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Multiplicity of forced oscillations for scalar retarded functional differential equations

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We find multiplicity results for forced oscillations of a periodically perturbed autonomous second order equation, the perturbing term possibly depending on the whole history of the system. The techniques that we employ are topological in nature but the technical details are hidden in the proofs and completely transparent to the reader only interested in the results. Copyright © 0000 John Wiley & Sons, Ltd.

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1. Introduction

In this paper we focus on a multiplicity result for the forced oscillations of a scalar *retarded functional differential equation* (RFDE for short). Namely, we consider the following parametrized scalar RFDE:

$$x''(t) = -\alpha x'(t) + g(x(t)) + \lambda F(t, x_t), \quad \lambda \geq 0, \tag{1}$$

where $\alpha \geq 0$, $g\colon \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function, and the map $F\colon \mathbb{R} \times BU((-\infty, 0], \mathbb{R}) \to \mathbb{R}$ is T-periodic in the first variable and locally Lipschitz in the second one. Here, $BU((-\infty, 0], \mathbb{R})$ denotes the Banach space of all $\mathbb{R}$-valued uniformly continuous bounded functions of $(-\infty, 0]$. Moreover, as usual in the setting of RFDEs, given $t \in \mathbb{R}$, by $x_t \in BU((-\infty, 0], \mathbb{R})$ we mean the function $\theta \mapsto x(t + \theta)$. We will prove that if the function g changes sign $n > 0$ times, and equation (1) for $\lambda = 0$ is non-T-isochronous (see Definition 3.3 and Remark 3.4 below) then there are at least n solutions of period T for $\lambda > 0$ sufficiently small. Notice that such a condition is not difficult to meet: it is automatically satisfied when $n > 1$ or when $\alpha > 0$. In fact non-T-isochronism holds even when equation $x''(t) = g\big(x(t)\big)$ has a local, but not global, T-isochronous center.

Our multiplicity result, Theorem 3.10 below, is mainly inspired by the paper [24] in which analogous theorems have been proved for an ODE, that is, in the undelayed case. It is in order to remark that here, unlike in [24], we allow the presence of friction; in fact, equation (1) can be interpreted as the motion equation of a particle (or of a system with one degree of freedom) subject to a conservative force plus a possible friction as well as to a periodic perturbation that may depend on the whole history of the process.

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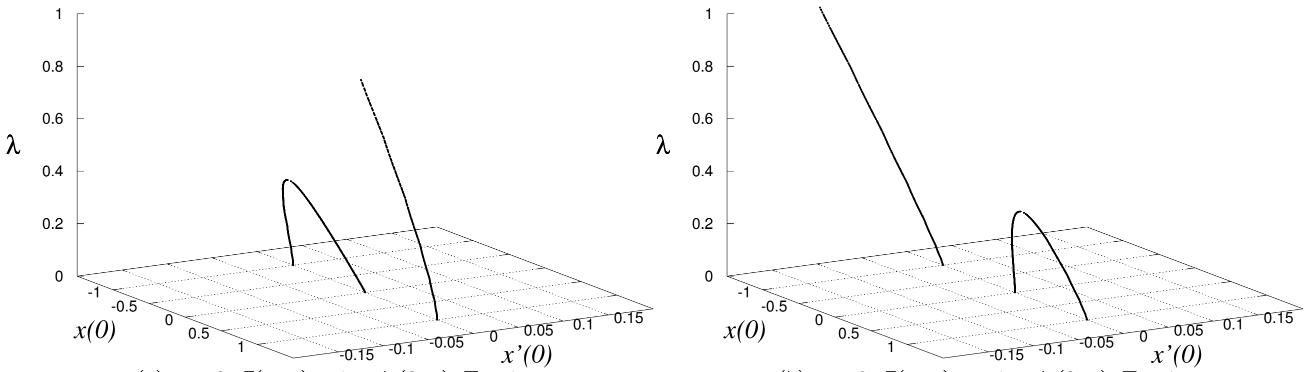
In the frictionless case, that is when $\alpha = 0$, the multiplicity result that we obtain here generalizes partially those of [24]. In fact, RFDEs include naturally ODEs as a particular case but, here, we need to assume that the forcing term does not depend on the velocity. The reason for that choice is that we do not have yet available results on the branches of T-periodic solutions in the case when the forcing term depends on the velocity. We leave this case to further investigation. Also, for technical reasons, unlike [24], we have to require locally Lipschitz continuity on the function g as well as on F. For a comprehensive discussion on the general properties of higher order RFDEs in Euclidean spaces see e.g. [6].

With regard to the ODE case, the problems of existence and multiplicity of periodic solutions for periodically forced second-order scalar autonomous differential equations, although classical and apparently well-investigated, still represent a field actively researched by mathematicians. Even if we confine ourselves to multiplicity results, it is impossible to give here an exhaustive list of the many approaches that have been successfully pursued. Let us only mention [10, 11, 14, 19, 37], the book [20], the survey papers [32, 33] and the references therein.

