Adaptive isogeometric methods with hierarchical splines: Optimality and convergence rates

ADAPTIVE ISOGEOMETRIC METHODS WITH HIERARCHICAL SPLINES: OPTIMALITY AND CONVERGENCE RATES

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We consider an adaptive isogeometric method (AIGM) based on (truncated) hierarchical B-splines and continue the study of its numerical properties. We prove that our AIGM is optimal in the sense that delivers optimal convergence rates as soon as the solution of the underlying partial differential equation belongs to a suitable approximation class. The main tool we use is the theory of adaptive methods, together with a local upper bound for the residual error indicators based on suitable properties of a well selected quasi-interpolation operator on hierarchical spline spaces.

1. Introduction

The use of adaptivity in the approximation of partial differential equations has a long tradition. Adaptive schemes are particularly important in all those problems where the solution we try to approach is not regular, or develop singularities along the simulation. When looking at isogeometric methods,[18] or more generally at methods based on splines, adaptivity and the ability to locally refine the resolution is of paramount importance since the tensor-product structure of the underlying spline construction is far too restrictive in the context of approximation of partial differential equations (PDEs).

Within the isogeometric framework, it is then natural to concentrate on locally refinable splines that can be suitably used both as design tool in geometric modelling and for the approximation of solutions of PDEs. In this context, hierarchical splines are one of the most powerful tool for the local refinement in both geometry and analysis, and for this reason they are gaining more and more importance in the field of isogeometric analysis. Thanks to the definition of the truncated basis,[16] hierarchical splines enjoys the main ingredients needed for locally refinable splines

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to be used as an effective geometric modeling tool and their use in design has been recently proposed and developed, see Ref. 19 and 15.

On the side of spline-based numerical methods for PDEs, the structure of hierarchical splines was first studied in Ref. 20 (see also Ref. 8), and is really suitable for defining multi-resolution methods and local refinement approaches. Adaptive methods using hierarchical splines were first tested in Ref. 29, and subsequently in Ref. 15 by exploiting the truncated basis. In our recent paper Ref. 10, an adaptive method based on hierarchical splines is described and its properties first analysed in mathematical terms.

Other approaches to locally refinable splines exist and are (in some case extensively) used as geometric modelling tools. Among them, surely T-splines[26] are the most used and their use in isogeometric analysis has been object of several studies,[1,13] bringing to the definition of analysis suitable[21] or equivalently dual compatible T-splines.[2,3] T-splines present a structure which is in principle more flexible than hierarchical splines, but this flexibility makes it difficult to develop an error-indicator-based adaptive framework. Only some partial results exist, see Ref. 23, 22.

The present paper is a natural continuation of Ref. 10, where an adaptive isogeometric method (AIGM) based on the following three ingredients was presented:

- the definition of *admissible* meshes as the class of meshes where a finite number of truncated hierarchical B-splines are non-zero on an element of the mesh;
- a residual error indicator, for which we prove an upper and a lower bound for hierarchical splines, of at least C^1 regularity and defined on admissible meshes;
- a refinement routine that, once a few marked elements are refined, recursively refines the neighboring elements in order to restore the admissibility of the mesh along the refinement.

In Ref. 10, we proved that the designed fully adaptive strategy converges and enjoys a contraction property. By following the steps of Ref. 5, 28, we subsequently proved a complexity estimate for the hierarchical refinement routine.[11]

The adaptivity analysis of hierarchical isogeometric methods is further extended in the present paper, where we prove that the method delivers *optimal* approximation estimates for the solution of our model problem (described in Section 2) that belong to suitable approximation classes. Our analysis follows the ideas and the framework proposed in Ref. 25, see also Ref. 12, 24. One of the main ingredient is the proof of a *local upper bound of the error* by the weighted residual error estimator. The derivation of this local version of the upper bound for the error is here presented by exploiting quasi-interpolation constructions in hierarchical spline spaces. In particular, a suitable operator (among the class of stable quasi-interpolators recently discussed in Ref. 9) onto the space of splines on tensor-product meshes is considered at any hierarchical level, and then suitably combined with THB-spline constructions by following the general approach presented in Ref. 27. It is important to notice that the choice of the truncated basis plays a key role in the construction of efficient hierarchical quasi-interpolants, and consequently, in the derivation of the certified bound. The theoretical foundations of optimal adaptive isogeometric methods based on hierarchical splines are in line with the established theory of adaptivity developed in the finite element setting. The challenging issues needed to properly frame hierarchical spline constructions in this machinery are strictly related with a propere choice of the basis functions and their suitable use in the definition of the adaptive scheme.

The paper is organized as follows. The adaptive isogeometric method and its properties are introduced in Section 2. In order to derive a local upper bound of the error, Section 3 shows how to combine a class of stable quasi-interpolation operators onto the space of splines on tensor-product meshes with the hierarchical construction. Finally, Section 4 presents the theoretical framework to link the adaptive method with optimal meshes to related approximation classes by proving the quasi-optimal cardinality of the AIGM. Our concluding remarks are presented in Section 5, while Appendix A collects the proofs of some auxiliary results.

2. The adaptive isogeometric method

We introduce a comprehensive theoretical framework for the analysis of adaptive isogeometric methods by focusing on hierarchical spline constructions. A selection of key results needed to prove the optimal convergence of the method will be reviewed. These include the efficiency and reliability of simple residual based error estimators, the contraction of the so-called *quasi-error*, as well as the complexity of the mesh refinement module.[10,11]

2.1. *Hierarchical refinement with linear complexity*

We consider a sequence of tensor-product d-variate spline spaces $V^{\ell-1} \subset V^{\ell}$, for $\ell = 1, \dots, N$, defined on a closed hypercube D in $\mathbb{R}^d$. Let $\widehat{\mathcal{B}}^{\ell}$ be the normalized tensor-product B-spline basis of degree $\mathbf{p} = (p_1, \dots, p_d)$ for the spline space V^{ℓ} defined on the grid $\widehat{G}^{\ell}$. Each grid value at level ℓ with respect to any coordinate direction i, for $i = 1, \dots, d$, appears in the corresponding knot vector as many times as specified by a certain multiplicity, that may vary from one to $p_i - 1$.[a] At level $\ell = 0$, we assume that the knot sequences are *open*, i.e., in direction i the first and the last knots are repeated $p_i + 1$ times, and a quasi-uniform tensor-product mesh. In addition, every knot of level $\ell - 1$ is also present at level ℓ at least with the same multiplicity in the corresponding coordinate direction, so that the given sequence of spline spaces is nested.

[a]The requirement of C^1 regularity is not strictly necessary, see also Ref. 10, and the C^0 case can be addressed analogously to the adaptive finite element theory.

A quadrilateral element $\widehat{Q}$ of $\widehat{G}^\ell$ is given by the Cartesian product of d open intervals between adjacent grid values. We assume that the element size $h_{\widehat{Q}} := |\widehat{Q}|^{1/d}$ satisfies

$$h_{\widehat{Q}} \lesssim \operatorname{diam}(\widehat{Q}) \lesssim h_{\widehat{Q}} \tag{2.1}$$

where we consider the symbol $\lesssim$ for any inequality which does not depend on the number N of hierarchical levels. We also consider a nested sequence of domains $\Omega^{\ell-1} \supseteq \Omega^\ell$, for $\ell = 1, \ldots, N$, that are closed subsets of D and are defined as the union of the closure of elements that belong to the tensor-product grid of the previous level. An element of level ℓ is active if it is a subset of $\widehat{\Omega}^\ell$ and does not contain any refined element at subsequent levels included in $\widehat{\Omega}^{\ell^*}$, with $\ell+1 \leq \ell^* \leq N-1$. Let

$$\widehat{\mathcal{Q}} := \left\{ \widehat{Q} \in \widehat{\mathcal{G}}^\ell, \ell = 0, \ldots, N-1 \right\} \quad \text{with} \quad \widehat{\mathcal{G}}^\ell := \left\{ \widehat{Q} \in \widehat{G}^\ell : \widehat{Q} \subset \widehat{\Omega}^\ell \wedge \widehat{Q} \not\subset \widehat{\Omega}^{\ell+1} \right\}$$

be the *hierarchical mesh* defined by the set of active elements at all levels. A mesh $\widehat{\mathcal{Q}}^*$ is a refinement of $\widehat{\mathcal{Q}}$, indicated as $\widehat{\mathcal{Q}}^* \succeq \widehat{\mathcal{Q}}$, if $\widehat{Q}^*$ is obtained from $\widehat{\mathcal{Q}}$ by splitting some of its elements via "q-adic" refinement, for some integer $q \geq 2$. For simplicity, we will consider the case of standard dyadic refinement with $q = 2$.

We consider the construction of THB-splines, whose theory was developed in Ref. 16, 17. Let

$$s = \sum_{\widehat{\beta} \in \widehat{\mathcal{B}}^{\ell+1}} c_{\widehat{\beta}}^{\ell+1}(s) \widehat{\beta},$$

be the representation of $s \in V^\ell \subset V^{\ell+1}$ with respect to the basis $\widehat{\mathcal{B}}^{\ell+1}$. The truncation of s with respect to $\widehat{\mathcal{B}}^{\ell+1}$ is defined as

$$\operatorname{trunc}^{\ell+1} s := \sum_{\widehat{\beta} \in \widehat{\mathcal{B}}^{\ell+1},\, \operatorname{supp} \widehat{\beta} \not\subseteq \widehat{\Omega}^{\ell+1}} c_{\widehat{\beta}}^{\ell+1}(s) \widehat{\beta}.$$

Definition 1. The truncated hierarchical B-spline (THB-spline) basis $\widehat{\mathcal{T}}$ with respect to the mesh $\widehat{\mathcal{Q}}$ is defined as

$$\widehat{\mathcal{T}}(\widehat{\mathcal{Q}}) := \left\{ \operatorname{Trunc}^{\ell+1} \widehat{\beta} : \widehat{\beta} \in \widehat{\mathcal{B}}^\ell \cap \widehat{\mathcal{H}}(\mathcal{Q}),\, \ell = 0, \ldots, N-1 \right\},$$

where $\operatorname{Trunc}^{\ell+1} \widehat{\beta} := \operatorname{trunc}^{N-1}(\operatorname{trunc}^{N-2}(\ldots(\operatorname{trunc}^{\ell+1}(\widehat{\beta}))\ldots))$, for any $\widehat{\beta} \in \widehat{\mathcal{B}}^\ell \cap \widehat{\mathcal{H}}(\mathcal{Q})$, and

$$\widehat{\mathcal{H}}(\mathcal{Q}) := \left\{ \widehat{\beta} \in \widehat{\mathcal{B}}^\ell : \operatorname{supp} \widehat{\beta} \subseteq \widehat{\Omega}^\ell \wedge \operatorname{supp} \widehat{\beta} \not\subseteq \widehat{\Omega}^{\ell+1},\, \ell = 0, \ldots, N-1 \right\}$$

is the hierarchical B-spline basis.

