



# Monotonicity conditions in oscillation to superlinear differential equations

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**Abstract.** We consider the second order differential equation

$$(a(t)|x'|^\alpha \operatorname{sgn} x')' + b(t)|x|^\beta \operatorname{sgn} x = 0$$

in the super-linear case  $\alpha < \beta$ . We prove the existence of the so-called intermediate solutions and we discuss their coexistence with other types of nonoscillatory and oscillatory solutions. Our results are new even for the Emden–Fowler equation ( $\alpha = 1$ ).

**Keywords:** second order nonlinear differential equation, nonoscillatory solution, oscillatory solution, intermediate solution.

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## 1 Introduction

Consider the second order superlinear differential equation

$$(a(t)|x'|^\alpha \operatorname{sgn} x')' + b(t)|x|^\beta \operatorname{sgn} x = 0, \quad (1.1)$$

where  $\alpha, \beta$  are positive constants such that  $\alpha < \beta$ , the functions  $a, b$  are positive continuously differentiable on  $[0, \infty)$  and

$$I_a = \int_0^\infty a^{-1/\alpha}(s) ds = \infty, \quad I_b = \int_0^\infty b(s) ds < \infty. \quad (1.2)$$

Equation (1.1) is the so-called generalized Emden–Fowler differential equation and it is widely studied in the literature, see, e.g., [15, 18, 27] and references therein. Equation (1.1) appears also in the study for searching spherically symmetric solutions of certain nonlinear elliptic differential equations with  $p$ -Laplacian, see, e.g., [6].

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In this paper, by a solution of (1.1) we mean a function  $x$ , defined on some ray  $[\tau_x, \infty)$ ,  $\tau_x \geq 0$ , such that its quasiderivative  $x^{[1]}$ , i.e. the function

$$x^{[1]}(t) = a(t)|x'(t)|^\alpha \operatorname{sgn} x'(t), \quad (1.3)$$

is continuously differentiable and satisfies (1.1) for any  $t \geq \tau_x$ . Since  $\alpha < \beta$ , the initial value problem associated to (1.1) has a unique local solution, that is, a solution  $x$  such that  $x(\bar{t}) = x_0, x'(\bar{t}) = x_1$  for arbitrary numbers  $x_0, x_1$  and any  $\bar{t} \geq 0$ . Moreover, in view of the regularity of the functions  $a, b$ , any local solution of (1.1) is continuable to infinity, see, e.g., [27, Section 3] or [18, Theorem 9.4].

As usual, a solution  $x$  of (1.1) is said to be *nonoscillatory* if  $x(t) \neq 0$  for large  $t$  and *oscillatory* otherwise. Equation (1.1) is said to be *nonoscillatory* if any solution is nonoscillatory. Since  $\alpha < \beta$ , nonoscillatory solutions of (1.1) may coexist with oscillatory ones, while this fact is impossible when  $\alpha = \beta$ , see, e.g., [18, Chapter III, Section 10].

Set

$$A(t) = \int_0^t a^{-1/\alpha}(s) ds.$$

In view of (1.2), any eventually positive solution of (1.1) is nondecreasing for large  $t$ . Moreover, the class  $\mathcal{S}$  of all eventually positive solutions of (1.1) can be divided into three subclasses, according as their asymptotic growth at infinity, see, e.g., [8]. More precisely, any solution  $x \in \mathcal{S}$  satisfies one of the following asymptotic properties:

$$\lim_{t \rightarrow \infty} x(t) = c_x, \quad (1.4)$$

$$\lim_{t \rightarrow \infty} x(t) = \infty, \quad \lim_{t \rightarrow \infty} \frac{x(t)}{A(t)} = 0, \quad (1.5)$$

$$\lim_{t \rightarrow \infty} \frac{x(t)}{A(t)} = c_x, \quad (1.6)$$

where  $c_x$  is a positive constant depending on  $x$ . Let  $x, y, z \in \mathcal{S}$  satisfy (1.4), (1.5), (1.6), respectively. Then we have for large  $t$

$$x(t) < y(t) < z(t).$$

In virtue of this fact, solutions satisfying (1.4), or (1.5), or (1.6) are referred also as *subdominant solutions*, *intermediate solutions* and *dominant solutions*, respectively, see, e.g., [4, 11, 24].

