10)$$

We have the following:

Proposition 3.4. *Let $\hat{J} = \begin{pmatrix} H & \lambda g^{-1} \\ g & -H^* \end{pmatrix}$ be the pseudo calibrated generalized complex structure on M defined by a pseudo Riemannian metric g and a g -symmetric operator H of TM such that $H^2 = (-1 - \lambda)I$. Let $N^{\nabla}(\hat{J})$ be the generalized Nijenhuis tensor of \hat{J} defined by (8), then for all $X, Y \in C^{\infty}(T(M))$ we have:*

$$N^\nabla(\widehat{J})(X, Y) = \{N(H)(X, Y) - \lambda g^{-1}((d^\nabla g)(X, Y))\} + \{(d^\nabla g)(HX, Y) + (d^\nabla g)(X, HY) + g((\nabla_Y H)(X) - (\nabla_X H)(Y))\} \quad (11)$$

$$\begin{aligned} N^\nabla(\widehat{J})(X, g(Y)) &= \lambda\{(\nabla_X H)(Y) - (\nabla_Y H)(X)\} + \\ &\quad -\lambda\{g^{-1}((\nabla_{HX}g)(Y) - (\nabla_Xg)(HY))\} \\ &\quad + \lambda g(T^\nabla(HX, Y) - HT^\nabla(X, Y)) + \\ &\quad + \lambda\{(d^\nabla g)(X, Y)\} - g((\nabla_{HX}H)(Y) - H(\nabla_X H)(Y)) \end{aligned} \quad (12)$$

$$N^\nabla(\widehat{J})(g(X), g(Y)) = -\lambda^2 g^{-1}((d^\nabla g)(X, Y)) - \lambda g((\nabla_Y H)X - (\nabla_X H)Y). \quad (13)$$

where $N(H)$ is the Nijenhuis tensor of H defined by:

$$N(H)(X, Y) = [HX, HY] - H[HX, Y] - H[X, HY] + H^2[X, Y]. \quad (14)$$

Proof. Direct computations give:

$$\begin{aligned} N^\nabla(\widehat{J})(X, Y) &= [HX + g(X), HY + g(Y)]_\nabla + \\ &\quad - \widehat{J}[HX + g(X), Y]_\nabla - \widehat{J}[X, HY + g(Y)]_\nabla - [X, Y]_\nabla \\ &= [HX, HY] + \nabla_{HX}g(Y) - \nabla_{HY}g(X) + \\ &\quad - \widehat{J}([HX, Y] - \nabla_Yg(X) + [X, HY] + \nabla_Xg(Y)) - [X, Y] \\ &= [HX, HY] - H[HX, Y] - H[X, HY] + \lambda g^{-1}(\nabla_Yg(X) - \nabla_Xg(Y)) + \\ &\quad - [X, Y] + \nabla_{HX}g(Y) - \nabla_{HY}g(X) - g([HX, Y]) + \\ &\quad - H^*(\nabla_Yg(X)) - g([X, HY]) + H^*(\nabla_Xg(Y)) \\ &= N(H)(X, Y) + \lambda[X, Y] + \lambda g^{-1}((\nabla_Yg)X - g(\nabla_XY)) + \\ &\quad + (\nabla_{HX}g)(Y) - (\nabla_{HY}g)(X) + g((\nabla_YH)X - (\nabla_XH)Y) + \\ &\quad - H^*((\nabla_Yg)(X)) + H^*((\nabla_Xg)(Y)) + g(T^\nabla(HX, Y) + T^\nabla(X, HY)) \\ &= N(H)(X, Y) + \lambda g^{-1}((\nabla_Yg)X - (\nabla_Xg)Y - T^\nabla(X, Y)) + \\ &\quad + (\nabla_{HX}g)(Y) - (\nabla_{HY}g)(X) + g((\nabla_YH)X - (\nabla_XH)Y) + \\ &\quad - ((\nabla_Yg)(HX)) + ((\nabla_Xg)(HY)) + g(T^\nabla(HX, Y) + T^\nabla(X, HY)) \\ &= N(H)(X, Y) - \lambda g^{-1}((d^\nabla g)(X, Y)) + \\ &\quad + (d^\nabla g)(HX, Y) + (d^\nabla g)(X, HY) + g((\nabla_YH)X - (\nabla_XH)Y). \end{aligned}$$

$$\begin{aligned}
N^\nabla(\widehat{J})(g(X), g(Y)) &= [\lambda X - g(HX), \lambda Y - g(HY)]_\nabla + \\
&\quad - \widehat{J}[\lambda X - g(HX), Y]_\nabla - \widehat{J}[g(X), \lambda Y - g(HY)]_\nabla \\
&= \lambda^2 [X, Y] - \lambda \{ \lambda g^{-1}((\nabla_X g(Y) - \nabla_Y g(X))) \} + \\
&\quad - \lambda \{ -H^*(\nabla_X g(Y)) + H^*(\nabla_Y g(X)) \} \\
&= \lambda^2 (-T^\nabla(X, Y) - g^{-1}(((\nabla_X g)(Y) - (\nabla_Y g)(X)))) + \\
&\quad - \lambda (g((\nabla_Y H)X - (\nabla_X H)Y)) \\
&= -\lambda^2 g^{-1}((d^\nabla g)(X, Y)) - \lambda g((\nabla_Y H)X - (\nabla_X H)Y).
\end{aligned}$$