We denote the B-spline $\widehat{\beta}$ that originates the THB-spline $\widehat{\tau} = \operatorname{Trunc}^{\ell+1} \widehat{\beta}$ via the truncation mechanism as the *mother* B-spline of $\widehat{\tau}$. It will be indicated as $\widehat{\beta} := \operatorname{mot} \widehat{\tau}$.

In order to properly exploit the reduced support of THB-splines with respect to standard hierarchical B-splines, we consider the notion of *(strictly) admissible meshes* introduced in Ref. 10.

Definition 2. A mesh $\widehat{\mathcal{Q}}$ is admissible of class m if the truncated basis functions in $\widehat{\mathcal{T}}(\widehat{\mathcal{Q}})$ which take non-zero values over any element $\widehat{Q} \in \widehat{\mathcal{Q}}$ belong to at most m successive levels.

Note that the number of THB-splines which are non-zero on any element of an admissible mesh is bounded. In addition, when admissible meshes are considered, the size of the support of any truncated basis function is comparable with the size of any mesh element that overlaps its support. These two properties play an important role in the adaptivity analysis of isogeometric methods, see e.g., the proof of the a posteriori upper bound for the error in Ref. 10.

To identify a subset of admissible meshes with a certain underlying structure, we consider the auxiliary subdomains

$$\widehat{\omega}^{\ell-m+1} := \bigcup \left\{ \overline{\widehat{Q}} : \widehat{Q} \in \widehat{G}^{\ell-m+1} \wedge S(\widehat{Q}, \ell-m+1) \subseteq \widehat{\Omega}^{\ell-m+1} \right\},$$

for $\ell = m, m+1, \ldots, N-1$, defined in terms of the support extension $S(\widehat{Q}, k)$ of an element $\widehat{Q} \in \widehat{G}^{\ell}$ with respect to level k:

$$S(\widehat{Q}, k) := \left\{ \widehat{Q}' \in \widehat{G}^k : \exists\, \widehat{\beta} \in \widehat{\mathcal{B}}^k,\ \operatorname{supp} \widehat{\beta} \cap \widehat{Q}' \neq \emptyset \wedge \operatorname{supp} \widehat{\beta} \cap \widehat{Q} \neq \emptyset \right\},$$

with $0 \leq k \leq \ell$.

Definition 3. The mesh $\widehat{\mathcal{Q}}$ of active elements with respect to the domain hierarchy $\widehat{\Omega}^{\ell-1} \supseteq \widehat{\Omega}^{\ell}$, for $\ell = 1, \ldots, N$, is strictly admissible of class m if $\widehat{\Omega}^{\ell} \subseteq \widehat{\omega}^{\ell-m+1}$.

2.2. *The geometric map*

Given a quasi-uniform tensor-product $\widehat{\mathcal{Q}}_0$, we consider THB-splines on the physical domain Ω parametrized by the map

$$\mathbf{x} \in \overline{\Omega}, \quad \mathbf{x} = \mathbf{F}(\widehat{\mathbf{x}}) = \sum_{\widehat{\tau} \in \widehat{\mathcal{T}}_0} \mathbf{C}_{\widehat{\tau}} \widehat{\tau}(\widehat{\mathbf{x}}), \qquad \widehat{\mathbf{x}} \in \widehat{\Omega}^0,$$

in terms of the truncated basis functions in $\widehat{\mathcal{T}}_0$ and a corresponding set of control points $\mathbf{C}_{\widehat{\tau}} \in \mathbb{R}^d$. We assume the mapping $\mathbf{F} : \widehat{\Omega}^0 \to \overline{\Omega}$ to be a bi-Lipschitz homeomorphism:

$$\|D^\alpha \mathbf{F}\|_{L^\infty(\widehat{\Omega}^0)} \leq C_{\mathbf{F}}, \quad \|D^\alpha \mathbf{F}^{-1}\|_{L^\infty(\Omega)} \leq c_{\mathbf{F}}^{-1}, \qquad |\alpha| \leq 1, \tag{2.2}$$

where $c_{\mathbf{F}}$ and $C_{\mathbf{F}}$ are independent constants bounded away from infinity.

Given a hierarchical mesh $\widehat{\mathcal{Q}}$, we denote by $\mathcal{Q}$ its image through $\mathbf{F}$, i.e.:

$$\mathcal{Q} = \{ Q = \mathbf{F}(\widehat{Q}) : \widehat{Q} \in \widehat{\mathcal{Q}} \}.$$

We say that a mesh $\mathcal{Q}$ is (strictly) admissible if its pre-image $\widehat{\mathcal{Q}}$ is (strictly) admissible. Moreover we denote by $\mathcal{T}(\mathcal{Q})$ the set of all functions τ defined as:

$$\tau(\mathbf{x}) = \widehat{\tau}(\widehat{\mathbf{x}}), \qquad \mathbf{x} = \mathbf{F}(\widehat{\mathbf{x}}), \qquad \widehat{\tau} \in \widehat{\mathcal{T}}(\widehat{\mathcal{Q}}).$$

For further use, we split $\mathcal{T}(\mathcal{Q})$ in two parts: $\mathcal{T}(\mathcal{Q}) = \mathcal{T}_\partial(\mathcal{Q}) \cup \mathcal{T}_D(\mathcal{Q})$,

$$\mathcal{T}_\partial(\mathcal{Q}) = \{\tau \in \mathcal{T}(\mathcal{Q}) \ : \ \tau|_{\partial\Omega} \neq 0\}, \qquad \mathcal{T}_D(\mathcal{Q}) = \{\tau \in \mathcal{T}(\mathcal{Q}) \ : \ \tau|_{\partial\Omega} = 0\}.$$

In view of (2.1) and (2.2), for any element $Q \in \mathcal{Q}$, if we indicate as $h_Q = |Q|^{1/d}$, where $|Q|$ is the volume of Q, we have that $h_Q \lesssim \mathrm{diam}(Q) \lesssim h_Q$.

In order to develop the adaptivity analysis of the AIGM, we extend all the notation previously introduced to the physical domain by simply removing the $\widehat{\cdot}$. For any hierarchical level ℓ, we then consider the domains $\Omega^\ell = \mathbf{F}(\widehat{\Omega}^\ell)$ and $\omega^\ell = \mathbf{F}(\widehat{\omega}^\ell)$, as well as the mapped grid and set of active elements, $\mathcal{G}^\ell = \{Q \in \mathcal{Q} \ : \ \widehat{Q} \in \widehat{\mathcal{G}}^\ell\}$ and $G^\ell = \{Q \subset \Omega \ : \ \widehat{Q} \in \widehat{G}^\ell\}$. The support extension with respect to level k is defined as

$$S(Q,k) = \{Q' \in G^k \ : \ \widehat{Q}' \in S(\widehat{Q},k)\}\,, \tag{2.3}$$

for all $Q \in \mathcal{G}^\ell$. A mesh $\mathcal{Q}^\star$ is a refinement of $\mathcal{Q}$, indicated as $\mathcal{Q}^\star \succeq \mathcal{Q}$, when their pre-images $\widehat{\mathcal{Q}}^\star$ and $\widehat{\mathcal{Q}}$ verifies $\widehat{\mathcal{Q}}^\star \succeq \widehat{\mathcal{Q}}$.

2.3. *Model setting*

We consider the elliptic model problem:

$$-\mathrm{div}(\mathbf{A}\nabla u) = f \quad \text{in } \Omega, \qquad u\big|_{\partial\Omega} = 0, \tag{2.4}$$

where $\Omega \subset \mathbb{R}^d$, $d \geq 1$, is a bounded domain with Lipschitz boundary $\partial\Omega$ and f is any square integrable function. We assume that $\mathbf{A} = \mathbf{A}(\mathbf{x})$ is a bounded, symmetric and positive definite matrix, and, more precisely, that there exist two constants η_1, η_2 with $0 < \eta_1 \leq \eta_2$ so that

$$\forall \mathbf{x} \in \Omega,\ \boldsymbol{\xi} \in \mathbb{R}^d\ \eta_1 |\boldsymbol{\xi}|^2 \leq \mathbf{A}(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \quad \text{ and } \quad |\mathbf{A}(\mathbf{x})\boldsymbol{\xi}| \leq \eta_2 |\boldsymbol{\xi}|.$$