Necessary and sufficient conditions for the existence of subdominant solutions and dominant solutions depend on the convergence of the following integrals

$$J = \int_0^\infty \frac{1}{a^{1/\alpha}(s)} \left( \int_s^\infty b(r) dr \right)^{1/\alpha} ds,$$

$$Y = \int_0^\infty b(s) \left( \int_0^s \frac{1}{a^{1/\alpha}(r)} dr \right)^\beta ds.$$

**Theorem A.** *The following hold.*

- (i<sub>1</sub>) Equation (1.1) has dominant solutions if and only if  $Y < \infty$ . Moreover, for any  $c, 0 < c < \infty$ , there exists a dominant solution  $x$  such that

$$\lim_{t \rightarrow \infty} \frac{x(t)}{A(t)} = c.$$

( $i_2$ ) Equation (1.1) has subdominant solutions if and only if  $J < \infty$ . Moreover, for any  $c$ ,  $0 < c < \infty$ , there exists a subdominant solution  $x$  such that  $\lim_{t \rightarrow \infty} x(t) = c$ .

( $i_3$ ) Equation (1.1) is oscillatory if and only if  $J = \infty$ .

Claims ( $i_1$ ), ( $i_2$ ) can be found in [4,8,11,18] or [1, Theorems 3.13.11, 3.13.12]. Claim ( $i_3$ ) is the particular case of the Atkinson–Mirzov theorem, see [18, Theorem 11.4] or [19].

Hence, the interesting question arises: can these three types of nonoscillatory solutions of (1.1) simultaneously coexist? This problem has a long history. For equation

$$x'' + b(t)|x|^\beta \operatorname{sgn} x = 0, \quad \beta > 1, \quad (1.7)$$

it started sixty years ago by Moore–Nehari [21] in which it is proved that this triple coexistence is impossible and intermediate solutions of (1.7) cannot coexist with dominant solutions or subdominant ones.

For the more general equation (1.1), this study was continued in the nineties and recently, see, e.g., [3,4,8,22] and completely solved in [7, Corollary 4].

**Theorem B.** Equation (1.1) does not admit simultaneously dominant, intermediate and subdominant solutions.

Concerning intermediate solutions, in spite of many examples of equations of type (1.1), having solutions satisfying the asymptotic behavior (1.5), which can be easily produced, until now no general necessary and sufficient conditions for their existence are known. This is a difficult problem due to the lack of sharp upper and lower bounds for intermediate solutions, see, e.g., [1, page 241], [12, page 3], [17, page 2].

Our goal here is to study the existence of intermediate solutions to equation (1.1) when the function

$$F(t) = A^\gamma(t)a^{1/\alpha}(t)b(t), \quad (1.8)$$

where

$$\gamma = \frac{1 + \alpha\beta + 2\alpha}{\alpha + 1}, \quad (1.9)$$

is monotone for large  $t$ . To more understand the meaning on this assumption in the theory of oscillation to (1.1), let us consider the prototype

$$x'' + b(t)|x|^\lambda \operatorname{sgn} x = 0, \quad \lambda > 1. \quad (1.10)$$

Jasny [10] and Kurzweil [16] have proved that if for large  $t$  the function

$$F_1(t) = t^{(\lambda+3)/2}b(t) \quad \text{is nondecreasing,} \quad (1.11)$$

then every solution  $x$  of (1.10), with  $x(t_0) = 0$  and  $|x'(t_0)|$  sufficiently large,  $t_0 \geq 0$ , is oscillatory. Observe that for equation (1.10) we have  $\gamma = (\lambda + 3)/2$  and the function  $F$  in (1.8) reads as  $F_1$ . Moore and Nehari [21] have posed the question as to whether it is possible the coexistence of oscillatory solutions with nonoscillatory solutions having at least one zero.

Kiguradze [13] negatively answered this question, by proving that: *if  $F_1$  is nondecreasing for  $t \geq T$  and  $\lim_{t \rightarrow \infty} F_1(t) = \infty$ , then every solution of (1.10), with a zero at some  $\tau \geq T$ , is oscillatory.*

Later on, other criteria for the existence of an oscillatory solution to (1.10) under the assumption (1.11) are given by Kiguradze [14], Coffman and Wong [5] and Heidel and Hinton [9]. The sharpness of the monotonicity condition for  $t^{(\lambda+3)/2}b(t)$  has been noticed by

Kiguradze [13], see also Nehari [25], which have shown that (1.10) does not have (nontrivial) oscillatory solutions if  $b(t)(t \ln t)^{(\lambda+3)/2}$  is nonincreasing. Finally, concerning the nonoscillation of (1.10), Kiguradze in [13] proved that: *if the function  $t^\varepsilon F_1(t)$  is nonincreasing for large  $t$  and some  $\varepsilon > 0$ , then (1.10) is nonoscillatory.* For more details on these topics, we refer to the monographs [1, 15] and to the survey [27].