$$\begin{aligned}
N^\nabla(\widehat{J})(X, g(Y)) &= [HX + g(X), \lambda Y - g(HY)]_\nabla - \widehat{J}[HX + g(X), g(Y)]_\nabla + \\
&\quad - \widehat{J}[X, \lambda Y - g(HY)]_\nabla - \nabla_X g(Y) \\
&= \lambda [HX, Y] - \nabla_{HX} g(HY) - \lambda \nabla_Y g(X) - \widehat{J}\lambda [X, Y] + \\
&\quad + \widehat{J}\nabla_X g(HY) - \widehat{J}\nabla_{HX} g(Y) - \nabla_X g(Y) \\
&= \lambda [HX, Y] - \nabla_{HX} g(HY) - \lambda \nabla_Y g(X) - \widehat{J}\lambda [X, Y] + \\
&\quad - \lambda H [X, Y] - \lambda g([X, Y]) + \lambda g^{-1}(\nabla_X g(HY) - \lambda g^{-1}((\nabla_{HX} g)(Y) - \nabla_Y g(X))) \\
&\quad + H^*(\nabla_{HX} g(Y)) - H^*(\nabla_X g(HY)) - \nabla_X g(Y) \\
&= \lambda((\nabla_X H)Y - (\nabla_Y H)X - g^{-1}((\nabla_{HX} g)Y - (\nabla_X g)HY) + \\
&\quad + g(T^\nabla(HX, Y) - HT^\nabla(X, Y))) + \lambda(d^\nabla g)(X, Y) - g((\nabla_{HX} H)Y - (\nabla_X H)Y) \quad \square
\end{aligned}$$

In particular we get:

Theorem 3.5. For $\lambda(\lambda+1) \neq 0$ the pseudo calibrated generalized complex structure $\widehat{J} = \begin{pmatrix} H & \lambda g^{-1} \\ g & -H^* \end{pmatrix}$ is ∇ -integrable if and only if the following conditions are satisfied:

$$\begin{cases} d^\nabla g = 0 \\ N(H) = 0 \\ \nabla H = 0 \end{cases} \quad (15)$$

If $\lambda = 0$, $\widehat{J} = \begin{pmatrix} H & O \\ g & -H^* \end{pmatrix}$ is ∇ -integrable if and only if the following conditions hold:

$$\begin{cases} N(H) = 0 \\ (\nabla_{HX} H) = H(\nabla_X H) \\ (d^\nabla g)(HX, Y) + (d^\nabla g)(X, HY) - g((\nabla_X H)(Y) - (\nabla_Y H)(X)) = 0 \end{cases} \quad (16)$$

Finally, for $\lambda = -1$, $\hat{J} = \begin{pmatrix} H & -g^{-1} \\ g & -H^* \end{pmatrix}$ is ∇ -integrable if and only if the following conditions are satisfied:

$$\begin{cases} d^\nabla g = 0 \\ N(H) = 0 \\ (\nabla_X H)(Y) = (\nabla_Y H)(X) \\ (\nabla_{HX} H) = H(\nabla_X H) \end{cases} . \quad (17)$$

In particular if $H = O$, from (17) it follows immediately that $\hat{J} = \begin{pmatrix} O & -g^{-1} \\ g & O \end{pmatrix}$ is ∇ -integrable if and only if

$$d^\nabla g = 0. \quad (18)$$

Proof. If $\lambda \neq 0$ then from (13) and (11) we get immediately the first and second condition in (15) and (17) and the third in (17). Moreover:

$$\begin{aligned} & (\nabla_{HX} g)(Y) - (\nabla_X g)(HY) + g(T^\nabla(HX, Y) - HT^\nabla(X, Y)) \\ = & (d^\nabla g)(HX, Y) + (\nabla_Y g)(HX) - (\nabla_X g)(HY) - g(HT^\nabla(X, Y)) \\ = & (d^\nabla g)(HX, Y) + H^*((\nabla_Y g)(X) - (\nabla_X g)(Y) - g(T^\nabla(X, Y))) \\ = & (d^\nabla g)(HX, Y) - H^*((d^\nabla g)(X, Y)). \end{aligned}$$

then we get (17).

In order to obtain (15) remark that if $\lambda \neq -1$ then we get

$$\begin{aligned} (\nabla_X H)(Y) &= H^{-1}((\nabla_{HX} H)(Y)) = \frac{1}{-1-\lambda} H((\nabla_{HX} H)(Y)) \\ &= \frac{1}{-1-\lambda} H((\nabla_Y H)(HX)) \\ &= \frac{1}{-1-\lambda} H(\nabla_Y H^2 X - H\nabla_Y HX) \\ &= \frac{1}{-1-\lambda} H^2(H\nabla_Y X - \nabla_Y HX) \\ &= -(\nabla_Y H)(X) \\ &= -(\nabla_X H)(Y) \end{aligned}$$

thus the third condition in (15) is obtained.

Finally (16) and (18) follow immediately.

On the other hand if (15), respectively (16), (17), (18), hold then $N^\nabla(\hat{J}) = 0$, then the proof is complete. \square

4 Examples

Examples of integrable structures with $H = 0$ can be found in the context of quasi statistical manifolds.

Definition 4.1. ([1], ([6]), ([22]) Let (M, g, ∇) be a pseudo Riemannian manifold with a torsion free linear connection, if ∇g is symmetric then (M, g, ∇) is called a *statistical manifold*.