By considering the space of functions in $H^1(\Omega)$ with vanishing trace on $\partial\Omega$

$$\mathbb{V} := H^1_0(\Omega) := \left\{ v \in H^1(\Omega) : v\big|_{\partial\Omega} = 0 \right\},$$

and the bilinear form

$$a : \mathbb{V} \times \mathbb{V} \to \mathbb{R}, \qquad a(u,v) := \int_\Omega \mathbf{A}\nabla u \nabla v, \qquad \forall\, u, v \in \mathbb{V},$$

a weak solution of (2.4) is a function $u \in \mathbb{V}$ satisfying

$$u \in \mathbb{V} : \quad a(u,v) = \langle f, v\rangle, \quad \forall\, v \in \mathbb{V}, \tag{2.5}$$

where $\langle \cdot, \cdot \rangle$ stands for the $L^2(\Omega)$ scalar product, and we assume that $f \in \mathbb{V}^*$, the dual space to $\mathbb{V}$. The spaces $\mathbb{V}$ and $\mathbb{V}^*$ are endowed with their standard norms

$$\|u\|^2_{\mathbb{V}} = \|\nabla v\|^2_{L^2(\Omega)^d} + \|v\|^2_{L^2(\Omega)}, \quad ||r||_{\mathbb{V}^*} := \sup\left\{\langle r, v\rangle \ : \ v \in \mathbb{V}, ||v||_{\mathbb{V}} \leq 1\right\}.$$

Coercivity and continuity of $a(u,v)$, namely

$$a(u,u) \geq \alpha_1 \|u\|^2_{\mathbb{V}}, \qquad a(u,v) \leq \alpha_2 \|u\|_{\mathbb{V}} \|v\|_{\mathbb{V}}\,, \qquad u\,,v \in \mathbb{V}\,.$$

with constant α_1 and α_2, respectively, ensure the existence and uniqueness of the weak solution (2.5) through the Lax-Milgram theorem. For further use, we also define the energy norm as $|||u|||_\Omega = \sqrt{a(u,u)}$.

Given a hierarchical mesh $\mathcal{Q}$, we set $\mathbb{S}(\mathcal{Q}) = \mathrm{span}\mathcal{T}(\mathcal{Q})$ and we construct the corresponding inner approximation for $H^1_0(\Omega)$ as follows:

$$\mathbb{S}_D(\mathcal{Q}) = \mathbb{S}(\mathcal{Q}) \cap H^1_0(\Omega) = \mathrm{span}\mathcal{T}_D(\mathcal{Q}).$$

The Galerkin approximation of (2.5) is given by

$$U \in \mathbb{S}_D(\mathcal{Q}): \quad a(U,V) = \langle f, V\rangle, \quad \forall\, V \in \mathbb{S}_D(\mathcal{Q}). \tag{2.6}$$

2.4. *Refinement strategy*

In this section we recall the refinement strategy adopted in Ref. 10. This strategy is constructed in order to allow refinement while guaranteeing that the (strictly) admissibility property is preserved under refinement.

Given a strictly admissible mesh $\mathcal{Q}$, a set of marked elements $\mathcal{M}$, and the class of admissibility m, the call $\mathcal{Q}^* = \mathrm{REFINE}(\mathcal{Q}, \mathcal{M}, m)$ returns a strictly admissible mesh $\mathcal{Q}^* \succeq \mathcal{Q}$ of class m. The REFINE module consists of the commands

for all $Q \in \mathcal{Q} \cap \mathcal{M}$
    $\mathcal{Q} = \mathrm{REFINE\_RECURSIVE}(\mathcal{Q}, Q, m)$
end
$\mathcal{Q}^\star = \mathcal{Q}$

with the internal $\mathrm{REFINE\_RECURSIVE}(\mathcal{Q}, Q, m)$ procedure defined by

for all $Q' \in \mathcal{N}(\mathcal{Q}, Q, m)$
    $\mathcal{Q} = \mathrm{REFINE\_RECURSIVE}(\mathcal{Q}, Q', m)$
end
subdivide Q and
update $\mathcal{Q}$ by replacing Q with its children

and

$$\mathcal{N}(\mathcal{Q}, Q, m) := \left\{Q' \in \mathcal{G}^{\ell-m+1} : \exists\, Q'' \in S(Q, \ell-m+2), Q'' \subseteq Q'\right\},$$

when $\ell - m + 1 \geq 0$, and $\mathcal{N}(\mathcal{Q}, Q, m) = \emptyset$ for $\ell - m + 1 < 0$.

By assuming that $\widehat{G}^0$ consists of open hypercubes with side length 1, and consequently $h_{\widehat{Q}} := 2^{-\ell}$ for all $\widehat{Q} \in \widehat{G}^\ell$, the following complexity estimate was recently proved.[11] Let $\mathcal{M} := \bigcup_{j=0}^{J-1} \mathcal{M}_j$ be the set of marked elements used to generate the sequence of strictly admissible meshes $\mathcal{Q}_0, \mathcal{Q}_1, \ldots, \mathcal{Q}_J$ starting from $\mathcal{Q}_0 = G^0$, namely

$$\mathcal{Q}_j = \text{REFINE}(\mathcal{Q}_{j-1}, \mathcal{M}_{j-1}, m), \quad \mathcal{M}_{j-1} \subseteq \mathcal{Q}_{j-1} \quad \text{for } j \in \{1, \ldots, J\}.$$

Then, there exists a constant $\Lambda > 0$ so that

$$\#\mathcal{Q}_J - \#\mathcal{Q}_0 \leq \Lambda \sum_{j=0}^{J-1} \#\mathcal{M}_j , \tag{2.7}$$

with $\Lambda = \Lambda(d, p, m) := 4(4\tilde{C} + 1)^d$, where $\tilde{C} := \left(2^{-1} + \frac{2}{1-2^{1-m}} C_s\right)$ and $C_s := 2^{m-2}(2\,p+1)$. Note, however, that this result can also be generalized to the current setting by suitably taking into account the corresponding maximum local mesh size.

Finally, the *overlay* mesh $\mathcal{Q}_* := \mathcal{Q}_1 \otimes \mathcal{Q}_2$ of two meshes $\mathcal{Q}_1, \mathcal{Q}_2$ is obtained as the coarsest common refinement of $\mathcal{Q}_1$ and $\mathcal{Q}_2$. It was recently shown that the overlay of two strictly admissible meshes is still a strictly admissible mesh.[11] The following inequality holds, see e.g., Ref. 6, 23,

$$\#\mathcal{Q}_* = \#(\mathcal{Q}_1 \otimes \mathcal{Q}_2) \leq \#\mathcal{Q}_1 + \#\mathcal{Q}_2 - \mathcal{Q}_0, \tag{2.8}$$

where $\mathcal{Q}_0$ is the initial mesh configuration.

Remark 4. It should be noted that both the complexity estimate (2.7) and the overlay inequality (2.8) were obtained on the parametric domain $\widehat{\Omega}$, in Ref. 11. Clearly, they hold verbatim also on physical meshes since they are just images of parametric meshes.

2.5. *Residual based error estimates, marking, and contraction property*

Let the functional in $\mathbb{V}^*$ defined by

$$\langle r, v \rangle := \langle f, v \rangle - a(U, v),$$

so that

$$\langle r, v \rangle = a(u - U, v) \quad \forall\, v \in \mathbb{V} \qquad \text{and} \qquad a(u - U, V) = \langle r, V \rangle = 0 \quad \forall\, V \in \mathbb{S}_D,$$

be the *residual* associate to $U \in \mathbb{S}_D$. By considering the error indicator

$$\varepsilon^2_{\mathcal{Q}}(U, \mathcal{Q}) = \sum_{Q \in \mathcal{Q}} \varepsilon^2_{\mathcal{Q}}(U, Q) \qquad \text{with} \qquad \varepsilon^2_{\mathcal{Q}}(U, Q) = h^2_Q ||r||^2_{L^2(Q)}, \tag{2.9}$$

defined in terms of the quantity $r = f + \text{div}(\mathbf{A}\nabla U)$ on any element $Q \in \mathcal{Q}$, the derivation of upper and lower bounds for the Galerkin error associated to the adaptive isogeometric method here considered leads to[10]

$$\frac{1}{C_{\text{gub}}} |||u - U|||^2_\Omega \leq \varepsilon^2_{\mathcal{Q}}(U, \mathcal{Q}) \leq \frac{1}{C_{\text{glb}}} \left(|||u - U|||^2_\Omega + \text{osc}^2_{\mathcal{Q}}(U, \mathcal{Q}) \right), \tag{2.10}$$

where

$$\operatorname{osc}^2_{\mathcal{Q}}(U,\mathcal{Q}) := \sum_{Q\in\mathcal{Q}} \operatorname{osc}^2(U,Q) \quad \text{with} \quad \operatorname{osc}_{\mathcal{Q}}(U,Q) := h_Q\|r-\Pi_{\mathbf{n}} r\|_{L^2(Q)}, \tag{2.11}$$

and $\Pi_{\mathbf{n}} : L^2(Q) \to \mathbb{P}_{\mathbf{n}}$, $\mathbf{n} = (n_1,\ldots,n_d)$, denotes the L^2 projector onto the space of polynomials of degree n_j in the space direction j. Note that the degrees n_j, $j = 1,\ldots,d$ can be fixed large enough so that the oscillation are "smaller" than the error.[6] The upper bound can be obtained by suitably combining classical inequalities considered in the (adaptive) finite element setting with two key properties of the truncated basis — the partition of unity property and the bound for the number of nonzero basis functions on any mesh element that holds within the class of admissible meshes. The lower bound instead can be directly derived according to classical finite element estimates.

While Theorem 12 of Ref. 10 also provides a local version of the lower bound in (2.10), namely

$$\varepsilon_{\mathcal{Q}}(U,Q) \lesssim ||u-U||_{\mathbb{V}(Q)} + \operatorname{osc}_{\mathcal{Q}}(U,Q),$$

a local upper bound for the error will be berived in the next section.

As marking strategy, we consider the Dörfler marking[14] that identifies the set of marked elements $\mathcal{M} = \text{MARK}\left(\{\varepsilon_{\mathcal{Q}}(U,Q)\}_{Q\in\mathcal{Q}}, \mathcal{Q}\right)$, by collecting all elements with largest error indicator until

$$\varepsilon_{\mathcal{Q}}(U,\mathcal{M}) \ge \theta\, \varepsilon_{\mathcal{Q}}(U,\mathcal{Q}) \tag{2.12}$$

for a given parameter $\theta \in (0,1]$.