Sufficient conditions for the existence of intermediate solutions in the case here considered, can be found in [24], where the special equation (1.7) is considered when, roughly speaking, the function  $b$  is close to the function  $t^{-\nu}$ ,  $\nu > 0$  or in [23], in which the equation (1.1) is considered with  $a \equiv 1$  and  $b(t) = kt^{-\mu}(1 + \eta(t))$  for  $t \geq t_0 > 0$ , where  $\mu, k$  are positive constants and  $\eta$  is a continuous function such that  $1 + \eta(t) > 0$  for  $t \geq t_0$ .

In this paper, we present two existence results for intermediate solutions, according to the function  $F$  is for large  $t$  either nonincreasing or nondecreasing, respectively. The proof of the first result is based on certain monotonicity properties of an energy-type function, jointly with a suitable transformation. The second one is proved by using a topological limit process which enables, to obtain intermediate solutions of (1.1) as the limit of a suitable sequence of subdominant solutions. This second criterion improves an analogous one in [7, Theorem 5]. Finally, we study the claimed question in [21] on the possible coexistence between oscillatory and nonoscillatory solutions, too. This study is achieved by using the obtained existence results, jointly with some known results, which are analogue ones of Kurzweil and Kiguradze oscillation criteria. Some examples complete the paper.

## 2 Existence of intermediate solutions

In [7, Theorem 4] it is proved that if  $Y < \infty$ , then (1.1) does not have intermediate solutions. Hence, in virtue of Theorem A, a necessary condition for the existence of intermediate solutions to (1.1) is

$$J < \infty, \quad Y = \infty. \quad (2.1)$$

Using certain monotonicity properties of an energy-type function, jointly with a suitable transformation we have the following existence results.

**Theorem 2.1.** *Equation (1.1) has infinitely many intermediate solutions if  $Y = \infty$  and for  $t \geq T > 0$*

$$\text{the function } F \text{ in (1.8) is nonincreasing.} \quad (2.2)$$

**Theorem 2.2.** *Equation (1.1) has intermediate solutions if  $J < \infty$  and for  $t \geq T > 0$*

$$\text{the function } F \text{ in (1.8) is nondecreasing.} \quad (2.3)$$

**Remark 2.3.** The assumption (2.2) in Theorem 2.1 yields  $J < \infty$ . Indeed, in virtue of (2.2) we get  $F(t) \leq F(T)$  on  $[T, \infty)$ , that is

$$b(t) \leq F(T)A^{-\gamma}(t)a^{-1/\alpha}(t).$$

Since

$$(1 - \gamma) \frac{d}{dt} A^{1-\gamma}(t) = A^{-\gamma}(t)a^{-1/\alpha}(t),$$

and  $\gamma > 1$  we have

$$\int_s^\infty b(r) dr \leq (1 - \gamma)F(T) \int_s^\infty \frac{d}{dt} A^{1-\gamma}(t) dt = (\gamma - 1)F(T)A^{1-\gamma}(s)$$

or

$$\left( \int_s^\infty b(r) dr \right)^{1/\alpha} \leq hA^{(1-\gamma)/\alpha}(s) = hA^{-\beta/\alpha}(s),$$

where  $h = (\gamma - 1)^{1/\alpha} F^{1/\alpha}(T)$ . Hence

$$\int_T^\infty \frac{1}{a^{1/\alpha}(s)} \left( \int_s^\infty b(r) dr \right)^{1/\alpha} ds \leq h \int_T^\infty \frac{1}{a^{1/\alpha}(s)} A^{-\beta/\alpha}(s) ds = \frac{\alpha}{\beta - \alpha} h A^{(\alpha-\beta)/\alpha}(T),$$

which yields  $J < \infty$ .

Similarly, as we show below in the proof of Theorem 2.2, the assumption (2.3) implies  $Y = \infty$ .

**Remark 2.4.** Theorem 2.2 extends Corollary 3 and Theorem 5 of [7] where it is assumed (2.1) and an additional assumption needed for the topological limit process.