The concept of statistical manifold can be generalized to the case of *statistical manifolds admitting torsion* or *quasi statistical manifolds* [13]:

Definition 4.2. Let (M, g) be a pseudo Riemannian manifold and let ∇ be a linear connection on M with torsion T^∇ then (M, g, ∇) is called a *quasi statistical manifold* or *statistical manifold admitting torsion* if, for all $X, Y \in TM$, the following formula holds:

$$(\nabla_X g)Y - (\nabla_Y g)X + g(T^\nabla(X, Y)) = 0. \quad (19)$$

As a direct consequence of (10) and (18) we get the following:

Corollary 4.3. Let (M, g) be a pseudo Riemannian manifold and let ∇ be a linear connection on M with torsion T^∇ , let

$$\hat{J} = \begin{pmatrix} O & -g^{-1} \\ g & O \end{pmatrix} \quad (20)$$

be the generalized complex structure on M defined by g , \hat{J} is ∇ -integrable if and only if (M, g, ∇) is a quasi statistical manifold.

Examples of integrable structures with $H^2 = -I$ can be found in the context of Norden manifolds.

Norden manifolds were introduced by A. P. Norden in [23] and then studied also under the names of almost complex manifolds with B-metric and anti Kählerian manifolds, [3], [12]. They have applications in mathematics and in theoretical physics.

Definition 4.4. Let (M, H) be an almost complex manifold of real dimension $2n$ and let g be a pseudo Riemannian metric on M , if H is a g -symmetric operator then g is called *Norden metric* and (M, H, g) is called *Norden manifold*.

Let (M, H, g) be a complex Norden manifold, that is a Norden manifold with H integrable, the following holds:

Theorem 4.5. ([12]) On a complex manifold with Norden metric (M, H, g) there exists a unique linear connection D with torsion T such that:

$$(D_X g)(Y, Z) = 0 \quad (21)$$

$$T(HX, Y) = -T(X, HY) \quad (22)$$

$$g(T(X, Y), Z) + g(T(Y, Z), X) + g(T(Z, X), Y) = 0 \quad (23)$$

for all vector fields X, Y, Z on M . D is called the natural canonical connection of the Norden manifold or B -connection and it is defined by:

$$D_X Y = \nabla_X Y - \frac{1}{2} H(\nabla_X H) Y \quad (24)$$

where ∇ is the Levi-Civita connection of g .

We remark that if D is the natural canonical connection of the complex Norden manifold (M, H, g) then

$$DH = 0. \quad (25)$$

Corollary 4.6. Let (M, H, g) be a complex Norden manifold and let D be the natural canonical connection on M , let

$$\hat{J} = \begin{pmatrix} H & O \\ g & -H^* \end{pmatrix} \quad (26)$$

be the generalized complex structure on M defined by H and g , \hat{J} is D -integrable.

Definition 4.7. Let (M, H, g) be a Norden manifold and let ∇ be the Levi-Civita connection of g , if $\nabla H = 0$ then (M, H, g) is called *Kähler Norden manifold*.

We remark that for a Kähler Norden manifold (M, H, g) the structure H is integrable and the natural canonical connection is the Levi Civita connection.

Examples of integrable structures with $H^2 = I$ are given by para Norden manifolds, [4], [26].

Definition 4.8. An *almost product structure* on a differentiable manifold M is a $(1, 1)$ tensor field H on M such that $H^2 = I$. The pair (M, H) is called an *almost product manifold*.

Definition 4.9. An *almost paracomplex manifold* is an almost product manifold (M, H) such that the two eigenbundles, $T^+ M, T^- M$, associated to the two eigenvalues, $+1$ and -1 of H respectively, have the same rank.

Definition 4.10. An *almost paracomplex Norden manifold* (M, H, g) is a real $2n$ -dimensional smooth manifold with an almost paracomplex structure H and a pseudo Riemannian metric g such that H is a g -symmetric operator.

Definition 4.11. A *paraholomorphic Norden manifold*, or *para Kähler Norden manifold*, is an almost paracomplex Norden manifold (M, H, g) such that $\nabla H = 0$, where ∇ is the Levi Civita connection of g .

We remark that for an almost paracomplex structure H the vanishing of the Nijenhuis tensor $N(H)$ is equivalent to the existence of a torsion free linear connection ∇ such that $\nabla H = 0$, [26]. In particular from (15) we get immediately the following:

Corollary 4.12. *Let (M, H, g) be a paraholomorphic Norden manifold and let ∇ be the Levi Civita connection of g , let*

$$\widehat{J} = \begin{pmatrix} H & -2g^{-1} \\ g & -H^* \end{pmatrix} \quad (27)$$

be the generalized complex structure on M defined by H and g , \widehat{J} is ∇ -integrable.

5 Complex Lie algebroids

Lie algebroids were introduced by J. Pradines in [24]; we recall here the definition and the main properties.

Definition 5.1. A complex Lie algebroid is a complex vector bundle L over a smooth real manifold M such that: a Lie bracket $[\cdot, \cdot]$ is defined on $C^\infty(L)$, a smooth bundle map $\rho : L \rightarrow T(M) \otimes \mathbb{C}$, called *anchor*, is defined and, for all $\sigma, \tau \in C^\infty(L)$, for all $f \in C^\infty(M)$ the following conditions hold:

1. $\rho([\sigma, \tau]) = [\rho(\sigma), \rho(\tau)]$
2. $[f\sigma, \tau] = f([\sigma, \tau]) - (\rho(\tau)(f))\sigma$.