Let $\{\mathcal{Q}_k, \mathbb{S}_D(\mathcal{Q}_k), U_k\}_{k\ge 0}$ be the sequence of strictly admissible meshes, hierarchical spline spaces, and discrete solution computed by the AIGM for the model problem (2.4). Then, there exist $\gamma > 0$ and $0 < \alpha < 1$, independent of k such that, for all $k > 0$, the so-called *contraction property of the quasi-error*, defined as the sum of the energy error and the scaled error estimator, holds:[10]

$$|||u-U_{k+1}|||^2_\Omega + \gamma\,\varepsilon^2_{\mathcal{Q}_{k+1}}(U_{k+1},\mathcal{Q}_{k+1}) \le \alpha^2\left[|||u-U_k|||^2_\Omega + \gamma\,\varepsilon^2_{\mathcal{Q}_k}(U_k,\mathcal{Q}_k)\right]. \tag{2.13}$$

3. Localized upper bound

This section extends the analysis of the AIGM in order to provide a local version of the upper bound for the error. To this aim, a class of quasi-interpolation operators onto the space of splines on tensor-product meshes of a certain level ℓ is considered. Subsequently, these operators are suitably combined with the hierarchical construction by exploiting THB-spline representations.

3.1. *Quasi-interpolant of level ℓ*

We follow the construction of a class of L^2-stable quasi-interpolation operators onto the space of splines on tensor-product meshes recently introduced in Ref. 9. For each level $\ell = 0, \ldots, N-1$, we can consider a quasi-interpolant $\mathcal{I}^\ell : L^2(\Omega) \to V^\ell$,

$$\mathcal{I}^\ell v = \sum_{i \in I^\ell} \lambda^\ell_{\beta_i}(v)\beta_i \quad \text{with} \quad I^\ell := \left\{ i : \beta_i \in \mathcal{B}^\ell \right\}, \tag{3.1}$$

where the functionals $\{\lambda^\ell_{\beta_i}\}_{\beta_i \in \mathcal{B}^\ell}$ are defined as convex combinations of local projections onto elements that belong to the support of corresponding B-splines, see Ref. 9 for the details. One possible choice relies in choosing just one element Q_{β_i} in the support of β_i whose size is equivalent to the size of $\operatorname{supp} \beta_i$, namely

$$\frac{|\operatorname{supp} \beta_i|}{|Q_{\beta_i}|} \le C, \tag{3.2}$$

for some constant C that depends on the degree $\mathbf{p}$, and define the functional $\lambda^\ell_{\beta_i}$ as the local projection onto this element. The L^2-stability of the functionals $\{\lambda^\ell_{\beta_i}\}_{\beta_i \in \mathcal{B}^\ell}$,

$$|\lambda_{\beta_i}(v)| \lesssim |Q_{\beta_i}|^{-1/2} ||v||_{L^2(Q_{\beta_i})}, \tag{3.3}$$

follows from Theorem 5.3 in Ref. 9. In addition, $\mathcal{I}^\ell$ is a projector onto $V^\ell = \operatorname{span} \mathcal{B}^\ell$,

$$\mathcal{I}^\ell s = s, \quad \forall s \in V^\ell, \tag{3.4}$$

and

$$||\mathcal{I}^\ell v||_{L^2(Q)} \lesssim ||v||_{L^2(S(Q,\ell))}, \qquad ||v - \mathcal{I}^\ell v||_{L^2(Q)} \lesssim h_Q ||v||_{H^1(S(Q,\ell))}, \tag{3.5}$$

where $S(Q, \ell)$ is given by (2.3) when $k = \ell$.

3.2. *Hierarchical quasi-interpolant*

By considering the truncated basis for hiearchical splines, in virtue of the so-called *preservation of coefficients*,[17] hierarchical quasi-interpolants are simply defined in terms of the coefficients associated to corresponding mother B-spline functions at different hierarchical levels.[27] Consequently, if $\mathcal{I}^\ell$ is the operator defined by (3.1) in terms of the functionals $\{\lambda^\ell_{\beta_i}\}_{\beta_i \in \mathcal{B}^\ell}$, the hierarchical quasi-interpolant $\mathcal{I}_\mathcal{Q} : L^2(\Omega) \to \operatorname{span} \mathcal{T}_D(\mathcal{Q})$ can be defined as

$$\mathcal{I}_\mathcal{Q} v := \sum_{\ell=0}^{N-1} \sum_{i \in I^\ell_\mathcal{Q}} \lambda^\ell_{\beta_i}(v)\tau_i, \tag{3.6}$$

where

$$I^\ell_\mathcal{Q} := \left\{ i : \beta_i \in H^1_0(\Omega),\ \beta_i \in \mathcal{B}^\ell \cap \mathcal{H}(\mathcal{Q}) \right\}$$

is the index set of active (T)HB-splines at level ℓ that vanish at the boundary of Ω and $\beta_i = \operatorname{mot} \tau_i$.

Note that, when considering an admissible hierarchical mesh $\mathcal{Q}$, an element Q_{β_i} that satisfies (3.2) may be chosen between the active elements of level ℓ that belongs to the support of β_i. Being β_i and τ_i active basis functions in $\mathcal{H}(\mathcal{Q})$ and $\mathcal{T}(\mathcal{Q})$, respectively, at least one element of this kind exists. Since $\tau_i|_{\Omega^\ell \setminus \Omega^{\ell+1}} = \beta_i|_{\Omega^\ell \setminus \Omega^{\ell+1}}$, the element Q_{β_i} also belongs to $\operatorname{supp} \tau_i$.

By verifying (3.4), the quasi-interpolant $\mathcal{I}^\ell$, for $\ell = 0, \dots, N-1$, preserves splines on V^ℓ. Moreover, each $\lambda^\ell_{\beta_i}(s)$ used in (3.6) is locally supported on $\Omega^\ell \setminus \Omega^{\ell+1}$. Consequently, according to Theorem 4 in Ref. 27, the hierarchical quasi-interpolant $\mathcal{I}_\mathcal{Q}$ is a projector onto span $\mathcal{T}_D(\mathcal{Q})$:

$$\mathcal{I}_\mathcal{Q} s = s, \quad \forall s \in \operatorname{span} \mathcal{T}_D(\mathcal{Q}).$$

Stability and approximation properties analogous to the ones in (3.5) for the tensor-product case, can be proven also in the hierarchical setting on admissible meshes of class m.

Proposition 5. *Let $\mathcal{Q}$ be an admissible mesh of class m and $\mathcal{I}_\mathcal{Q}$ the operator defined by* (3.6)*. We have*

$$||\mathcal{I}_\mathcal{Q} v||_{L^2(Q)} \lesssim ||v||_{L^2(S(Q,\ell(Q)-m+1))}, \qquad \forall v \in L^2(\Omega), \tag{3.7}$$

$$||v - \mathcal{I}_\mathcal{Q} v||_{L^2(Q)} \lesssim h_Q ||v||_{H^1(S(Q,\ell(Q)-m+1))}, \qquad \forall v \in H^1_0(\Omega), \tag{3.8}$$

where $S(Q, \ell(Q)-m+1)$ is the support extension of $Q \in G^\ell$ with respect to level $\ell(Q)-m+1$ defined in (2.3)*.*

Proof. In view of the admissibility of the mesh and (3.3), we have

$$\begin{aligned}
||\mathcal{I}_\mathcal{Q} v||_{L^2(Q)} &\leq \sum_{\ell=\ell(Q)-m+1}^{\ell(Q)} || \sum_{i \in I^\ell_\mathcal{Q}} \lambda^\ell_{\beta_i}(v) \tau_i ||_{L^2(Q)} \\
&\lesssim \sum_{\ell=\ell(Q)-m+1}^{\ell(Q)} \sum_{i \in I^\ell_\mathcal{Q},\, \operatorname{supp} \tau_i \cap Q \neq \emptyset} |\lambda^\ell_{\beta_i}(v)| \, ||\tau_i||_{L^2(Q)} \\
&\lesssim \sum_{\ell=\ell(Q)-m+1}^{\ell(Q)} \sup_{i \in I^\ell_\mathcal{Q},\, \operatorname{supp} \tau_i \cap Q \neq \emptyset} |\lambda^\ell_{\beta_i}(v)| \, h_Q \\
&\lesssim \sum_{\ell=\ell(Q)-m+1}^{\ell(Q)} h_Q^{-1} ||v||_{L^2(S(Q,\ell(Q)))} \, h_Q \lesssim m ||v||_{L^2(S(Q,\ell(Q)-m+1))},
\end{aligned}$$

which leads to (3.7). In order to obtain (3.8) instead, we consider first the Q such that $Q \cap \partial\Omega = \emptyset$. By taking into account (3.7), we may consider a generic constant $c \in \mathbb{R}$ and proceed as follows

$$\begin{aligned}
||v - \mathcal{I}_\mathcal{Q} v||_{L^2(Q)} &\leq ||v - c - \mathcal{I}_\mathcal{Q}(v-c)||_{L^2(Q)} \leq ||v-c||_{L^2(Q)} + ||\mathcal{I}_\mathcal{Q}(v-c)||_{L^2(Q)} \\
&\lesssim ||v-c||_{L^2(Q)} + ||\mathcal{I}_\mathcal{Q}(v-c)||_{L^2(S(Q,\ell(Q)-m+1))} \\
&\lesssim h_Q ||v||_{H^1(Q)} + h_{S(Q,\ell(Q)-m+1)} ||v||_{H^1(S(Q,\ell(Q)-m+1))}.
\end{aligned}$$