Delay differential equations, as well as RFDEs, represent nowadays a well-studied subject in view of many applications (see e.g. [2, 12, 18]). However, despite the (apparent) simplicity of the model, it seems that the extension of the quoted above existence and multiplicity results for scalar ODEs to the case of a functional, or delayed, forcing term – like in equation (1) – has not yet been considered by the mathematical community. A possible reason for that is that some techniques may be difficult (or impossible) to extend to the delayed case. On the other hand, a topological approach can be fruitfully pursued even in this extended framework (see, e.g., [4, 5]).

Although the technicalities are hidden in the proofs, to obtain our multiplicity result we apply topological methods. In fact, a key step for our result is a Rabinowitz-type theorem on branches of periodic solutions for first-order parametrized RFDEs obtained by M. Furi and the last two authors in [26] (see Theorem 3.2 below). The proof of this theorem is based on the notion of degree of a tangent vector field (see, e.g., [25]). Another crucial instrument is a point-set topology result of [24], Theorem 2.5 below, that gathers some known connectivity results (see, e.g., [1, 21, 30]) Moreover, a "local" counterpart in our approach is the notion of ejecting set or point ([22], see Definition 2.1) which, broadly speaking, is analogous to the concept of bifurcation point. All these notions and results are suitably combined with the already mentioned concept of T-isochronism of equation (1) for $\lambda = 0$.

We point out that a slightly different strategy, based on the connectivity result Theorem 2.2 below, could be pursued leading again to a different kind of multiplicity theorems, where the key notion is that of *T-resonance*. Such results are founded on a comparison between the local behaviour and the global structure of the set of T-periodic solutions (see e.g. [16, Ch. 7] see also [7]). A similar idea can be traced back to Poincaré (see [33] for an exposition). The latter approach has been used in our paper [13] for the investigation of forced oscillations of the spherical pendulum and, in the past, in [22, 23, 24] in the ODE case. In the present setting, however, due to the simpler structure of the underlying space, it seems that the notion of T-isochronism is more convenient as it leads to simpler statements.



(a) $\alpha = 0$, $F(t, x_t) = 1 + \sin(2\pi t)$, $T = 1$.

(b) $\alpha = 0$, $F(t, x_t) = -1 + \sin(2\pi t)$, $T = 1$.

Figure 1. Initial conditions in the box $[-1.5, 1.5] \times [-0.2, 0.2] \times [0, 1]$ of 1-periodic solutions of the double-well oscillator for two different 1-periodic forcing terms.

As a paradigm of the kind of result that we obtain, consider the well-known example of the double-well oscillator (see, e.g., [28, 34, 40]) subject to a periodic perturbative forcing that may depend on the whole history of the system:

$$x'' = -\alpha x' + x - x^3 + \lambda F(t, x_t), \quad \lambda \geq 0, \tag{2}$$

with F of period T in t. Our result shows that, for any choice of F, when $\lambda > 0$ is sufficiently small (*how* small depends, of course, on F) one has three T-periodic solutions of (2).

In the simpler case when the perturbation does not depend on the history of the solution one can visualize to some extent the set of T-periodic solutions (see, e.g., [7, 8] and, also, [39]). The idea is to draw, for a selected box in $\mathbb{R}^2 \times [0, \infty)$, the set of triples (p, v, λ) for which the equation under scrutiny admits a T-periodic solution with initial conditions $x(0) = p$ and $x'(0) = v$. In Figure 1 we apply this technique to equation (2), for particular choices of F and T, showing (as a black curve) the set of triples (p, v, λ) for which the initial conditions $x(0) = p$ and $x'(0) = v$ lead to a T-periodic solution corresponding to λ. As it is suggested by this picture, there exists $\lambda_0 > 0$ such that for any $\bar\lambda \in [0, \lambda_0)$ the plane $\lambda = \bar\lambda$ intersects this set in three different points. Such points are initial conditions of T-periodic solutions of (2) corresponding to $\bar\lambda$.

Notice that, when $g^{-1}(0)$ is a discrete set and g changes sign at infinitely many zeros, our result implies that, given $n \in \mathbb{N}$, there exists $\delta_n > 0$ such that (1) admits at least n forced oscillations for any $\lambda \in [0, \delta_n)$. It is important to realize, however, that one may not have infinitely many T-periodic solutions even for small values of λ, as illustrated by Example 3.11.

Observe finally that our result is neither directly deducible from the standard implicit function theorem nor by arguments like those of [22, 23], as illustrated by Examples 3.12 and 3.13 below.

2. Preliminaries

2.1. Ejecting sets

Let Y be a metric space and X a subset of $[0, +\infty) \times Y$. Given $\lambda \geq 0$, we denote by X_λ the slice $\{y \in Y : (\lambda, y) \in X\}$.

Definition 2.1 ([22]) *We say that $E \subseteq X_0$ is* ejecting *(for X) if it is relatively open in X_0 and there exists a connected subset of X which meets $\{0\} \times E$ and is not contained in $\{0\} \times X_0$.*

We give here a more detailed version of the statement of Theorem 3.3 in [22].

Theorem 2.2 *Let Y be a metric space and let X be a locally compact subset of $[0, +\infty) \times Y$. Assume that X_0 contains n pairwise disjoint ejecting subsets $E_1, \dots, E_n$. Suppose that $n - 1$ of them are compact. Then, there are open neighborhoods $U_1, \dots, U_n$ in Y of $E_1, \dots, E_n$, respectively, with pairwise disjoint closure, and a positive number λ_* such that for $\lambda \in [0, \lambda_*)$*

$$X_\lambda \cap U_i \neq \emptyset, \quad i = 1, \dots, n.$$

In particular, we have that the cardinality of X_λ is greater than or equal to n for any $\lambda \in [0, \lambda_*)$.