Let L and its dual vector bundle L^* be Lie algebroids; on sections of $\wedge L$, respectively $\wedge L^*$, the *Schouten bracket* is defined by:

$$[\cdot, \cdot]_L : C^\infty(\wedge^p L) \times C^\infty(\wedge^q L) \longrightarrow C^\infty(\wedge^{p+q-1} L) \quad (28)$$

$$\begin{aligned} & [X_1 \wedge \dots \wedge X_p, Y_1 \wedge \dots \wedge Y_q]_L = \\ & = \sum_{i=1}^p \sum_{j=1}^q (-1)^{i+j} [X_i, Y_j]_L \wedge X_1 \wedge \dots \widehat{i} \dots \wedge X_p \wedge Y_1 \wedge \dots \widehat{j} \dots \wedge Y_q \end{aligned} \quad (29)$$

and, for $f \in C^\infty(M)$, $X \in C^\infty(L)$

$$[X, f]_L = -[f, X]_L = \rho(X)(f); \quad (30)$$

respectively, by:

$$[\cdot, \cdot]_{L^*} : C^\infty(\wedge^p L^*) \times C^\infty(\wedge^q L^*) \longrightarrow C^\infty(\wedge^{p+q-1} L^*) \quad (31)$$

$$\begin{aligned} & [X_1^* \wedge \dots \wedge X_p^*, Y_1^* \wedge \dots \wedge Y_q^*]_{L^*} = \\ & = \sum_{i=1}^p \sum_{j=1}^q (-1)^{i+j} [X_i^*, Y_j^*]_{L^*} \wedge X_1^* \wedge \dots \widehat{i} \dots \wedge X_p^* \wedge Y_1^* \wedge \dots \widehat{j} \dots \wedge Y_q^* \end{aligned} \quad (32)$$

and, for $f \in C^\infty(M)$, $X \in C^\infty(L^*)$

$$[X, f]_{L^*} = -[f, X]_{L^*} = \rho(X)(f). \quad (33)$$

Moreover the *exterior derivatives* d and d_* associated with the Lie algebroid structure of L and L^* are defined respectively by:

$$d : C^\infty(\wedge^p L^*) \longrightarrow C^\infty(\wedge^{p+1} L^*) \quad (34)$$

$$\begin{aligned} & (d\alpha)(\sigma_0, \dots, \sigma_p) = \\ & = \sum_{i=0}^p (-1)^i \rho(\sigma_i) \alpha(\sigma_0, \dots, \widehat{\sigma}_i, \dots, \sigma_p) + \sum_{i < j} (-1)^{i+j} \alpha([\sigma_i, \sigma_j]_L, \sigma_0, \dots, \widehat{\sigma}_i, \widehat{\sigma}_j, \dots, \sigma_p) \end{aligned} \quad (35)$$

for $\alpha \in C^\infty(\wedge^p L^*)$, $\sigma_0, \dots, \sigma_p \in C^\infty(L)$,
and:

$$d_* : C^\infty(\wedge^p L) \longrightarrow C^\infty(\wedge^{p+1} L) \quad (36)$$

$$\begin{aligned} & (d_*\alpha)(\sigma_0, \dots, \sigma_p) = \\ & = \sum_{i=0}^p (-1)^i \rho(\sigma_i) \alpha(\sigma_0, \dots, \widehat{\sigma}_i, \dots, \sigma_p) + \sum_{i < j} (-1)^{i+j} \alpha([\sigma_i, \sigma_j]_{L^*}, \sigma_0, \dots, \widehat{\sigma}_i, \widehat{\sigma}_j, \dots, \sigma_p) \end{aligned} \quad (37)$$

for $\alpha \in C^\infty(\wedge^p L)$, $\sigma_0, \dots, \sigma_p \in C^\infty(L^*)$.

Let M be a smooth manifold and let $\widehat{J} = \begin{pmatrix} H & \lambda g^{-1} \\ g & -H^* \end{pmatrix}$ be the generalized pseudo calibrated complex structure on M defined by a pseudo Riemannian metric g and a g -symmetric operator H of $T(M)$ such that $H^2 = (-1 - \lambda)I$.

Let

$$E^{\mathbb{C}} = (T(M) \oplus T^*(M)) \otimes \mathbb{C} \quad (38)$$

be the complexified generalized tangent bundle. The splitting in $\pm i$ eigenspaces of \widehat{J} is denoted by:

$$E^{\mathbb{C}} = E_{\widehat{J}}^{1,0} \oplus E_{\widehat{J}}^{0,1} \quad (39)$$

with

$$E_{\widehat{J}}^{0,1} = \overline{E_{\widehat{J}}^{1,0}}. \quad (40)$$

A direct computation gives:

$$E_{\widehat{J}}^{1,0} = \{Z - iHZ + g(W + iHW - iZ) + i(-\lambda)W \mid Z, W \in C^\infty(T(M) \otimes \mathbb{C})\}, \quad (41)$$

equivalently $E_{\widehat{J}}^{1,0}$ is generated by elements of the following type:

$$Z - iHZ - ig(Z) \text{ with } Z \in C^\infty(T(M)), \quad (42)$$

$$-\lambda iW + g(W + iHW) \text{ with } W \in C^\infty(T(M)). \quad (43)$$

Analogously we have:

$$E_{\hat{J}}^{0,1} = \{Z + iHZ + g(W - iHW + iZ) - i(-\lambda)W \mid Z, W \in C^\infty(T(M) \otimes \mathbb{C})\} \quad (44)$$

and $E_{\hat{J}}^{0,1}$ is generated by elements of the following type:

$$Z + iHZ + ig(Z) \text{ with } Z \in C^\infty(T(M)), \quad (45)$$

$$\lambda iW + g(W - iHW) \text{ with } W \in C^\infty(T(M)). \quad (46)$$

Lemma 5.2. *For $\lambda \neq 0$ the map*

$$\psi : T(M) \otimes \mathbb{C} \rightarrow T(M) \otimes \mathbb{C}$$

defined by:

$$\psi(Z) = Z + iHZ \quad (47)$$

is an isomorphism and

$$\psi(Z - iHZ) - ig(\psi(Z)) = -\lambda Z - ig(Z + iHZ) = -i(-\lambda iZ + g(Z + iHZ)). \quad (48)$$