First, we prove both theorems for the particular case in which  $a \equiv 1$ , that is for the equation

$$(|x'|^\alpha \operatorname{sgn} x')' + b(t)|x|^\beta \operatorname{sgn} x = 0. \quad (2.4)$$

Later on, we extend the result to (1.1) by means of a suitable change of the independent variable. Observe that for (2.4), the integral  $Y$  is

$$\bar{Y} = \int_0^\infty s^\beta b(s) ds$$

and the function  $F$  in (1.8) becomes

$$\bar{F}(t) = t^\gamma b(t).$$

For any solution  $x$  of (2.4), define the energy function

$$E_x(t) = (tx'(t) - x(t))x^{[1]}(t) + ktb(t)|x(t)|^{\beta+1} \quad (2.5)$$

where  $x^{[1]}$  is defined in (1.3) and

$$k = \frac{\alpha + 1}{\alpha(\beta + 1)}. \quad (2.6)$$

The following lemmas are needed for proving Theorem 2.1.

**Lemma 2.5.** *We have*

$$\alpha x^{[1]}(t) \frac{d}{dt} x'(t) = x'(t) \frac{d}{dt} x^{[1]}(t).$$

Consequently, for any solution  $x$  of (2.4) we have

$$x^{[1]}(t)x''(t) = -\frac{1}{\alpha} b(t)x'(t)|x(t)|^\beta \operatorname{sgn} x(t). \quad (2.7)$$

*Proof.* Clearly, if  $f$  is a continuously differentiable function on  $[t_0, \infty)$ ,  $t_0 \geq 0$ , then for any positive constant  $\mu$  the function  $|f(t)|^{\mu+1}$  is continuously differentiable and

$$\frac{d}{dt} |f(t)|^{\mu+1} = (\mu + 1) |f(t)|^\mu f'(t) \operatorname{sgn} f(t). \quad (2.8)$$

If  $x'(t) = 0$ , the identity (2.7) is valid. Now, assume  $x'(t) \neq 0$ . We have

$$\frac{d}{dt} x^{[1]}(t) = \alpha |x'(t)|^{\alpha-1} x''(t)$$

or

$$x'(t) \frac{d}{dt} x^{[1]}(t) = \alpha x^{[1]}(t) x''(t).$$

Thus, in view of (2.4), we obtain the assertion.  $\square$

**Lemma 2.6.** *Assume that*

$$\bar{F}(t) = t^\gamma b(t) \text{ is nonincreasing for } t \geq T > 0, \quad (2.9)$$

where  $\gamma$  is given by (1.9). Then for any solution  $x$  of (2.4) we have for  $t \geq T$

$$\frac{d}{dt} E_x(t) \leq 0.$$

*Proof.* From (2.4) and (2.8) we obtain

$$\begin{aligned} \frac{d}{dt} E_x(t) &= tx''(t)x^{[1]}(t) - (tx'(t) - x(t))b(t)|x(t)|^\beta \operatorname{sgn} x(t) + kb(t)|x(t)|^{\beta+1} \\ &\quad + ktb'(t)|x(t)|^{\beta+1} + k(\beta + 1)tb(t)x'(t)|x(t)|^\beta \operatorname{sgn} x(t) \end{aligned}$$

or, in view (2.6) and Lemma 2.5

$$\begin{aligned} \frac{d}{dt} E_x(t) &= - \left( \frac{1}{\alpha} + 1 \right) tb(t)x'(t)|x(t)|^\beta \operatorname{sgn} x(t) \\ &\quad + (1 + k)b(t)|x(t)|^{\beta+1} + ktb'(t)|x(t)|^{\beta+1} + \frac{\alpha + 1}{\alpha} tb(t)x'(t)|x(t)|^\beta \operatorname{sgn} x(t) \\ &= \frac{\alpha + 1}{\alpha(\beta + 1)} \left( \gamma b(t)|x(t)|^{\beta+1} + tb'(t)|x(t)|^{\beta+1} \right). \end{aligned}$$

Using the identity

$$t^{-\gamma+1}|x(t)|^{\beta+1} \frac{d}{dt} (t^\gamma b(t)) = \gamma b(t)|x(t)|^{\beta+1} + tb'(t)|x(t)|^{\beta+1}$$

and (2.9), the assertion follows.  $\square$

**Lemma 2.7.** *Assume (2.9). If  $x$  is a subdominant solution or an oscillatory solution of (2.4), then we have*

$$\lim_{t \rightarrow \infty} E_x(t) > 0.$$

*Proof.* Let  $x$  be an oscillatory solution of (2.4) such that  $x'$  vanishes at some  $t^* > T$ , that is  $x'(t^*) = 0$ . Hence  $E_x(t^*) = kt^*b(t^*)|x(t^*)|^{\beta+1}$ . Since  $b(t) > 0$ , we get  $E_x(t^*) > 0$ . Then, the assertion follows from Lemma 2.6.