As shown by examples in [22], despite the apparent simplicity of the assertion of Theorem 2.2, none of its assumptions can be dropped.

We will also need the following fact (inspired by Lemma 3.6 in [13]) concerning compact ejecting sets:

Lemma 2.3 *Let X and Y be as in Theorem 2.2. Let $E \subseteq X_0$ be a compact ejecting set. Then for any sufficiently small open neighborhood V of E in Y there exists a positive number $\lambda_\#$ such that*

$$([0, \lambda_\#] \times \partial V) \cap X = \emptyset.$$

In order to give the proof of this lemma we need to recall the notion of upper semicontinuous multivalued map. A multivalued map $\phi\colon \mathcal{X} \multimap \mathcal{Y}$ between two metric spaces is said to be *upper semicontinuous* if it has compact (possibly empty) values and for any open subset $\mathcal{V}$ of $\mathcal{Y}$ the upper inverse image of $\mathcal{V}$, i.e., the set $\phi^{-1}(\mathcal{V}) = \{x \in \mathcal{X} : \phi(x) \subseteq \mathcal{V}\}$, is open in $\mathcal{X}$.

Remark 2.4 *Given a compact subset K of $\mathcal{X} \times \mathcal{Y}$, the multivalued map that associates to $x \in \mathcal{X}$ the slice K_x (whose graph is K) is upper semicontinuous. To see this, let $\mathcal{V}$ be any open subset of $\mathcal{Y}$ and assume, by contradiction, that the set $\mathcal{U} = \{x \in \mathcal{X} : K_x \subseteq V\}$ is not open. Then, there exists a sequence $\{x_n\}$ in $\mathcal{X}\backslash\mathcal{U}$ which converges to some $x_0 \in \mathcal{U}$. For any $n \in \mathbb{N}$, choose $y_n \in K_{x_n} \cap (\mathcal{Y}\backslash\mathcal{V})$. Because of the compactness of K, we may assume $(x_n, y_n) \to (x_0, y_0) \in K$. Thus, y_0 belongs to K_{x_0} which is a subset of $\mathcal{V}$, contradicting the fact that y_0 also belongs to the closed set $\mathcal{Y}\backslash\mathcal{V}$.*

Proof of Lemma 2.3. Since $E \subseteq X_0$ is ejecting then it is relatively open. Since E is also compact, there exists an open neighborhood V of E in Y such that $X_0 \cap \mathrm{cl}\,(V) = E$, where $\mathrm{cl}\,(V)$ denotes the closure of V in Y. By assumption, X is locally compact. Hence, there exists an open neighborhood W of E in Y and a number $\mu > 0$ such that $([0,\mu] \times \mathrm{cl}\,(W)) \cap X$ is compact. By restricting V, if necessary, we may assume that $\mathrm{cl}\,(V) \subseteq W$. By Remark 2.4, the multimap $\Psi\colon [0,\mu] \multimap \mathrm{cl}\,(V)$ given by $\Psi(\lambda) = X_\lambda \cap \mathrm{cl}\,(V)$ is upper semicontinuous. Thus, since $\Psi(0) = X_0 \cap \mathrm{cl}\, V = E \subseteq V$, there exists $\lambda_\# > 0$ such that $\Psi([0,\lambda_\#]) \subseteq V$. Whence the assertion. ☐

A key step in our main result will be proving that certain sets are ejecting. Instrumental to this task will be Theorem 2.2 along with the following point-set topology result about connectivity [24, Lemma 2.6], which is in the spirit of a well-known result (see e.g. [1] and [30, chapter V]).

Theorem 2.5 *Let Y be a locally compact Hausdorff topological space and let $Y_1, \dots, Y_n$, $n \geq 1$, be pairwise disjoint compact subsets of Y. Then the following alternative holds:*

1. *there exist n pairwise disjoint compact open subsets $A_1, \dots, A_n$ of Y containing $Y_1, \dots, Y_n$ respectively;*
2. *there exists a connected set of $Y \setminus \bigcup_{i=1}^n Y_i$ whose closure in Y meets $\bigcup_{i=1}^n Y_i$ and has one of the following properties:*
   - *(a) it is not compact;*
   - *(b) meets at least two different Y_i's.*

2.2. Retarded functional differential equations

Here we specialize some definitions and properties of first order RFDEs with infinite delay on $\mathbb{R}^k$, which have been studied e.g. in [3] in the case of possibly noncompact differentiable manifolds. As a general reference on RFDEs with finite delay in Euclidean spaces, see the monograph [29].

Given an arbitrary subset A of $\mathbb{R}^k$, we denote by $BU((-\infty,0], A)$ the set of bounded and uniformly continuous maps from $(-\infty,0]$ into A. Notice that $BU((-\infty,0],\mathbb{R}^k)$ is a Banach space, being closed in the space $BC((-\infty,0],\mathbb{R}^k)$ of the bounded and continuous functions from $(-\infty,0]$ into $\mathbb{R}^k$ (endowed with the standard supremum norm $\|\cdot\|$).