Proof. We have that ψ injective if and only if $(I + iH)$ is invertible, or i is not an eigenvalue of H . Moreover a direct computation, by using the condition $H^2 = (-1 - \lambda)I$, gives (48). \square

Corollary 5.3. *If $\lambda \neq 0$ then:*

$$E_{\hat{J}}^{1,0} = \{-\lambda Z - ig(Z + iHZ) \mid Z \in T(M) \otimes \mathbb{C}\}, \quad (49)$$

$$E_{\hat{J}}^{0,1} = \{-\lambda Z + ig(Z - iHZ) \mid Z \in T(M) \otimes \mathbb{C}\}. \quad (50)$$

Moreover, for any linear connection ∇ , the following holds:

Lemma 5.4. $E_{\hat{J}}^{1,0}$ and $E_{\hat{J}}^{0,1}$ are $[\cdot, \cdot]_{\nabla}$ -involutive if and only if $N^\nabla(\hat{J}) = 0$.

Proof. Let $P_+ : E^{\mathbb{C}} \rightarrow E_{\hat{J}}^{1,0}$ and $P_- : E^{\mathbb{C}} \rightarrow E_{\hat{J}}^{0,1}$ be the projection operators:

$$P_{\pm} = \frac{1}{2}(I \mp i\hat{J}), \quad (51)$$

for all $\sigma, \tau \in C^\infty(E^{\mathbb{C}})$ we have:

$$\begin{aligned} P_{\mp} [P_{\pm}(\sigma), P_{\pm}(\tau)]_{\nabla} &= P_{\mp} \left[\frac{1}{2}(\sigma \mp i\hat{J}\sigma), \frac{1}{2}(\tau \mp i\hat{J}\tau) \right]_{\nabla} \\ &= -\frac{1}{8}(N^\nabla(\hat{J})(\sigma, \tau) \pm i\hat{J}N^\nabla(\hat{J})(\sigma, \tau)) = -\frac{1}{4}P_{\mp}(N^\nabla(\hat{J})(\sigma, \tau)). \end{aligned} \quad (52)$$

\square

Theorem 5.4. Let $\hat{J} = \begin{pmatrix} H & \lambda g^{-1} \\ g & -H^* \end{pmatrix}$ be the pseudo calibrated generalized complex structure on M defined by a pseudo Riemannian metric g and a g -symmetric operator H of $T(M)$ such that $H^2 = (-1 - \lambda)I$ with $\lambda \neq 0$, let ∇ be a linear connection on M , if \hat{J} is ∇ -integrable then $E_{\hat{J}}^{1,0}$ and $E_{\hat{J}}^{0,1}$ are complex Lie algebroids.

Proof. The anchor

$$\rho : E_{\hat{J}}^{1,0} \rightarrow T(M) \otimes \mathbb{C} \quad (53)$$

is defined by

$$\rho(-\lambda Z - ig(Z + iHZ)) = -\lambda Z. \quad (54)$$

Conditions 1. and 2. of the definition are trivially satisfied then we need to prove the the Jacobi identity holds.

Let $\sigma, \tau, v \in E_{\hat{J}}^{1,0}$, we denote *Jacobiator* with *Jac*, that is:

$$Jac [[\sigma, \tau]_{\nabla}, v]_{\nabla} = [[\sigma, \tau]_{\nabla}, v]_{\nabla} + [[\tau, v]_{\nabla}, \sigma]_{\nabla} + [[v, \sigma]_{\nabla}, \tau]_{\nabla}. \quad (55)$$

We may assume, without loss of generality:

$$\begin{aligned} \sigma &= -\lambda Z - ig(Z + iHZ) \\ \tau &= -\lambda W - ig(W + iHW) \\ v &= -\lambda U - ig(U + iHU) \end{aligned} \quad (56)$$

with $Z, W, U \in TM \otimes \mathbb{C}$ and use integrability conditions.

Compute

$$Jac [[-\lambda Z - ig(Z + iHZ), -\lambda W - ig(W + iHW)]_{\nabla}, -\lambda U - ig(U + iHU)]_{\nabla}.$$

We have

$$\begin{aligned} & [-\lambda Z - ig(Z + iHZ), -\lambda W - ig(W + iHW)]_{\nabla} \\ &= -\lambda^2 [Z, W] + i\lambda \nabla_Z g(W + iHW) - i\lambda \nabla_W g(Z + iHZ) \\ &= -i\lambda \{ -i\lambda [Z, W] - (\nabla_Z g)(W + iHW) + (\nabla_W g)(Z + iHZ) \} + \\ & \quad + i\lambda \{ g(\nabla_Z(W + iHW)) - g(\nabla_W(Z + iHZ)) \} \\ &= -i\lambda \{ -i\lambda [Z, W] - g([Z, W] - iH[Z, W]) \} + \\ & \quad - (d^{\nabla}g)(Z, W) - iH^*(d^{\nabla}g)(Z, W) \\ &= -i\lambda \{ -i\lambda [Z, W] + g([Z, W] + iH[Z, W]) \}. \end{aligned}$$