Now, let  $x$  be a subdominant solution of (2.4). Since  $\lim_{t \rightarrow \infty} x(t)x^{[1]}(t) = 0$  and  $tx'(t)x^{[1]}(t) = t|x'(t)|^{\alpha+1}$ , the assertion again follows.  $\square$

*Proof of Theorem 2.1. Step 1.* We prove the statement for equation (2.4). As already claimed, any solution of (2.4), which is defined in a neighborhood of  $T > 0$ , is continuable to infinity, see, e.g., [27].

We show that (2.4) has solutions  $y$  for which  $E_y(\bar{t}) < 0$  at some  $\bar{t} \geq T$ . Put

$$m = \left( \frac{1}{kTb(T)(\beta + 1)} \right)^{1/\beta},$$

and consider the solution  $y$  of (2.4) satisfying the initial conditions

$$y(T) = u, \quad y'(T) = \mu, \quad (2.10)$$

where  $\mu$  is a parameter such that

$$0 < \mu < \left( \frac{\beta m}{T(\beta + 1)} \right)^{\beta/(\beta-\alpha)}. \quad (2.11)$$

From (2.5) we get

$$E_y(T) = T\mu^{\alpha+1} - \mu^\alpha u + kTb(T)u^{\beta+1}. \quad (2.12)$$

Define

$$\varphi(u) = T\mu^{\alpha+1} - \mu^\alpha u + kTb(T)u^{\beta+1}. \quad (2.13)$$

A standard calculation shows that the point  $\bar{u}$  such that

$$\bar{u}^\beta = \frac{\mu^\alpha}{kTb(T)(\beta + 1)}, \quad (2.14)$$

is a point of minimum for  $\varphi$ . Moreover, we have

$$\begin{aligned} \varphi(\bar{u}) &= T\mu^{\alpha+1} - m \mu^{\alpha(\beta+1)/\beta} + \frac{m}{\beta + 1} \mu^{\alpha(\beta+1)/\beta} \\ &= T\mu^{\alpha+1} - \frac{m\beta}{\beta + 1} \mu^{\alpha(\beta+1)/\beta} \end{aligned}$$

or

$$\varphi(\bar{u}) = \mu^{\alpha+1} \left( T - \frac{m\beta}{\beta + 1} \mu^{(\alpha-\beta)/\beta} \right).$$

In view of (2.11), we have

$$T - \frac{m\beta}{\beta + 1} \mu^{(\alpha-\beta)/\beta} < 0,$$

thus  $\varphi(\bar{u}) < 0$ . Hence, in view of (2.12), (2.13) and Lemma 2.6, equation (2.4) has solutions  $y$  such that

$$E_y(t) < 0 \quad \text{on } [T, \infty). \quad (2.15)$$

Using Lemma 2.7,  $y$  is neither an oscillatory solution nor a subdominant solution. In view of (2.15), from (2.5) we obtain  $y(t) > 0$ ,  $y'(t) > 0$  on  $[T, \infty)$ . Since  $\bar{Y} = \infty$ , from Theorem A equation (2.4) does not have dominant solutions. Hence,  $y$  is an intermediate solution of (2.4). Since there are infinitely many solutions which satisfy (2.10) with the choice of  $\mu$  taken with (2.11), the assertion is proved for equation (2.4).

*Step 2.* Now, we extend the assertion to the more general equation (1.1). The change of the independent variable [11, Section 3]

$$r(t) = \int_0^t a^{-1/\alpha}(\tau) d\tau, \quad v(r) = x(t(r)), \quad (2.16)$$

transforms (1.1) to

$$\frac{d}{dr} (|\dot{v}|^\alpha \operatorname{sgn} \dot{v}) + c(r)|v|^\beta \operatorname{sgn} v = 0, \quad (2.17)$$

where  $t(r)$  is the inverse function of  $r(t)$ , the function  $c$  is given by

$$c(r) = a^{1/\alpha}(t(r))b(t(r))$$

and the symbol  $\dot{\phantom{v}}$  denotes the derivative with respect to the variable  $r$ .