Let U be an open subset of $\mathbb{R}^k$. Any continuous map

$$G\colon \mathbb{R} \times BU((-\infty,0],U) \to \mathbb{R}^k$$

will be called a *retarded functional map over U*. Let us consider a first order RFDE of the type

$$z'(t) = G(t, z_t), \tag{3}$$

where G is a retarded functional map over U. Here, as usual and whenever it makes sense, given $t \in \mathbb{R}$, by $z_t \in BU((-\infty,0],U)$ we mean the function $\theta \mapsto z(t+\theta)$.

A *solution* of (3) is a function $z\colon J \to U$, defined on an open real interval J with $\inf J = -\infty$, bounded and uniformly continuous on any closed half-line $(-\infty, b] \subset J$, and which verifies eventually the equality $z'(t) = G(t, z_t)$. That is, $z\colon J \to U$ is a solution of (3) if $z_t \in BU((-\infty,0],U)$ for all $t \in J$ and there exists $\tau \in J$ such that z is C^1 on the interval $(\tau, \sup J)$ and $z'(t) = G(t, z_t)$ for all $t \in (\tau, \sup J)$.

It can be proved (see e.g. [3]) that if the retarded functional map G is locally Lipschitz in the second variable, then two maximal solutions of equation (3) coinciding in the past must coincide also in the future.

3. Main results

We consider the following parametrized scalar RFDE:

$$x''(t) = -\alpha x'(t) + g(x(t)) + \lambda\, F(t, x_t), \quad \lambda \geq 0, \tag{4}$$

where $\alpha \geq 0$, $g\colon \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function, and the retarded functional map $F\colon \mathbb{R} \times BU((-\infty, 0], \mathbb{R}) \to \mathbb{R}$ is T-periodic in the first variable and locally Lipschitz in the second one, i.e., given $(\tau, \varphi) \in \mathbb{R} \times BU((-\infty, 0], \mathbb{R})$, there exist an open neighborhood U of (τ, φ) and $L \geq 0$ such that

$$|F(t, \varphi_1) - F(t, \varphi_2)| \leq L\, \|\varphi_1 - \varphi_2\|\,,$$

for all $(t, \varphi_1), (t, \varphi_2) \in U$.

We are interested in a multiplicity result for the T-periodic solutions of (4) when $\lambda > 0$ is small.

In order to clarify what we mean by a solution of (4), we introduce in a natural way a first order RFDE on $\mathbb{R}^2$. Let, for $(q, v) \in \mathbb{R}^2$ and $(t, (\varphi, \psi)) \in \mathbb{R} \times BU((-\infty, 0], \mathbb{R}^2)$:

- $\kappa(q, v) = (0, \alpha v)$
- $\hat{g}(q, v) = (v, g(q))$
- $\hat{F}(t, (\varphi, \psi)) = (0, F(t, \varphi))$.

Setting $\xi = (q, v)$, the following is a first order RFDE on $\mathbb{R}^2$:

$$\xi'(t) = -\kappa(\xi(t)) + \hat{g}(\xi(t)) + \lambda\, \hat{F}(t, \xi_t). \tag{5}$$

That is, (5) is of the form (3) with, for any $\lambda \in [0, \infty)$, $G\colon \mathbb{R} \times BU\big((-\infty, 0], \mathbb{R}^2\big) \to \mathbb{R}^2$ given by

$$G\big(t, (\varphi, \psi)\big) = \big(\psi(0), -\alpha\psi(0) + g\big(\varphi(0)\big) + \lambda F(t, \varphi)\big)\,. \tag{6}$$

We regard a solution of (4) as a map $x\colon J \to \mathbb{R}$, defined on an open real interval J with $\inf J = -\infty$, such that the pair $(x, x')\colon J \to \mathbb{R}^2$ is a solution of (5). For a different approach to the notion of solution of a second-order RFDE see [6].

We need some results taken mostly from [24, 26]. In what follows, we will mainly work with equation (5).

We will denote by $C_T(\mathbb{R}^2)$ the Banach space of the T-periodic continuous maps from $\mathbb{R}$ into $\mathbb{R}^2$ with the usual supremum norm.

A pair $(\lambda, \xi) \in [0, +\infty) \times C_T(\mathbb{R}^2)$, where ξ is a solution of (5) corresponding to λ, is called a *T-periodic pair* (for (5)). Those T-periodic pairs that are of the particular form $(0, \overline{\zeta})$, $\overline{\zeta}$ being the map constantly equal to ζ, are said to be *trivial*. Observe that any T-periodic pair $(0, \overline{\zeta})$ is trivial if and only if $\zeta = (q, 0)$ with $g(q) = 0$.

The following immediate consequence of Lemma 3.1 of [4] expresses a crucial property of the set of T-periodic pairs.