Then

$$\begin{aligned} & Jac [[-\lambda Z - ig(Z + iHZ), -\lambda W - ig(W + iHW)]_{\nabla}, -\lambda U - ig(U + iHU)]_{\nabla} \\ &= -i\lambda \{ -i\lambda Jac [[Z, W], U] + g(Jac [[Z, W], U] + iH Jac [[Z, W], U]) \} \\ &= O. \end{aligned}$$

Or

$$Jac[[\sigma, \tau]_{\nabla}, v]_{\nabla} = O.$$

Analogous computation for $E_{\hat{J}}^{0,1}$ gives the statement, then the proof is complete. \square

We remark that in the case $\lambda = 0$ ∇ -integrability of \hat{J} is not sufficient to have Jacobi identity on $E_{\hat{J}}^{1,0}$ and $E_{\hat{J}}^{0,1}$, namely in this case we get the following:

Proposition 5.5. ([21]) *Let (M, H, g) be a complex Norden manifold, the Jacobi identity holds on $E_{\hat{J}}^{1,0}$ and $E_{\hat{J}}^{0,1}$ if and only if the following conditions are satisfied:*

$$R^D(HY, HZ) - HR^D(HY, Z) - HR^D(Y, HZ) - R^D(Y, Z) = O \quad (57)$$

$$\begin{aligned} (R^D(HX, Y) + R^D(X, HY))Z + (R^D(HZ, X) + R^D(Z, HX))Y + \\ + (R^D(Y, HZ) + R^D(HY, Z))X = O \end{aligned} \quad (58)$$

for all $X, Y, Z \in C^\infty(TM)$, where R^D denotes the curvature operator of the natural canonical connection D on M ,

$$R^D(X, Y) = D_X D_Y - D_Y D_X - D_{[X, Y]}. \quad (59)$$

In particular we have the following, [21]:

Theorem 5.6. *Let (M, H, g) be a Kähler Norden manifold then $E_{\hat{J}}^{1,0}$ and $E_{\hat{J}}^{0,1}$ are complex Lie algebroids.*

6 Generalized $\bar{\partial}_{\hat{J}}$ -operator

The following holds:

Proposition 6.1. *Let M be a smooth manifold and let $\hat{J} = \begin{pmatrix} H & \lambda g^{-1} \\ g & -H^* \end{pmatrix}$ be the generalized pseudo calibrated complex structure on M defined by a pseudo Riemannian metric g and a g -symmetric operator H of $T(M)$ such that $H^2 = (-1 - \lambda)I$. The natural symplectic structure on E defines a canonical isomorphism between $E_{\hat{J}}^{0,1}$ and the dual bundle of $E_{\hat{J}}^{1,0}$, $(E_{\hat{J}}^{1,0})^*$.*

Proof. We define

$$\varphi : E_{\hat{J}}^{0,1} \rightarrow (E_{\hat{J}}^{1,0})^* \quad (60)$$

by:

$$\begin{aligned} & (\varphi(Z + iHZ + g(W - iHW + iZ) + i\lambda W)) \\ & (X - iHX + g(Y + iHY - iX) - i\lambda Y) = \\ & = (Z + iHZ + g(W - iHW + iZ) + i\lambda W, X - iHX + g(Y + iHY - iX) - i\lambda Y) \end{aligned} \quad (61)$$

for all $X, Y, Z, W \in T(M) \otimes \mathbb{C}$ and we extend by linearity.
A direct computation gives:

$$\begin{aligned} & (\varphi(Z + iHZ + g(W - iHW + iZ) + i\lambda W)) \\ & (X - iHX + g(Y + iHY - iX) - i\lambda Y) = \\ & = g(Y, Z) - g(W, X) + i(g(W, HX) + g(Y, HZ) - g(X, Z)). \end{aligned} \quad (62)$$

We have immediately that φ is injective and furthermore φ is an isomorphism. \square

The canonical isomorphism φ between $E_{\hat{J}}^{0,1}$ and the dual bundle $(E_{\hat{J}}^{1,0})^*$ allows us to define the $\bar{\partial}_{\hat{J}}$ -operator associated to the complex structure \hat{J} , and to the fixed linear connection ∇ on M , as in the following:

let $f \in C^\infty(M)$ and let $df \in C^\infty(T^*(M)) \hookrightarrow C^\infty(T(M) \oplus T^*(M))$, we pose

$$\bar{\partial}_{\hat{J}}f = 2(df)^{0,1} = df + i\hat{J}df \quad (63)$$

or:

$$\begin{aligned} \bar{\partial}_{\hat{J}}f &= df - iJ^*(df) \\ &= df - i(df)J; \end{aligned} \quad (64)$$

moreover we define:

$$\bar{\partial}_{\hat{J}} : C^\infty(E_{\hat{J}}^{0,1}) \rightarrow C^\infty(\wedge^2(E_{\hat{J}}^{0,1})) \quad (65)$$

via the natural isomorphism

$$E_{\hat{J}}^{0,1} \xrightarrow{\cong} (E_{\hat{J}}^{1,0})^* \quad (66)$$

as:

$$\bar{\partial}_{\hat{J}} : C^\infty((E_{\hat{J}}^{1,0})^*) \rightarrow C^\infty(\wedge^2(E_{\hat{J}}^{1,0})^*) \quad (67)$$

$$(\bar{\partial}_{\hat{J}}\alpha)(\sigma, \tau) = \rho(\sigma)\alpha(\tau) - \rho(\tau)\alpha(\sigma) - \alpha([\sigma, \tau]_{\nabla}) \quad (68)$$

for $\alpha \in C^\infty((E_{\hat{J}}^{1,0})^*)$, $\sigma, \tau \in C^\infty(E_{\hat{J}}^{1,0})$.