Thus, (2.16) transforms (1.1) into (2.17), i.e. into an equation of type (2.4). Moreover, (1.1) has intermediate solutions if and only if (2.17) has intermediate solutions and for (2.17) the condition (2.9) reads as (2.2). Thus, the assertion follows reasoning as in the first part of the proof. The details are left to the reader.  $\square$

The following lemma is needed for proving Theorem 2.2.

**Lemma 2.8.** *Assume that*

$$\bar{F}(t) = t^\gamma b(t) \quad \text{is nondecreasing for } t \geq T > 0, \quad (2.18)$$

where  $\gamma$  is given by (1.9). Then any subdominant solution  $x \in \mathbb{S}$  satisfies on the whole interval  $[T, \infty)$

$$x(t) > 0, \quad x'(t) > 0$$

and

$$0 < x(t) \leq \varphi(t), \quad 0 < x'(t) \leq M_\varphi, \quad (2.19)$$

where

$$\varphi(t) = c (F(t))^{-1/(\beta-\alpha)} t^{\alpha/(\alpha+1)} \quad (2.20)$$

$$M_\varphi = B^{1/\alpha}(T) \frac{\varphi(T)}{\int_T^{2T} B^{1/\alpha}(r) dr}, \quad (2.21)$$

$c$  is a suitable constant which depends on  $\alpha$ ,  $\beta$  and

$$B(t) = \int_t^\infty b(r) dr. \quad (2.22)$$

*Proof.* Let  $x \in \mathbb{S}$  be a subdominant solution. By a result in [2, Lemma 5 and Theorem 3], see also [7, Theorem 2], we obtain the boundedness of  $x$  by  $\varphi$  given by (2.20). It remains to prove the boundedness of  $x'$  by the constant  $M_\varphi$  given by (2.21).

First, we claim that the function

$$\frac{(x'(t))^\alpha}{B(t)} \quad \text{is nondecreasing for } t \geq T. \quad (2.23)$$

Indeed, following [24], we get from (2.4) for  $t \geq T$

$$(x'(t))^\alpha = \int_t^\infty b(r) x^\beta(r) dr \geq x^\beta(t) \int_t^\infty b(r) dr = B(t) x^\beta(t), \quad (2.24)$$

where  $B$  is given in (2.22), and using this,

$$\begin{aligned} \frac{d}{dt} \frac{(x'(t))^\alpha}{B(t)} &= \frac{-b(t) x^\beta(t) B(t) + b(t) (x'(t))^\alpha}{B^2(t)} \\ &= \frac{b(t) ((x'(t))^\alpha - x^\beta(t) B(t))}{B^2(t)} \geq 0. \end{aligned}$$

Now, using (2.23) we have

$$\begin{aligned} x(2T) - x(T) &= \int_T^{2T} x'(r) dr = \int_T^{2T} B^{1/\alpha}(r) \left( \frac{(x'(r))^\alpha}{B(r)} \right)^{1/\alpha} dr \\ &\geq \left( \frac{(x'(T))^\alpha}{B(T)} \right)^{1/\alpha} \int_T^{2T} B^{1/\alpha}(r) dr \end{aligned}$$

or

$$\frac{(x'(T))^\alpha}{B(T)} \leq \left( \frac{x(2T) - x(T)}{\int_T^{2T} B^{1/\alpha}(r) dr} \right)^\alpha.$$

Since  $x'$  is decreasing on  $[T, \infty)$ , that is  $x'(t) < x'(T)$  for  $t > T$ , we get

$$x'(t) < x'(T) \leq B^{1/\alpha}(T) \frac{\varphi(2T)}{\int_T^{2T} B^{1/\alpha}(r) dr},$$

which yields the constant in (2.21).  $\square$

*Proof of Theorem 2.2.* First, we prove the statement for equation (2.4). The argument is similar to the one in [7, Theorem 3]. Since  $J < \infty$ , in view of Theorem A, for any  $n > 0$  equation (2.4) has a subdominant solution  $x_n$  such that

$$\lim_{t \rightarrow \infty} x_n(t) = n,$$

In virtue of Lemma 2.8, the sequences  $\{x_n\}$ ,  $\{x'_n\}$  are equibounded and equicontinuous on every finite subinterval of  $[T, \infty)$ . Hence there exists a uniformly converging subsequence  $\{x_{n_j}\}$  to a function  $x$  such that  $\{x'_{n_j}\}$  uniformly converges on every finite subinterval of  $[T, \infty)$ . Clearly,  $x$  is an unbounded solution of (2.4).