Lemma 3.1 *The set of T-periodic pairs for* (5) *is closed and locally compact.*

For the sake of simplicity, we will identify $\mathbb{R}^2$ with its image in $[0, +\infty) \times C_T(\mathbb{R}^2)$ under the embedding which associates to any $\zeta \in \mathbb{R}^2$ the pair $(0, \overline{\zeta})$. In particular, given $q \in \mathbb{R}$, according to our convention $(q, 0)$ can be seen as an element of $[0, \infty) \times C_T(\mathbb{R}^2)$. Moreover, with a slight abuse of notation, if Ξ is a subset of $[0, +\infty) \times C_T(\mathbb{R}^2)$, by $\Xi \cap \mathbb{R}$ we mean the subset of $\mathbb{R}$ given by all $q \in \mathbb{R}$ such that the pair $(0, \overline{(q, 0)})$ belongs to Ξ. Observe that if $\Omega \subseteq [0, +\infty) \times C_T(\mathbb{R}^2)$ is open, then $\Omega \cap \mathbb{R}$ is open in $\mathbb{R}$.

The following consequence of [26, Corollary 4.4] yields the existence of a Rabinowitz-type branch of T- periodic pairs for (5). Its proof relies on the notion of degree of a tangent vector field and on some of its standard properties. For a quick outline see, e.g., [25].

Theorem 3.2 *Assume that g changes sign in $q \in g^{-1}(0)$. Then, there is a connected set Γ of nontrivial T-periodic pairs for* (5) *whose closure meets $\{q\}$ and either is unbounded or meets $g^{-1}(0) \setminus \{q\}$.*

Proof. Observe first that since g changes sign at q, the map $\hat{g}$ has nonzero degree in a conveniently small neighborhood of $(q,0)$. To see this, observe that there necessarily is $\delta > 0$ such that either $g(p)(p-q) > 0$ or $g(p)(p-q) < 0$, for $p \in [q-\delta, q+\delta]$. Correspondingly, either the map $H^+ : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^2$ or $H^- : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^2$, given by $H^{\pm}(\lambda, p, v) = \big(v, \lambda g(p) \pm (1-\lambda)(p-q)\big)$, is a homotopy, admissible in the square $Q := (q-\delta, q+\delta) \times (-\delta, \delta)$, and joining $\hat{g}$ and the map $(p,v) \mapsto (v, p-q)$ or $(p,v) \mapsto (v, q-p)$, respectively. Thus $\deg(\hat{g}, Q) = \pm 1$.

Since $(q,0)$ is isolated in $\hat{g}^{-1}(0,0)$, the set $\hat{g}^{-1}(0,0) \setminus \{(q,0)\}$ is closed in $\mathbb{R}^2$. Hence, $U := \mathbb{R}^2 \setminus \big(\hat{g}^{-1}(0,0) \setminus \{(q,0)\}\big)$ is open. The excision property of the degree yields $\deg(\hat{g}, U) = \pm 1$. Now the assertion follows from [26, Corollary 4.4]. ☐

Consider now the unperturbed equation

$$x''(t) = -\alpha x'(t) + g(x(t)) \tag{7}$$

which can be equivalently written as equation (5) with $\lambda = 0$, namely

$$\xi'(t) = -\kappa(\xi(t)) + \hat{g}(\xi(t)). \tag{8}$$

Let us now introduce the notion of T-isochronism for (7).

Definition 3.3 *We say that equation* (7) *is T-isochronous if all its solutions are T-periodic.*

In other words, (7) is T-isochronous if it has a global (not merely local) T-isochronous center.

Remark 3.4 *Define $\Phi : \mathbb{R}^2 \to \mathbb{R}$ by $\Phi(q,v) = \frac{1}{2}v^2 - \int_0^q g(s)\,ds$. Observe that, when $\alpha > 0$, the function Φ is always monotone decreasing along any nonconstant solution of* (8). *In fact, if ξ is any such solution with $\xi(t) = (x(t), x'(t))$,*

$$\frac{d}{dt}\Phi(\xi(t)) = \frac{d}{dt}\Phi\left(x(t), x'(t)\right) = x'(t)x''(t) - x'(t)g(x(t)) = -\alpha\left(x'(t)\right)^2 < 0.$$

Hence, if $\alpha > 0$, (7) *is not T-isochronous.*

In the next two key lemmas we link the property of non-T-isochronism of the unperturbed equation (7) to the existence of particular ejecting subsets of the set X of T-periodic pairs for (5). The first one, Lemma 3.5, is a consequence of Lemma 3.4 in [24], while Lemma 3.6 is related to Theorem 3.5 in [24].

Lemma 3.5 *Assume that the T-isochronism property does not hold for* (7). *Let $\mathcal{G} \subseteq C_T(\mathbb{R}^2)$ be a connected component of the set of T-periodic solutions of* (8) *containing a zero $(q,0)$ of $\hat{g}$ such that g changes sign in q. Then $\mathcal{G}$ is compact and does not intersect any zero of $\hat{g}$ different from $(q,0)$.*

Proof. Observe first that if $\xi = (x,y)$ is a (T-periodic) solution of (8), then $y = x'$ and x is a (T-periodic) solution of (7). Conversely, if x is a (T-periodic) solution of (7), then (x,x') is a (T-periodic) solution of (8). Then, the assertion of Lemma 3.5 is a consequence of Lemma 3.4 in [24]. ☐

Lemma 3.6 *Let X be the set of T-periodic pairs for* (5). *Assume that the T-isochronism property does not hold for* (7), *and that g changes sign at n zeros $q_1, \dots, q_n$. Then, there exist n ejecting sets $E_1, \dots, E_n \subseteq X_0$ which are compact, pairwise disjoint and such that $q_i \in E_i$ for $i = 1, \dots, n$.*

Proof. By Lemma 3.5 we have that the second alternative in Theorem 2.5 does not hold. Then, there exist n compact, pairwise disjoint, open subsets $E_1, \dots, E_n$ of X_0 such that $q_i \in E_i$ for $i = 1, \dots, n$. We need only show that $E_1, \dots, E_n$ are ejecting sets.