In general:

$$\bar{\partial}_{\hat{J}} : C^\infty(\wedge^p(E_{\hat{J}}^{1,0})^*) \rightarrow C^\infty(\wedge^{p+1}(E_{\hat{J}}^{1,0})^*) \quad (69)$$

is defined by:

$$\begin{aligned} & (\bar{\partial}_{\hat{J}}\alpha)(\sigma_0, \dots, \sigma_p) = \\ & = \sum_{i=0}^p (-1)^i \rho(\sigma_i) \alpha(\sigma_0, \dots, \hat{\sigma}_i, \dots, \sigma_p) + \sum_{i < j} (-1)^{i+j} \alpha([\sigma_i, \sigma_j]_{\nabla}, \sigma_0, \dots, \hat{\sigma}_i, \dots, \hat{\sigma}_j, \dots, \sigma_p) \end{aligned} \quad (70)$$

for $\alpha \in C^\infty \left(\wedge^p \left(E_{\hat{f}}^{1,0} \right)^* \right)$, $\sigma_0, \dots, \sigma_p \in C^\infty \left(E_{\hat{f}}^{1,0} \right)$.

Definition 6.2. $\bar{\partial}_{\hat{f}}$ is called *generalized $\bar{\partial}$ -operator* of (M, J, g, ∇) or *generalized $\bar{\partial}_{\hat{f}}$ -operator*.

We get the following:

Proposition 6.3. *If $E_{\hat{f}}^{1,0}$ and $E_{\hat{f}}^{0,1}$ are complex Lie algebroids then $(\bar{\partial}_{\hat{f}})^2 = 0$ and $(\partial_{\hat{f}})^2 = 0$.*

Proof. It follows from the fact that Jacobi identity holds on $E_{\hat{f}}^{1,0}$ and $\left(E_{\hat{f}}^{1,0} \right)^*$.

□

It turns out that $\bar{\partial}_{\hat{f}}$ is the exterior derivative, d_L , of the Lie algebroid $L = E_{\hat{f}}^{1,0}$. Moreover the exterior derivative d_{L^*} of $L^* = \left(E_{\hat{f}}^{1,0} \right)^*$ is given by the operator $\partial_{\hat{f}}$ defined by:

$$\partial_{\hat{f}} : C^\infty \left(\wedge^p \left(E_{\hat{f}}^{1,0} \right) \right) \rightarrow C^\infty \left(\wedge^{p+1} \left(E_{\hat{f}}^{1,0} \right) \right) \quad (71)$$

$$\begin{aligned} & (\partial_{\hat{f}}\sigma) (\alpha_0^*, \dots, \alpha_p^*) = \\ & = \sum_{i=0}^p (-1)^i \rho(\alpha_i^*) \sigma \left(\alpha_0^*, \dots, \hat{\alpha}_i^*, \dots, \alpha_p^* \right) + \sum_{i < j} (-1)^{i+j} \sigma \left([\alpha_i^*, \alpha_j^*]_{\nabla}, \alpha_0^*, \dots, \hat{\alpha}_i^*, \dots, \hat{\alpha}_j^*, \dots, \alpha_p^* \right) \end{aligned} \quad (72)$$

for $\sigma \in C^\infty \left(\wedge^p \left(E_{\hat{f}}^{1,0} \right) \right)$, $\alpha_0^*, \dots, \alpha_p^* \in C^\infty \left(\left(E_{\hat{f}}^{1,0} \right)^* \right)$.

In particular $\left(C^\infty \left(\wedge^\bullet \left(E_{\hat{f}}^{1,0} \right) \right), \wedge, \bar{\partial}_{\hat{f}}, [\ , \]_{\nabla} \right)$ is a differential Gerstenhaber algebra, where \wedge denotes the Schouten bracket, [14], [29].

7 Generalized holomorphic sections

Definition 7.1. Let $\alpha \in C^\infty \left(\wedge^p \left(E_{\hat{f}}^{1,0} \right)^* \right)$, α is called *generalized holomorphic section* if

$$\bar{\partial}_{\hat{f}}\alpha = 0. \quad (73)$$

We remark that for all $f \in C^\infty(M)$ we have $\bar{\partial}_{\hat{f}}f = 0$ if and only if $df = 0$, so the generalized holomorphic condition for functions gives only constant functions on connected components of M .

The following holds:

Proposition 7.2. Let $\hat{J} = \begin{pmatrix} H & \lambda g^{-1} \\ g & -H^* \end{pmatrix}$ be the pseudo calibrated generalized complex structure on M defined by a pseudo Riemannian metric g and a g -symmetric operator H of $T(M)$, assume that $H^2 = (-1-\lambda)I$ with $\lambda \neq 0$. Let ∇ be a linear connection on M such that \hat{J} is ∇ -integrable, let $W \in C^\infty(T(M))$ and let $\sigma = -\lambda W + ig(W - iHW) \in E_{\hat{J}}^{0,1}$, then $\bar{\partial}_{\hat{J}}\sigma = 0$ if and only if $g(W)$ is a Lagrangian submanifold of $T^*(M)$ with respect to the standard symplectic structure.