For Q on the boundary, i.e., such that $Q \cap \partial\Omega \neq \emptyset$, then (3.8) can be easily obtained via Poincaré inequality, see Ref. 4. □

Let

$$\mathcal{R} := \mathcal{R}_{\mathcal{Q} \to \mathcal{Q}^*} \tag{3.9}$$

be the set of elements of $\mathcal{Q}$ that do not belong to $\mathcal{Q}^*$ and $\Omega_{\mathcal{R}} := \bigcup \{\overline{Q} : Q \in \mathcal{R}\}$. The index sets $I^\ell_{\mathcal{Q}}$ and $I^\ell_{\mathcal{Q}^*}$ of active THB-splines $\tau \in \mathcal{T}_D(\mathcal{Q})$ and $\tau^* \in \mathcal{T}_D(\mathcal{Q}^*)$, respectively, can be represented in terms of two disjoint sets as follows:

$$I^\ell_{\mathcal{Q}} = I^{\ell,1}_{\mathcal{Q}} \cup I^{\ell,2}_{\mathcal{Q}} \quad \text{and} \quad I^\ell_{\mathcal{Q}^*} = I^{\ell,1}_{\mathcal{Q}^*} \cup I^{\ell,2}_{\mathcal{Q}^*}, \tag{3.10}$$

where

$$I^{\ell,1}_{\mathcal{Q}} := \left\{ i \in I^\ell_{\mathcal{Q}} : \exists\, j \in I^\ell_{\mathcal{Q}^*} : \tau_i|_{\Omega_{\mathcal{Q}}} = \tau^*_j|_{\Omega_{\mathcal{Q}}} \right\}, \qquad I^{\ell,2}_{\mathcal{Q}} := I^\ell_{\mathcal{Q}} \setminus I^{\ell,1}_{\mathcal{Q}},$$

and, analogously,

$$I^{\ell,1}_{\mathcal{Q}^*} := \left\{ j \in I^\ell_{\mathcal{Q}^*} : \exists\, i \in I^\ell_{\mathcal{Q}} : \tau^*_j|_{\Omega_{\mathcal{Q}}} = \tau_i|_{\Omega_{\mathcal{Q}}} \right\}, \qquad I^{\ell,2}_{\mathcal{Q}^*} := I^\ell_{\mathcal{Q}^*} \setminus I^{\ell,1}_{\mathcal{Q}^*}.$$

While $I^{\ell,1}_{\mathcal{Q}}$, collects the indices of THB-splines defined over $\mathcal{Q}$ that either remain unchanged on $\mathcal{Q}^*$ or are further truncated on $\Omega_{\mathcal{R}}$, $I^{\ell,2}_{\mathcal{Q}}$ contains the indices of THB-splines in $\mathcal{T}_D(\mathcal{Q})$ that are no more active in $\mathcal{T}_D(\mathcal{Q}^*)$. The viceversa holds true for $I^{\ell,1}_{\mathcal{Q}^*}$ and $I^{\ell,2}_{\mathcal{Q}^*}$.

Let $w \in \mathbb{S}_D(\mathcal{Q}^*)$ be a hierarchical spline defined on the refined mesh $\mathcal{Q}^* \succeq \mathcal{Q}$. The approximation $\mathcal{I}_{\mathcal{Q}} w$ given by (3.6) coincides with w on the set of elements of $\mathcal{Q}$ that were not refined. This can be verified by observing that the effect of the truncation on basis functions of $\mathcal{T}_D(\mathcal{Q})$ to generate the new basis $\mathcal{T}_D(\mathcal{Q}^*)$, as well as the newly inserted basis functions, does not have influence on the set of elements of $\mathcal{Q}$ that are also present in $\mathcal{Q}^*$. This result is formalized in the following proposition.

Proposition 6. *Let $\mathcal{Q}$ and $\mathcal{Q}^*$ be two admissible meshes so that $\mathcal{Q}^* \succeq \mathcal{Q}$. If $w \in \mathbb{S}_D(\mathcal{Q}^*)$ and $\mathcal{R}$ is the set of elements of $\mathcal{Q}$ that are refined in $\mathcal{Q}^*$ as defined by* (3.9), *we have*

$$\mathcal{I}_{\mathcal{Q}} w = w \quad \text{in} \quad \Omega_{\mathcal{Q}} := \Omega \setminus \Omega_{\mathcal{R}}. \tag{3.11}$$

Proof. We consider the form (3.6) of the hierarchical quasi-interpolants

$$\mathcal{I}_{\mathcal{Q}} w = \sum_{\ell=0}^{N-1} \sum_{i \in I^\ell_{\mathcal{Q}}} \lambda^\ell_{\beta_i}(w)\tau_i \quad \text{and} \quad \mathcal{I}_{\mathcal{Q}^*} w = \sum_{\ell=0}^{M-1} \sum_{i \in I^\ell_{\mathcal{Q}^*}} \lambda^{\ell,*}_{\beta_i}(w)\tau_i, \tag{3.12}$$

defined on the meshes $\mathcal{Q}$ and $\mathcal{Q}^*$, in terms of the truncated bases

$$\mathcal{T}_D(\mathcal{Q}) = \left\{\tau_i : i \in I^\ell_{\mathcal{Q}}\right\}_{\ell=0,\ldots,N-1} \quad \text{and} \quad \mathcal{T}_D(\mathcal{Q}^*) = \left\{\tau^*_j : j \in I^\ell_{\mathcal{Q}^*}\right\}_{\ell=0,\ldots,M-1},$$

respectively. The inner sums in (3.12) can be subdivided according to (3.10) as follows:

$$\sum_{i \in I^{\ell,1}_{\mathcal{Q}}} \lambda^\ell_{\beta_i}(w)\tau_i + \sum_{i \in I^{\ell,2}_{\mathcal{Q}}} \lambda^\ell_{\beta_i}(w)\tau_i, \qquad \sum_{i \in I^{\ell,1}_{\mathcal{Q}^*}} \lambda^{\ell,*}_{\beta_i}(w)\tau_i + \sum_{i \in I^{\ell,2}_{\mathcal{Q}^*}} \lambda^{\ell,*}_{\beta_i}(w)\tau_i.$$

For any THB-spline of level ℓ that belongs to $\mathcal{T}_D(\mathcal{Q}) \cap \mathcal{T}_D(\mathcal{Q}^*)$ there exists at least one element $Q_{\beta_i} \in \mathcal{R}$ of level ℓ contained in its support. By defining $\lambda^\ell_{\beta_i}$ as the local projection onto this element, since THB-splines preserve the coefficients of their mother functions, for all τ_i of index $i \in I_{\mathcal{Q}}^{\ell,1}$, there exists $j \in I_{\mathcal{Q}^*}^{\ell,1}$ so that

$$\tau_i|_{\Omega_{\mathcal{Q}}} = \tau_j^*|_{\Omega_{\mathcal{Q}}} \quad \text{and} \quad \lambda_{\beta_i}^{\ell} = \lambda_{\beta_i}^{\ell,*}. \tag{3.13}$$

Moreover, any THB-spline whose index $i \in I_{\mathcal{Q}}^{\ell,2}$ or $i \in I_{\mathcal{Q}^*}^{\ell,2}$ is so that $\tau_i|_{\Omega_{\mathcal{Q}}} = 0$, and consequently,

$$\sum_{i \in I_{\mathcal{Q}}^{\ell,2}} \lambda_{\beta_i}^{\ell}(w)\tau_i|_{\Omega_{\mathcal{Q}}} = \sum_{j \in I_{\mathcal{Q}^*}^{\ell,2}} \lambda_{\beta_j}^{\ell,*}(w)\tau_j^*|_{\Omega_{\mathcal{Q}}} = 0. \tag{3.14}$$

In view of (3.13) and (3.14), since $\mathcal{I}_{\mathcal{Q}^*}$ is a projector onto the hierarchical spline space span $\mathcal{T}_D(\mathcal{Q}^*)$, we obtain

$$\mathcal{I}_{\mathcal{Q}} w|_{\Omega_{\mathcal{Q}}} = \sum_{\ell=0}^{N-1} \sum_{i \in I_{\mathcal{Q}}^{\ell,1}} \lambda_{\beta_i}^{\ell}(w)\tau_i|_{\Omega_{\mathcal{Q}}} = \sum_{\ell=0}^{N-1} \sum_{j \in I_{\mathcal{Q}^*}^{\ell,1}} \lambda_{\beta_j}^{\ell,*}(w)\tau_j^*|_{\Omega_{\mathcal{Q}}} = \mathcal{I}_{\mathcal{Q}^*} w|_{\Omega_{\mathcal{Q}}} = w|_{\Omega_{\mathcal{Q}}}.$$

□

Remark 3.1. It is worth noting that all the results in this section easily extend to the case where there are no Dirichlet conditions, or Dirichlet boundary conditions are imposed only on a collection of faces of the physical domain.

3.3. *A local upper bound for the error*

The main result of this section is formalized in the following lemma.

Lemma 7. *(Localized upper bound) Let $\mathcal{Q}$ and $\mathcal{Q}^*$ be two admissible meshes so that $\mathcal{Q}^* \succeq \mathcal{Q}$. The corresponding Galerkin solutions $U \in \mathbb{S}_D(\mathcal{Q})$ and $U^* \in \mathbb{S}_D(\mathcal{Q}^*)$ of problem* (2.6) *satisfy*

$$|||U - U^*|||_{\Omega}^2 \le C_{\text{lub}}\, \varepsilon_{\mathcal{Q}}^2(U, \mathcal{R}), \tag{3.15}$$

for some constant C_{lub}, where $\mathcal{R}$ is the refined set of elements defined in (3.9).