Since g changes sign in $q_1, \dots, q_n$, by Theorem 3.2, for $i = 1, \dots, n$, there exists a connected set Γ_i of nontrivial T-periodic pairs of (5) whose closure is noncompact or intersects $\hat{g}^{-1}(0,0) \setminus \{(q_i,0)\}$. Again by Lemma 3.5 this closure is not contained in E_i. Therefore E_i is ejecting for $i = 1, \dots, n$. ☐

If all the assumptions of Lemma 3.6 hold, then, by Theorem 2.2, there exists $\lambda_* > 0$ such that (5) has at least n solutions of period T for $\lambda \in [0, \lambda_*)$. These clearly correspond to T-periodic solutions of (4). We claim that, reducing $\lambda_* > 0$ if necessary, the image of these solutions are not pairwise coincident. To prove this claim we need a simple well-known fact whose proof we provide for the sake of completeness.

Lemma 3.7 *Let x_1 and x_2 be T-periodic solutions of the unperturbed equation* (7) *with the property that $x_1([0,T]) = x_2([0,T])$. Then, there exists $t_0 \in [0,T]$ such that $x_1(t+t_0) = x_2(t)$ for all $t \in [0,T]$. Furthermore, if X is the set of T-periodic pairs for* (5) *as in Lemma 3.6, then $t \mapsto (x_1(t), x_1'(t))$ and $t \mapsto (x_2(t), x_2'(t))$ are solutions of* (5) *for $\lambda = 0$ that belong to the same connected component of X_0.*

Proof. Let us first observe that if $\alpha > 0$ then the assertion is trivial because the only periodic solutions of (7) are necessarily constant. Thus one may assume for simplicity (but this is *not* necessary for our argument) that $\alpha = 0$ in the rest of the proof.

Let s_0 be a maximum point for $t \mapsto x_1(t)$. Since $x_1([0,T]) = x_2([0,T])$ there exists r_0 such that $x_2(r_0) = x_1(s_0)$ and, clearly r_0 is a maximum point for $t \mapsto x_2(t)$. Define, for all $t \in \mathbb{R}$, $\gamma(t) = x_1(t + s_0 - r_0)$. One has that γ is a solution of (7) and $\gamma(r_0) = x_1(s_0) = x_2(r_0)$ and $\gamma'(r_0) = x_1'(s_0) = 0 = x_2'(r_0)$. Thus x_2 coincides with γ being both maximal solutions of the same Cauchy problem. The first part of the assertion is proven by setting $t_0 = s_0 - r_0$.

To prove the validity of the second part of the assertion define, for $t \in \mathbb{R}$ and $s \in [0,1]$, $h(s,t) = x_1(t + st_0)$. Clearly, $h(s,\cdot)$ is a C^1_T function that lies in X_0. Thus, the curve $s \mapsto h(s,\cdot)$ connects x_1 and x_2. Whence the assertion. □

The next proposition is a crucial step in order to obtain our multiplicity result.

Proposition 3.8 *Assume that the T-isochronism property does not hold for* (7)*, and that g changes sign at n zeros $q_1, \dots, q_n$. Then there exists $\lambda_* > 0$ such that* (4) *has at least n solutions of period T for $\lambda \in [0, \lambda_*)$ that have pairwise not coincident images. Furthermore, for positive friction coefficient α, these solutions have mutually disjoint images.*

Proof. Let X be the set of T-periodic pairs for (5). Lemma 3.6 yields the existence of n ejecting sets $E_1, \dots, E_n \subseteq X_0$ which are compact, pairwise disjoint and such that $q_i \in E_i$ for $i = 1, \dots, n$. As in Theorem 2.2 we can find bounded neighborhoods $U_i \subseteq C_T(\mathbb{R}^2)$ of E_i, for $i = 1, \dots, n$, with pairwise disjoint closures (recall that, by Lemma 3.1, X is a locally compact subset of $[0, +\infty) \times C_T(\mathbb{R}^2)$). Now Lemma 2.3 implies that, reducing λ_* if necessary, we can assume $X \cap ([0, \lambda_*] \times \partial(U_i)) = \emptyset$. By Theorem 2.2 we have that, for any $\lambda \in [0, \lambda_*]$, $\{\lambda\} \times U_i$ contains at least a T-periodic pair (λ, ξ_i^λ) for (5). Let us put $\xi_i^\lambda = (x_i^\lambda, y_i^\lambda)$; clearly x_i^λ is a T-periodic solution of (4). We wish to prove that for each $i, j \in \{1, \dots, n\}$, $i \neq j$, the image of x_i^λ does not coincide with that of x_j^λ. We actually prove this assertion for $i = 1$, the proof being the same for $i = 2, \dots, n$. Assume by contradiction that there exists a sequence of positive numbers λ_n with $\lambda_n \to 0$ such that $x_1^{\lambda_n}([0,T]) = x_{j_n}^{\lambda_n}([0,T])$ for some $j_n \in \{2, \dots, n\}$ (not necessarily the same for all n). Passing to a subsequence (recall that $X \cap [0, \lambda_*] \times \cup_{i=1}^n U_i$ is compact) we can assume that the sequences $x_{j_n}^{\lambda_n}$ and $x_1^{\lambda_n}$ tend to T-periodic solutions of (7), say x_0^0 and x_1^0, respectively. Also one has $x_0^0([0,T]) = x_1^0([0,T])$.