Proof. Let $X, Y \in C^\infty(T(M))$, direct computations give:

$$\begin{aligned} (\bar{\partial}_{\hat{J}}\sigma)(-\lambda X - ig(X + iHX), -\lambda Y - ig(Y + iHY)) &= \\ &= -\lambda X(-2i\lambda g(W, Y)) + \lambda Y(-2i\lambda g(X, W)) + \\ &-\sigma([\lambda X, \lambda Y] + ig(X + iHY, \lambda Y) - ig(\lambda X, Y + iHY)) \\ &= 2i\lambda^2\{Xg(W, Y) - Yg(W, X) - g([X, Y], W)\}. \end{aligned} \quad (74)$$

In particular we have $(\bar{\partial}_{\hat{J}}\sigma) = 0$ if and only if:

$$(d(g(W)))(X, Y) = 0 \quad (75)$$

and then, by using a classical result in symplectic geometry, [16], we have that $\sigma = -\lambda W + ig(W - iHW)$ is a generalized holomorphic section of $E_{\hat{J}}^{0,1}$ if and only if $g(W)$ is a Lagrangian submanifold of $T^*(M)$ with respect to the standard symplectic structure. \square

Examples of generalized holomorphic sections can be obtained naturally in the field of Hessian geometry, [25], [27]. The concept of Hessian manifold was inspired by the Bergmann metric on bounded domains in \mathbb{C}^n and now is a very interesting topic, related to many other fields in mathematics and theoretical physics: Kähler and symplectic geometry, affine differential geometry, special manifolds, string theory and mirror symmetry, [7], [8], [9], [15], [27], [28].

Definition 7.3. Let (M, g) be a pseudo Riemannian manifold, g is called of *Hessian type* if there exists $u \in C^\infty(M)$ such that $g = Hess(u) = \nabla^2 u$, where ∇ is the Levi Civita connection of g . (M, g) is called *Hessian pseudo Riemannian manifold* if g is of Hessian type.

Proposition 7.4. Let (M, H, g) be a Hessian pseudo Riemannian manifold such that H is a g -symmetric operator and $H^2 = (-1-\lambda)I$, $\lambda \neq 0$. Let ∇ be the Levi Civita connection of g , assume $\hat{J} = \begin{pmatrix} H & \lambda g^{-1} \\ g & -H^* \end{pmatrix}$ is ∇ -integrable. Let $\{x_1, \dots, x_n\}$ be local coordinates on M , let $\left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right\}$ be local frames

for $T(M)$, if the curvature tensor of ∇ , R^∇ , vanishes, then for all $k = 1, \dots, n$ the local section

$$\sigma_k = -\lambda \frac{\partial}{\partial x_k} + ig \left(\frac{\partial}{\partial x_k} - iH \frac{\partial}{\partial x_k} \right) \in C^\infty \left(E_{\hat{f}}^{0,1} \right)$$

is $\bar{\partial}_{\hat{f}}$ -closed.

Proof. Let $g = \nabla^2 u$, then:

$$g_{jk} = \frac{\partial^2 u}{\partial x_j \partial x_k} - \sum_{l=1}^n \Gamma_{jk}^l \frac{\partial u}{\partial x_l}$$

where $\{\Gamma_{jk}^l\}$ are Christoffel's symbols of g . In particular $g \left(\frac{\partial}{\partial x_k} \right)$ is d closed if and only if for all $i, j, k = 1, \dots, n$:

$$\begin{aligned} & \frac{\partial^3 u}{\partial x_j \partial x_k \partial x_i} - \sum_{l=1}^n \frac{\partial \Gamma_{jk}^l}{\partial x_i} \frac{\partial u}{\partial x_l} - \sum_{l=1}^n \Gamma_{jk}^l \frac{\partial^2 u}{\partial x_l \partial x_i} + \\ & - \frac{\partial^3 u}{\partial x_i \partial x_k \partial x_j} + \sum_{l=1}^n \frac{\partial \Gamma_{ik}^l}{\partial x_j} \frac{\partial u}{\partial x_l} + \sum_{l=1}^n \Gamma_{ik}^l \frac{\partial^2 u}{\partial x_l \partial x_j} \\ & = \sum_{l=1}^n \left(\Gamma_{ik}^l \left(\frac{\partial^2 u}{\partial x_l \partial x_j} - \sum_{r=1}^n \Gamma_{lj}^r \frac{\partial u}{\partial x_r} \right) - \Gamma_{jk}^l \left(\frac{\partial^2 u}{\partial x_l \partial x_i} - \sum_{r=1}^n \Gamma_{li}^r \frac{\partial u}{\partial x_r} \right) \right) \\ & = 0 \end{aligned}$$

or:

$$\begin{aligned} & \sum_{l=1}^n \left(\frac{\partial \Gamma_{ik}^l}{\partial x_j} - \frac{\partial \Gamma_{jk}^l}{\partial x_i} + \sum_{r=1}^n (\Gamma_{ik}^r \Gamma_{rj}^l - \Gamma_{jk}^r \Gamma_{ri}^l) \right) \frac{\partial u}{\partial x_l} \\ & = \sum_{l=1}^n R_{ijk}^l \frac{\partial u}{\partial x_l} \\ & = 0 \end{aligned}$$

and thus, by using Proposition 32, we have the statement. \square

Holomorphic sections in the case $\lambda = 0$ have been studied in [21].

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