Now, let us show that  $Y = \infty$ . In virtue of (2.18) we have for  $t \geq T$

$$t^\gamma b(t) \geq T^\gamma b(T) = M_T.$$

Thus

$$\int_T^\infty b(r) r^\beta dr \geq M_T \int_T^\infty r^{\beta-\gamma} dr. \quad (2.25)$$

Since  $\beta > \alpha$ , we have

$$\beta - \gamma = \frac{\beta - 2\alpha - 1}{\alpha + 1} \geq -1.$$

Hence, from (2.25) we get  $Y = \infty$ . Hence, in virtue of Theorem A, the solution  $x$  is an intermediate solution of (2.4). Reasoning as in the proof of Theorem 2.1 – Step 2, we get the assertion for (1.1).  $\square$

### 3 Oscillatory and nonoscillatory solutions

The properties of the function  $F$ , given in (1.8), plays an important role also in studying the existence of oscillatory solutions and their coexistence with nonoscillatory ones. We recall the following two results.

**Theorem 3.1.** *The equation (1.1) has an oscillatory solution if the function  $F$ , given in (1.8), is non-decreasing for  $t \geq T > 0$ .*

**Theorem 3.2.** *The equation (1.1) is nonoscillatory if there exists  $\varepsilon > 0$  such that the function*

$$F(t)A^\varepsilon(t) \quad \text{is nonincreasing for } t \geq T > 0. \quad (3.1)$$

Theorem 3.1 is Kurzweil–Jasný–Mirzov theorem and it is proved in [20], see also [18, Theorem 13.1.]. Theorem 3.2 is Kiguradze–Skhalyakho theorem and it is proved in [26], see also [18, Theorem 14.3.].

Combining these results with Theorem A, Theorem 2.1 and Theorem 2.2, we get the following.

**Corollary 3.3.** *Assume that the function  $F$ , given in (1.8), is nonincreasing for large  $t \geq T > 0$ . Then the following statements hold.*

- (i<sub>1</sub>) *If  $Y = \infty$ , then (1.1) has both intermediate solutions and subdominant solutions and no dominant solutions.*
- (i<sub>2</sub>) *If  $Y < \infty$ , then (1.1) has both subdominant solutions and dominant solutions and no intermediate solutions.*

*In addition, if (3.1) is valid for some  $\varepsilon > 0$ , then (1.1) does not have oscillatory solutions.*

*Proof.* Claim (i<sub>1</sub>). In view of Remark 2.3, we have  $J < \infty$ . Hence, the assertion follows from Theorem A and Theorem 2.1. Claim (i<sub>2</sub>). Again from Remark 2.3, we have  $J < \infty$ . Hence the assertion follows from Theorem A and Theorem B. Finally, the last assertion follows from Theorem 3.2.  $\square$

**Corollary 3.4.** *Assume that the function  $F$ , given in (1.8), is nondecreasing for  $t \geq T > 0$ . If  $J < \infty$ , then (1.1) has intermediate solutions, subdominant solutions and oscillatory solutions. Moreover (1.1) does not have dominant solutions.*

*Proof.* The assertion follows from Theorem A, Theorem 2.2 and Theorem 3.1. The details are left to the reader.  $\square$

**Corollary 3.5.** *Assume that the function  $F$ , given in (1.8), is constant for  $t \geq T > 0$ . Then (1.1) has both oscillatory solutions and subdominant solutions. In addition (1.1) has also either dominant solutions or intermediate solutions, according to  $Y < \infty$  or  $Y = \infty$ .*

*Proof.* Since  $F$  is constant, say  $F(t) = h_0 > 0$ , we get for  $t \geq T$

$$b(t) = \frac{h_0}{a^{1/\alpha}(t)} A^{-\gamma}(t).$$

Since  $A'(t) = a^{-1/\alpha}(t)$ , reasoning as in Remark 2.3 we obtain  $J < \infty$ . From Theorem A and Theorem 3.1, equation (1.1) has both oscillatory solutions and subdominant solutions. The second assertion follows from Theorem A and Theorem 2.1.  $\square$

The following examples illustrate our results.