By Lemma 3.7, x_0^0 and x_1^0 belong to the same connected component of X_0, hence to the same ejecting set E_1. This means that the T-pair $\left(\lambda_n, \left(x_{j_n}^{\lambda_n}, (x_{j_n}^{\lambda_n})'\right)\right) \in [0, \lambda_*] \times U_{j_n}$ eventually belongs to $[0, \lambda_*] \times U_1$, but this is impossible by the choice of the U_i's. This contradiction proves our claim.

Observe also that when $\alpha > 0$ the ejecting sets $E_1, \dots, E_n$ of Lemma 3.6 consist merely of the constant functions $\overline{(q_1, 0)}, \dots \overline{(q_n, 0)}$. Thus, the same argument used in the proof of [13, Theorem 3.9] for non-T-resonant zeros shows that the images of the solutions x_i^λ must actually be mutually disjoint for sufficiently small $\lambda > 0$. □

We observe that the non-T-isochronism assumption in Lemma 3.6 and Proposition 3.8 is not very restrictive. For instance, when $\alpha > 0$ one cannot have T-isochronism. Also, when $\alpha = 0$, (see e.g. [31]) the unique odd continuous function g for which T-isochronism holds is

$$g(s) = -\left(\frac{2\pi}{T}\right)^2 s.$$

On the other hand, if we drop the oddness assumption on g one can easily find examples in which T-isochronism holds. For necessary and sufficient conditions for T-isochronism one can see, e.g., [31, 41, 42] and the more recent [9, 15, 27, 36, 38].

Remark 3.9 *We stress that even for $\alpha = 0$, if the function g changes sign at least two times, as an immediate consequence of Lemma 3.1 in [24] one gets that the T-isochronism property does not hold for* (7)*. In fact, suppose that there are two different isolated zeros, $q_1 < q_2$, of g. We can assume without loss of generality that $g(q) \neq 0$ for any $q \in (q_1, q_2)$. Then, exactly one of them, say q_1, is a maximum point of any primitive of $-g$. So, according to Lemma 3.1 in [24], there exists a neighborhood U of q_1 with the property that there are no T-periodic solutions of* (7) *with image in U different from the constant $\overline{q}(t) \equiv q_1$. The continuity with respect to data implies the existence of a smaller neighborhood $W \subseteq U$ of q_1 such that the solutions starting (at*

$t = 0$) in W remain in U for all $t \in [0, T]$. Thus, all the points of $W \setminus \{q_1\}$ cannot belong to the image of a T-periodic solutions of (7). This shows that equation (7) is not T-isochronous.

By Proposition 3.8 and Remark 3.9 we immediately deduce our main multiplicity result.

Theorem 3.10 *Assume that g changes sign at $n > 1$ zeros. Then there exists $\lambda_* > 0$ such that, for $\lambda \in [0, \lambda_*)$, equation (4) has at least n solutions of period T that have pairwise not coincident images. If the friction coefficient α is positive, these solutions actually have mutually disjoint images.*

The following examples show that, yet in the undelayed case, even if g changes sign infinitely many times, one not necessarily gets a value of λ that yields infinitely many T-periodic solutions.

Example 3.11 *Consider the following equation with $T = 1$:*

$$x'' = -\sin x + \lambda x, \quad \lambda \geq 0, \tag{9}$$

We claim that, for any $\lambda > 0$, (9) has a finite number of 1-periodic solutions. In fact consider, for $\lambda > 0$, the interval $I(\lambda) := [-\frac{1}{\lambda}, \frac{1}{\lambda}]$ and observe that for a given $\bar{\lambda} > 0$, no 1-periodic solution can enter the region $\mathbb{R} \setminus I(\bar{\lambda})$ (in the event, it would keep accelerating away from the origin). Thus, all 1-periodic solutions corresponding to $\bar{\lambda}$ are contained in $I(\bar{\lambda})$. This "spatial" bound immediately yields one for the speed: Let x be a 1-periodic solution of (9) and let $t_0 \in \mathbb{R}$ a time when x attains its minimum. Then, for $t \in [t_0, t_0 + 1]$,

$$\left|x'(t)\right| \leq \int_{t_0}^{t} \left|x''(s)\right| \, ds \leq \int_{t_0}^{t} \left|-\sin(x(s)) + \lambda x(s)\right| \, ds \leq 2(t - t_0) \leq 2.$$

Hence, the first order system associated to (9) for $\lambda = \bar{\lambda}$,

$$\begin{cases} x'(t) = y(t), \\ y'(t) = -\sin(x(t)) + \bar{\lambda} x(t) \end{cases} \tag{10}$$

has the property that all 1-periodic solutions are contained in the box $I(\bar{\lambda}) \times [-2, 2]$. Also, it is not difficult to prove that the set $S(\bar{\lambda})$ of all initial conditions of 1-periodic solutions of (10) consists of isolated points in $I(\bar{\lambda}) \times [-2, 2]$. Thus, as a discrete compact set, the set $S(\bar{\lambda})$ is finite, whence the claim.