Proof. Let $\mathcal{I}_{\mathcal{Q}} : \text{span}\, \mathcal{T}_D(\mathcal{Q}) \to \text{span}\mathcal{T}_{\mathcal{D}}(\mathcal{Q}^*)$ be the operator defined in Section 3.2 and $E^* = U - U^*$. In view of (3.11) we can consider the approximation $V \in \mathbb{S}_D(\mathcal{Q})$ defined as

$$V = \begin{cases} \mathcal{I}_{\mathcal{Q}} E^* & \text{in } \Omega_{\mathcal{R}}, \\ E^* & \text{in } \Omega_{\mathcal{Q}}, \end{cases}$$

so that

$$E^* - V = \begin{cases} E^* - \mathcal{I}_{\mathcal{Q}} E^* & \text{in } \Omega_{\mathcal{R}}, \\ 0 & \text{in } \Omega_{\mathcal{Q}}. \end{cases} \tag{3.16}$$

By combining

$$a(E^*, E^*) = a(U, E^*) - a(U^*, E^*)$$

with $a(E^*, E^*) = a(E^*, E^* - V)$ and taking into account (3.16), we have

$$a(E^*, E^*) \le \sum_{Q\in\mathcal{R}} ||r(U)||_{L^2(Q)} ||E^* - \mathcal{I}_\mathcal{Q} E^*||_{L^2(Q)},$$

which in turn, due to (2.9) and (3.8), reduces to

$$\begin{aligned} |||E^*|||_\Omega^2 = a(E^*, E^*) &\lesssim \sum_{Q\in\mathcal{R}} \varepsilon_\mathcal{Q}(U,Q) ||E^*||_{H^1(S(Q,\ell(Q)-m+1))} \\ &\lesssim \left(\sum_{Q\in\mathcal{R}} \varepsilon_\mathcal{Q}^2(U,Q)\right)^{1/2} \left(\sum_{Q\in\mathcal{R}} ||E^*||^2_{H^1(S(Q,\ell(Q)-m+1))}\right)^{1/2} \\ &\lesssim \varepsilon_\mathcal{Q}(U,\mathcal{R})\, |||E^*|||_\Omega, \end{aligned}$$

which directly implies (3.15). □

4. Approximation classes and optimality

By exploting the different ingredients presented in the previous sections, after introducing the notion of *total errror* and *approximation class*, we conclude the analysis of the adaptive method by proving the quasi-optimalilty result.

4.1. *Total error and approximation classes*

Let $\mathbb{Q}^m$ be the set of all possible strictly admissible refinements of class m of an initial quasi-uniform tensor-product configuration $\mathcal{Q}_0$. We consider the set $\mathbb{Q}^m_M \subset \mathbb{Q}^m$ of refinements of $\mathcal{Q}_0$ whose number of elements differs at most M by the one of $\mathcal{Q}_0$, namely

$$\mathbb{Q}^m_M := \{\mathcal{Q} \in \mathbb{Q}^m : \#\mathcal{Q} - \#\mathcal{Q}_0 \le M\}.$$

By following the optimality analysis of adaptive finite element methods, see e.g. Ref. 12, we consider the notion of *total error* for $U \in \mathbb{S}_D(\mathcal{Q})$

$$|||u - U|||_\Omega^2 + \mathrm{osc}^2_\mathcal{Q}(U, \mathcal{Q}) \tag{4.1}$$

in order to define the *approximation class* $\mathbb{A}_s$ as

$$\mathbb{A}_s := \left\{(v, f, \mathbf{A}) : |v, f, \mathbf{A}|_s := \sup_{M>0} (M^s \sigma(M; v, f, \mathbf{A})) < \infty\right\},$$

for $s > 0$, where

$$\sigma(M; u, f, \mathbf{A}) := \inf_{\mathcal{Q}\in\mathbb{Q}^m_M} \sigma_e(\mathcal{Q}; u, f, \mathbf{A})^{1/2}$$

characterizes the quality of the best approximation in $\mathbb{Q}^m_M$ with respect to the so-called *best total error*

$$\sigma_e(\mathcal{Q}; u, f, \mathbf{A}) := \inf_{V\in\mathbb{S}_D(\mathcal{Q})} \left(|||u - V|||_\Omega^2 + \mathrm{osc}^2_\mathcal{Q}(V, \mathcal{Q})\right).$$

Remark 8. A deep understanding of these classes is due and we leave this to future work. Indeed, how the approximation classes depend on the regularity of the spline spaces and how approximation classes for classical finite elements and splines compare are relevant questions that deserve attention. Another parameter hidden in the definition of admissible meshes which may affect the approximation classes is the class m. How the approximation classes depend (or not) on m remains an open issue.

Remark 9. Note that an alternative definition of $\mathbb{A}_s$ can be derived as follows, see also Ref. 25 . For any given $\epsilon > 0$, let consider the set of strictly admissible meshes

$$\left\{\mathcal{Q} : \inf_{V \in \mathbb{S}_D(\mathcal{Q})} \left(|||u - V|||_\Omega^2 + \operatorname{osc}_\mathcal{Q}^2(V, \mathcal{Q}) \right) \le \epsilon^2 \right\}.$$

Note that this is a non-empty set in view of the convergence of the considered AIGM. Let $\mathcal{Q}_\epsilon$ be the coarsest mesh of this kind. We have

$$\inf_{V \in \mathbb{S}_D(\mathcal{Q}_\epsilon)} \left(|||u - V|||_\Omega^2 + \operatorname{osc}_\mathcal{Q}^2(V, \mathcal{Q}) \right) \le \epsilon^2$$

and

$$\inf_{V \in \mathbb{S}_D(\mathcal{Q})} \left(|||u - V|||_\Omega^2 + \operatorname{osc}_\mathcal{Q}^2(V, \mathcal{Q}) \right) > \epsilon^2$$

for all $\mathcal{Q}$ such that $\#\mathcal{Q} - \mathcal{Q}_0 \le \#\mathcal{Q}_\epsilon - \mathcal{Q}_0 - 1$. Let $(v, f, \mathbf{A}) \in \mathbb{A}_s$, i.e., there exists a constant Λ_{cls} such that $|v, f, \mathbf{A}|_s \le \Lambda_{\text{cls}}$. We have

$$\begin{aligned}
\Lambda_{\text{cls}} &= \sup_{M>0} \left(M^s \sigma(M; v, f, \mathbf{A}) \right) \\
&= \sup_{M>0} \left(M^s \inf_{\mathcal{Q} \in \mathbb{Q}_M^m} \left(\inf_{V \in \mathbb{S}_D(\mathcal{Q})} \left(|||u - V|||_\Omega^2 + \operatorname{osc}_\mathcal{Q}^2(V, \mathcal{Q}) \right) \right)^{1/2} \right) \\
&> \sup_{M>0} \left(M^s \epsilon \right) = \left(\#\mathcal{Q}_\epsilon - \mathcal{Q}_0 - 1 \right)^s \epsilon \quad \Rightarrow \quad \#\mathcal{Q}_\epsilon - \mathcal{Q}_0 - 1 < \Lambda_{\text{cls}}^{1/s} \, \epsilon^{-1/s}.
\end{aligned}$$

We conclude that $(v, f, \mathbf{A}) \in \mathbb{A}_s$ if and only if there exists a constant Λ_{cls} such that for all $\epsilon > 0$ there exist a strictly admissible mesh $\mathcal{Q}_\epsilon \succeq \mathcal{Q}_0$ and $V_\epsilon \in \mathbb{S}_D(\mathcal{Q}_\epsilon)$ such that the corresponding total error is less or equal to ϵ^2 and the number of elements of $\mathcal{Q}_\epsilon$ differs at most $\Lambda_{\text{cls}}^{1/s} \epsilon^{-1/s}$ by the one of $\mathcal{Q}_0$, namely

$$|||u - V_\epsilon|||_\Omega^2 + \operatorname{osc}_{\mathcal{Q}_\epsilon}^2(V_\epsilon, \mathcal{Q}_\epsilon) \le \epsilon^2 \quad \text{and} \quad \#\mathcal{Q}_\epsilon - \#\mathcal{Q}_0 \le \Lambda_{\text{cls}}^{1/s} \epsilon^{-1/s}.$$

In other words, the inequality on the right provides an upper bound for the number of elements of the coarsest mesh $\mathcal{Q}_\epsilon$ whose total error is less or equal than ϵ^2. This characterization of the approximation class $\mathbb{A}_s$ is exploited in the proof of Lemma 12 in Appendix A.

Lemma 10. *(Quasi-optimality of total error) Let $\mathcal{Q} \in \mathbb{Q}^m$ be a strictly admissible mesh. The total error associated to the Galerking solution $U \in \mathbb{S}_D(\mathcal{Q})$ of problem* (2.6) *on $\mathbb{S}_D(\mathcal{Q})$ satisfies*

$$|||u - U|||_\Omega^2 + \operatorname{osc}_\mathcal{Q}^2(U, \mathcal{Q}) \lesssim \inf_{V \in \mathbb{S}_D(\mathcal{Q})} \left(|||u - V|||_\Omega^2 + \operatorname{osc}_\mathcal{Q}^2(V, \mathcal{Q}) \right).$$

Proof. This proof follows the classical proof of quasi-optimality of total error in the theory of adaptive finite element methods, see Lemma 5.2 in 12 or Lemma 21 in 25. □

4.2. *Quasi-optimality result*

In order to prove the quasi-optimality of the AIGM in Theorem 13 below, we need the two following preliminary lemmas whose proofs, analogous to the ones of the finite element setting, are postponed to Appendix A.

If the Dörfler property (2.12) is sufficient for proving the convergence of the adaptive method[10], the design of a quasi-optimal AIGM requires to shrink the interval of admissible values for the marking parameter θ, see e.g., Ref. 12.