**Example 3.6.** Consider for  $t \geq 0$  the equation

$$(\sqrt{t+2}|x'|^{1/2} \operatorname{sgn} x')' + \frac{1}{(t+2)(\log(t+2) - \log 2)^{5/3}} x = 0. \quad (3.2)$$

For this equation  $\alpha = 1/2$ ,  $\beta = 1$ , so  $\gamma = 5/3$ ,  $A(t) = \log(t+2) - \log 2$  and  $F(t) = 1$ . Moreover, we have

$$Y = \int_0^\infty \frac{1}{(s+2)(\log(s+2) - \log 2)^{2/3}} ds = \infty.$$

Hence, in view of Corollary 3.5, equation (3.2) has oscillatory solutions, subdominant solutions and intermediate solutions. Further, (3.2) does not admit dominant solutions.

**Example 3.7.** Consider for  $t \geq 0$  the equation

$$(\sqrt{t+2}|x'|^{1/2} \operatorname{sgn} x')' + \frac{1}{(t+2)^2(\log(t+2) - \log 2)^{5/3}} x = 0. \quad (3.3)$$

In this case, as in Example 3.6, we have  $\gamma = 5/3$  and  $A(t) = \log(t+2) - \log 2$ . Thus  $F(t) = (t+2)^{-1}$  and

$$F(t)A^\varepsilon(t) = \frac{(\log(t+2) - \log 2)^\varepsilon}{t+2}.$$

Hence, (3.1) is valid. By Corollary 3.3, any solution of (3.3) is nonoscillatory. Moreover, (3.3) has subdominant solutions. A standard calculation shows that  $Y < \infty$ . Consequently, (3.3) has also dominant solutions and, in virtue of Theorem B, does not admit intermediate solutions.

**Example 3.8.** Consider for  $t \geq 0$  the equation

$$x'' + \frac{1}{4(t+1)^{(\beta+3)/2}} |x|^\beta \operatorname{sgn} x = 0, \quad \beta > 1. \quad (3.4)$$

Equation (3.4) has been carefully investigated in [21], especially as regards the coexistence between oscillatory and nonoscillatory solutions. For (3.4) we have  $\gamma = (\beta+3)/2$ ,  $A(t) = t$  and

$$F(t) = \frac{1}{4} \left( \frac{t}{t+1} \right)^{(\beta+3)/2}.$$

Moreover, we obtain

$$Y = \frac{1}{4} \int_0^\infty \frac{s^\beta}{(s+1)^{(\beta+3)/2}} ds, \quad J = \frac{1}{2(\beta+1)} \int_0^\infty (s+1)^{-(\beta+1)/2} ds$$

Since  $\beta > 1$ , a standard calculation gives  $Y = \infty$ ,  $J < \infty$ . From Corollary 3.4, equation (3.4) has oscillatory solutions, subdominant solutions and intermediate solutions (one of them is  $x(t) = \sqrt{t+1}$ ). Moreover, (3.4) does not have dominant solutions.

**Example 3.9.** Consider for  $t \geq 0$  the equation

$$x'' + (t+2)^{-5/2} \frac{1}{4 \log(t+2)} x^2 \operatorname{sgn} x = 0. \quad (3.5)$$

For (3.5) we have  $\alpha = 1$ ,  $\beta = 2$ , so  $\gamma = 5/2$ ,  $A(t) = t$  and

$$F(t) = \left( \frac{t}{t+2} \right)^{5/2} \frac{1}{4 \log(t+2)}.$$

Thus,  $F$  is eventually nonincreasing. A standard calculation shows that  $Y = \infty$ . Hence, in virtue of Corollary 3.3, equation (3.5) has subdominant solutions and intermediate solutions. One can check that one of intermediate solutions is  $x(t) = \sqrt{t+2} \log(t+2)$ . Moreover, (3.5) does not have dominant solutions. Since the function

$$F(t)A^\varepsilon(t) = \left( \frac{t}{t+2} \right)^{5/2} \frac{t^\varepsilon}{4 \log(t+2)}$$

is eventually increasing for any  $\varepsilon > 0$ , the last statement in Corollary 3.3 cannot be applied and so the existence of an oscillatory solution of (3.5) is an open problem.

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