Lemma 11. *(Optimal marking) Let* $\mathcal{Q}$ *and* $\mathcal{Q}^*$ *be two strictly admissible meshes so that* $\mathcal{Q}^* \succeq \mathcal{Q}$*. If the total error associated to the Galerkin solution* $U^* \in \mathbb{S}_D(\mathcal{Q}^*)$ *of problem* (2.6) *on* $\mathbb{S}_D(\mathcal{Q}^*)$ *satisfies*

$$|||u - U^*|||^2_\Omega + \mathrm{osc}^2_{\mathcal{Q}^*}(U^*, \mathcal{Q}^*) \le \mu \left(|||u - U|||^2_\Omega + \mathrm{osc}^2_{\mathcal{Q}}(U, \mathcal{Q})\right), \tag{4.2}$$

with

$$\mu := \frac{1}{2}\left(1 - \frac{\theta^2}{\theta_*^2}\right), \qquad \theta_* := \sqrt{\frac{C_{\mathrm{glb}}}{1 + C_{\mathrm{lub}}(1 + \Lambda_{\mathrm{osc}})}}, \qquad \theta \in (0, \theta_*), \tag{4.3}$$

and C_{glb}*,* C_{lub}*,* Λ_{osc} *introduced in* (2.10)*, Lemma 7, and* (A.4)*, respectively, the refined set of elements* $\mathcal{R} = \mathcal{R}_{\mathcal{Q}\to\mathcal{Q}^*}$ *satisfies the Dörfler property*

$$\varepsilon_{\mathcal{Q}}(U, \mathcal{R}) \ge \theta\, \varepsilon_{\mathcal{Q}}(U, \mathcal{Q}). \tag{4.4}$$

Lemma 11 establishes the interplay of a suitable total error reduction given by (4.2), when moving from $\mathcal{Q}$ to $\mathcal{Q}^*$, with a Dörfler property (4.4) on the refined set of elements $\mathcal{R}_{Q\to\mathcal{Q}^*}$ associated to the error indicator defined over $\mathcal{Q}$. The local upper bound for the error derived in the previous section is taken into account in the proof of this lemma, see Appendix A.

As final ingredient needed for the optimality result, the following lemma also requires the marked set of elements $\mathcal{M}_k$ at step k of the adaptive loop to be of *minimal* cardinality.[12,28] As detailed in Appendix A, in order to provide a suitable bound for the cardinality of $\mathcal{M}_k$, the number of elements of the overlay mesh is bounded through (2.8). In addition, Lemmas 10 and 11 are properly exploited.

Lemma 12. *(Cardinality of* $\mathcal{M}_k$*) Let the marking parameter* θ *satisfy* $\theta \in (0, \theta_*)$ *with* θ_* *defined as in* (4.3)*, and assume that the module MARK select a set* $\mathcal{M}_k$ *of marked elements with minimal cardinality. Let* u *be the solution of the model problem* (2.4)*. If* $(u, f, \mathbf{A}) \in \mathbb{A}_s$*, the AIGM generates a sequence* $\{\mathcal{Q}_k, \mathbb{S}_D(\mathcal{Q}_k), U_k\}_{k\ge0}$ *of strictly admissible meshes, hierarchical spline spaces, and discrete solutions so that*

$$\#\mathcal{M}_k \lesssim |u, f, \mathbf{A}|_s^{1/s} \left[|||u - U_k|||^2_\Omega + \mathrm{osc}^2_{\mathcal{Q}_k}(U_k, \mathcal{Q}_k)\right]^{-\frac{1}{2s}},$$

for any $k \geq 0$*.*

The two previous lemmas can finally be used together with the complexity estimate (2.7), the (global) lower bound in (2.10), and the contraction property (2.13) to prove the quasi-optimality result, see also Ref. 12 or 25 in the finite element setting.

Theorem 13. *Let the marking parameter* θ *satisfy* $\theta \in (0, \theta_*)$ *with* θ_* *defined as in* (4.3)*, and assume that the module MARK select a set* $\mathcal{M}_k$ *of marked elements with minimal cardinality. Let* u *be the solution of the model problem* (2.4)*. If* $(u, f, \mathbf{A}) \in \mathbb{A}_s$*, the AIGM generates a sequence* $\{\mathcal{Q}_k, \mathbb{S}_D(\mathcal{Q}_k), U_k\}_{k\geq 0}$ *of strictly admissible meshes, hierarchical spline spaces, and discrete solutions so that*

$$\left[|||u - U_k|||_\Omega^2 + \mathrm{osc}^2_{\mathcal{Q}_k}(U_k, \mathcal{Q}_k)\right]^{\frac{1}{2}} \lesssim |u, f, \mathbf{A}|_s (\#\mathcal{Q}_k - \#\mathcal{Q}_0)^{-s},$$

for any $k \geq 1$*.*

Proof. First of all, by exploiting the complexity estimate (2.7) together with Lemma 12, we obtain

$$\#\mathcal{Q}_k - \#\mathcal{Q}_0 \lesssim \sum_{j=0}^{k-1} \mathcal{M}_j \lesssim |u, f, \mathbf{A}|^{\frac{1}{s}} \sum_{j=0}^{k-1} \left[|||u - U_j|||_\Omega^2 + \mathrm{osc}^2_{\mathcal{Q}_j}(U_j, \mathcal{Q}_j)\right]^{-\frac{1}{2s}}. \quad (4.5)$$

Second, the contraction inequality (2.13) implies that there exist $\gamma > 0$ and $0 < \alpha < 1$ satisfying

$$|||u - U_k|||_\Omega^2 + \gamma\, \varepsilon^2_{\mathcal{Q}_k}(U_k, \mathcal{Q}_k) \leq \alpha^{2(k-j)} \left[|||u - U_j|||_\Omega + \gamma\, \varepsilon^2_{\mathcal{Q}_j}(U_j, \mathcal{Q}_j)\right], \quad (4.6)$$

for any $0 \leq j \leq k-1$. In addition, since $\mathrm{osc}_{\mathcal{Q}_j}(U_j, \mathcal{Q}_j) \leq \varepsilon_{\mathcal{Q}_j}(U_j, \mathcal{Q}_j)$, in view of the (global) lower bound in (2.10), we have

$$\begin{aligned} |||u - U_j|||_\Omega^2 + \gamma\, \mathrm{osc}^2_{\mathcal{Q}_j}(U_j, \mathcal{Q}_j) &\leq |||u - U_j|||_\Omega^2 + \gamma\, \varepsilon_{\mathcal{Q}_j}(U_j, \mathcal{Q}_j) \\ &\leq \left(1 + \frac{\gamma}{C_{\mathrm{glb}}}\right) \left[|||u - U_j|||_\Omega^2 + \mathrm{osc}^2_{\mathcal{Q}_j}(U_j, \mathcal{Q}_j)\right], \end{aligned} \quad (4.7)$$

and, consequently

$$\begin{aligned} \sum_{j=0}^{k-1} \left[|||u - U_j|||_\Omega^2 + \mathrm{osc}^2_{\mathcal{Q}_j}(U_j, \mathcal{Q}_j)\right]^{-\frac{1}{2s}} &\lesssim \sum_{j=0}^{k-1} \left[|||u - U_j|||_\Omega^2 + \gamma\, \varepsilon^2_{\mathcal{Q}_j}(U_j, \mathcal{Q}_j)\right]^{-\frac{1}{2s}} \\ &\lesssim \left[|||u - U_k|||_\Omega^2 + \gamma\, \varepsilon^2_{\mathcal{Q}}(U_k, \mathcal{Q}_k)\right]^{-\frac{1}{2s}} \sum_{j=0}^{k-1} \alpha^{\frac{k-j}{s}} \\ &= \left[|||u - U_k|||_\Omega^2 + \gamma\, \varepsilon^2_{\mathcal{Q}}(U_k, \mathcal{Q}_k)\right]^{-\frac{1}{2s}} \sum_{j=1}^{k} \alpha^{\frac{j}{s}} \\ &\lesssim \left[|||u - U_k|||_\Omega^2 + \gamma\, \varepsilon^2_{\mathcal{Q}}(U_k, \mathcal{Q}_k)\right]^{-\frac{1}{2s}} \end{aligned}$$

in view of (4.6) and observing that $\sum_{j=1}^{k} \alpha^{\frac{j}{s}} < \infty$ since $\alpha < 1$. By taking into account the above inequality and combining (4.5), (4.6), and (4.7), we obtain

$$\begin{aligned}\#\mathcal{Q}_k - \#\mathcal{Q}_0 &\lesssim |u, f, \mathbf{A}|_s^{\frac{1}{s}} \left[|||u - U_k|||_\Omega^2 + \operatorname{osc}^2_{\mathcal{Q}_j}(U_j, \mathcal{Q}_j)\right]^{-\frac{1}{2s}} \\ &\lesssim |u, f, \mathbf{A}|_s^{\frac{1}{s}} \left[|||u - U_k|||_\Omega^2 + \gamma\, \varepsilon^2_{\mathcal{Q}_j}(U_j, \mathcal{Q}_j)\right]^{-\frac{1}{2s}} \\ &\lesssim |u, f, \mathbf{A}|_s^{\frac{1}{s}} \left[|||u - U_k|||_\Omega^2 + \gamma\, \operatorname{osc}^2_{\mathcal{Q}_j}(U_j, \mathcal{Q}_j)\right]^{-\frac{1}{2s}}\end{aligned}$$

for any $k \geq 1$, which directly leads to conclude the proof. □

5. Conclusions

This paper, together with Ref. 10 and 11, provides a comprehensive analysis of adaptive isogeometric methods based on hierarchical splines and residual-based error estimator. As it is natural, at each adaptive step, in order to restore the properties of the mesh, refinement is done "around" marked elements. Although the complexity estimates derived in Ref. 11 proves that this procedure still enjoys optimal complexity, in practice, we add a non-negligible number of degrees of freedom and the question wether this is really needed remains open. Our first numerical experiments (collected in a forthcoming paper) show that indeed a larger value of the parameter m seems to reduce this effect and produce "better meshes". In any case, we believe that all effort for the future should go in relaxing the condition of admissibility in order to obtain a "weaker" refine routine, alleviating the proliferation of degrees of freedom, while guaranteeing certified error bounds. To this respect, one possibility may be to change also the error indicator and consider error indicators related to functions and not elements (as the one proposed in Ref. 7) but for which a complete convergence analysis is a far from simple question.

Appendix A. Proofs of Lemmas 11 and 12

We now detail the proofs of Lemmas 11 and 12, in line with the adaptive finite element theory, see e.g., Ref. 12, 25.

Proof. (**Lemma 11**) By considering the global lower bound in (2.10) together with (4.2), we have

$$\begin{aligned}(1 - 2\mu)\, C_{\mathrm{glb}}\, \varepsilon^2_{\mathcal{Q}}(U, \mathcal{Q}) &\leq (1 - 2\mu) \left(|||u - U|||_\Omega^2 + \operatorname{osc}^2_{\mathcal{Q}}(U, \mathcal{Q})\right) \\ &\leq |||u - U|||_\Omega^2 - |||u - U^*|||_\Omega^2 + \operatorname{osc}^2_{\mathcal{Q}}(U, \mathcal{Q}) - 2\operatorname{osc}^2_{\mathcal{Q}}(U^*, \mathcal{Q}^*). \qquad \text{(A.1)}\end{aligned}$$

The contribution of the two errors on the right-hand side of (A.1) may be bounded through the orthogonality relation, see e.g., Lemma A.1 in Ref. 10, and the local upper bound (3.15), as follows

$$|||u - U|||_\Omega^2 - |||u - U^*|||_\Omega^2 = |||U^* - U|||_\Omega^2 \leq C_{\mathrm{lub}}\, \varepsilon^2_{\mathcal{Q}}(U, \mathcal{R}). \qquad \text{(A.2)}$$

In order to analyse the contribution of the two oscillation terms on the right-hand side of (A.1) instead, we consider the decomposition $\mathcal{Q} = \mathcal{R} \cup \{\mathcal{Q} \setminus \mathcal{R}\}$. Since $\mathrm{osc}^2_{\mathcal{Q}}(U,\mathcal{Q}) \le \varepsilon^2_{\mathcal{Q}}(U,\mathcal{Q})$ for any $Q \in \mathcal{R}$, see Remark 2.1 in Ref. 12, we have

$$\mathrm{osc}^2_{\mathcal{Q}}(U,\mathcal{R}) - 2\,\mathrm{osc}_{\mathcal{Q}^*}(U,\mathcal{R}) \le \mathrm{osc}^2_{\mathcal{Q}}(U,\mathcal{R}) \le \varepsilon^2_{\mathcal{Q}}(U,\mathcal{R}). \tag{A.3}$$

Since $\mathcal{Q} \setminus \mathcal{R} = \mathcal{Q} \cap \mathcal{Q}^*$ and, according to Corollary 3.5 in Ref. 12, there exists a constant Λ_{osc} so that

$$\mathrm{osc}^2_{\mathcal{Q}}(U,\mathcal{Q}\cap\mathcal{Q}^*) \le 2\,\mathrm{osc}^2_{\mathcal{Q}^*}(U^*,\mathcal{Q}\cap\mathcal{Q}^*) + \Lambda_{\mathrm{osc}}\,|||U-U^*|||_\Omega, \tag{A.4}$$

in view of the local upper bound (3.15), we have

$$\mathrm{osc}^2_{\mathcal{Q}}(U,\mathcal{Q}\cap\mathcal{Q}^*) - 2\,\mathrm{osc}^2_{\mathcal{Q}^*}(U^*,\mathcal{Q}\cap\mathcal{Q}^*) \le \Lambda_{\mathrm{osc}}\,|||U-U^*|||_\Omega \le \Lambda_{\mathrm{osc}}\,C_{\mathrm{lub}}\,\varepsilon^2_{\mathcal{Q}}(U,\mathcal{R}). \tag{A.5}$$

By combining (A.3) and (A.5), we obtain

$$\mathrm{osc}^2_{\mathcal{Q}}(U,\mathcal{Q}) - 2\,\mathrm{osc}_{\mathcal{Q}^*}(U,\mathcal{Q}^*) \le (1+\Lambda_{\mathrm{osc}}\,C_{\mathrm{lub}})\,\varepsilon^2_{\mathcal{Q}}(U,\mathcal{R}),$$

and inequality (A.1) reduces then to

$$\begin{aligned}(1-2\,\mu)\,C_{\mathrm{glb}}\,\varepsilon^2_{\mathcal{Q}}(U,\mathcal{Q}) &\le C_{\mathrm{lub}}\,\varepsilon^2_{\mathcal{Q}}(U,\mathcal{R}) + (1+\Lambda_{\mathrm{osc}}\,C_{\mathrm{lub}})\,\varepsilon^2_{\mathcal{Q}}(U,\mathcal{R})\\ &= (1+C_{\mathrm{lub}}(1+\Lambda_{\mathrm{osc}}))\,\varepsilon^2_{\mathcal{Q}}(U,\mathcal{R}).\end{aligned}$$

By taking into account the definition of θ_* in (4.3) we may conclude that $\varepsilon^2_{\mathcal{Q}}(U,\mathcal{R}) \ge (1-2\,\mu)\,\theta_*^2\,\varepsilon^2_{\mathcal{Q}}(U,\mathcal{Q})$ and, consequently, the Dörfler inequality (4.4) is satisfied for $\theta \in (0,\theta_*)$ and $\theta^2 = (1-2\mu)\,\theta_*^2$ according to (4.3). □

Proof. (**Lemma 12**) In view of Remark 9, since $(u,f,\mathbf{A}) \in \mathbb{A}_s$, by choosing

$$\epsilon^2 := \mu\,\left[|||u-U_k|||^2_\Omega + \mathrm{osc}^2_{\mathcal{Q}_k}(U_k,\mathcal{Q}_k)\right] \tag{A.6}$$

with μ defined as in Lemmas 11, there exists a strictly admissible mesh $\mathcal{Q}_\epsilon$ and $U_\epsilon \in \mathbb{S}_D(\mathcal{Q}_\epsilon)$ such that

$$|||u-U_\epsilon|||^2_\Omega + \mathrm{osc}^2_{\mathcal{Q}_\epsilon}(U_\epsilon,\mathcal{Q}_\epsilon) \le \epsilon^2 \quad\text{and}\quad \#\mathcal{Q}_\epsilon - \mathcal{Q}_0 \lesssim |u,f,\mathbf{A}|^{\frac{1}{s}}\,\epsilon^{-\frac{1}{s}}. \tag{A.7}$$

In order to take into account both the refinement of $\mathcal{Q}_\epsilon$ and $\mathcal{Q}_k$ (that may or may not be related to each other), we consider the overlay mesh $\mathcal{Q}_* := \mathcal{Q}\epsilon \otimes \mathcal{Q}_k$ and the discrete solution $U_* \in \mathbb{S}_D(\mathcal{Q}_*)$. Since $\mathcal{Q}_* \succeq \mathcal{Q}_\epsilon$, and consequently $\mathbb{S}_D(\mathcal{Q}_*) \supseteq \mathbb{S}_D(\mathcal{Q}_\epsilon)$, Lemma 10 — together with (A.6) and (A.7) — implies

$$\begin{aligned}|||u-U_*|||^2_\Omega + \mathrm{osc}^2_{\mathcal{Q}_*}(U_*,\mathcal{Q}_*) &\lesssim \left[|||u-U_\epsilon|||^2_\Omega + \mathrm{osc}^2_{\mathcal{Q}_\epsilon}(U_\epsilon,\mathcal{Q}_\epsilon)\right]\\ &\le \epsilon^2 = \mu\left[|||u-U_k|||^2_\Omega + \mathrm{osc}^2_{\mathcal{Q}_k}(U_k,\mathcal{Q}_k)\right].\end{aligned}$$

By Lemma 11, the set $\mathcal{R}_k = \mathcal{R}_{\mathcal{Q}_k\to\mathcal{Q}_*}$, that collects the elements of $\mathcal{Q}_k$ which do not belong to $\mathcal{Q}_*$, satisfies the Dörfler property $\varepsilon_{\mathcal{Q}_k}(U_k,\mathcal{R}_k) \gtrsim \theta\,\varepsilon_{\mathcal{Q}_k}(U_k,\mathcal{Q}_k)$ for $\theta < \theta_*$. Thanks to the assumpition that the module MARK selects a minimal set

$\mathcal{M}_k \subseteq \mathcal{Q}_k$ that also satisfies the same property, by also considering (2.8), (A.6) and (A.7), we have

$$\begin{aligned}\#\mathcal{M}_k &\le \#\mathcal{R}_k \le \mathcal{Q}_k - \mathcal{Q}_0 \le \#\mathcal{Q}_\epsilon - \mathcal{Q}_0 \lesssim |u, f, \mathbf{A}|^{\frac{1}{s}} \epsilon^{-\frac{1}{s}} \\ &= \mu^{-\frac{1}{2s}} \; |u, f, \mathbf{A}|^{\frac{1}{s}} \left[|||u - U_k|||_\Omega^2 + \mathrm{osc}^2_{\mathcal{Q}_k}(U_k, \mathcal{Q}_k)\right]^{-\frac{1}{2s}}.\end{aligned}$$

This concludes the proof. □

Acknowledgments

Annalisa Buffa has been partially supported by the ERC Advanced Grant "CHANGE" (694515, 2016-2020), by the Istituto di Matematica Applicata e Tecnologie Informatiche del CNR, Pavia, Italy and by the Istituto Nazionale di Alta Matematica (INdAM) through Finanziamenti Premiali MATHTECH. Carlotta Giannelli has been partially supported by the MIUR "Futuro in Ricerca" project DREAMS (RBFR13FBI3), and by INdAM through Finanziamenti Premiali SUNRISE.

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