Figure 2 shows a part of the set of triples (p, v, λ) for which (9) admits a T-periodic solution with initial conditions $x(0) = p$ and $x'(0) = v$.

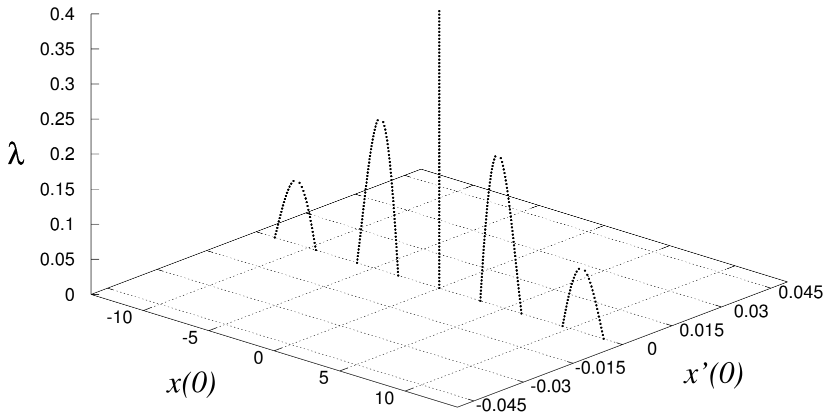


Figure 2. Initial conditions in the box $[-14, 14] \times [-0.07, 0.02] \times [0, 0.5]$ of 1-periodic solutions of equation (9).

We conclude the paper with the following two examples that illustrate two main issues that hamper a more direct "standard" implicit function theorem approach, or the use of arguments like those of [22, 23]. The first concern is regularity. In fact our results only require g and F to be Lipschitz. Example 3.12 illustrates such a situation. The second problem is more serious: even for g regular, Theorem 3.10 does not require anything on the linearized equation, for $\lambda = 0$, at the zeros of g. Indeed, one could have, as in Example 3.13 that g' is the zero map at all zeros of g.

Example 3.12 *Take $g(x) := \min\{|x| - 1\,,\, 2(|x| - 1)\}$. Then, by our result, fixed any T-periodic perturbative force F, equation* (1) *admits two T-periodic solutions for sufficiently small values of λ. In order to illustrate our point, we proceed as above restricting our attention, for graphical reasons, to an undelayed perturbation. in Figure 3 we choose $f(t, x_t) = 2|x(t)| + \sin(2\pi t)$ and show a portion of the set of triples (p, v, λ) for which* (2) *admits a T-periodic solution with initial conditions $x(0) = p$ and $x'(0) = v$.*

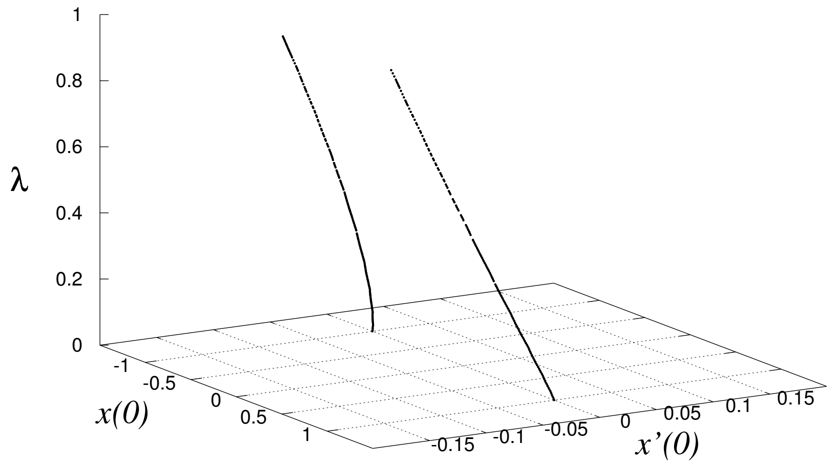


Figure 3. Initial conditions in the box $[-1.5, 1.5] \times [-0.2, 0.2] \times [0, 1]$ of 1-periodic solutions of equation $x''(t) = \min\{|x| - 1\,,\, 2(|x| - 1)\} + \lambda\,(2|x| + \sin(2\pi t))$.

Example 3.13 *Consider equation* (1) *with $g(x) = -\frac{(x^2-1)|x^2-1|}{1+x^4}$ and $T = 1$. Our result yields the existence of two 1-periodic solutions for small values of $\lambda > 0$. Figure 4 shows part of the set of triples (p, v, λ) for which* (2) *with a given undelayed (for graphical reasons) perturbation admits a 1-periodic solution with initial conditions $x(0) = p$ and $x'(0) = v$.*

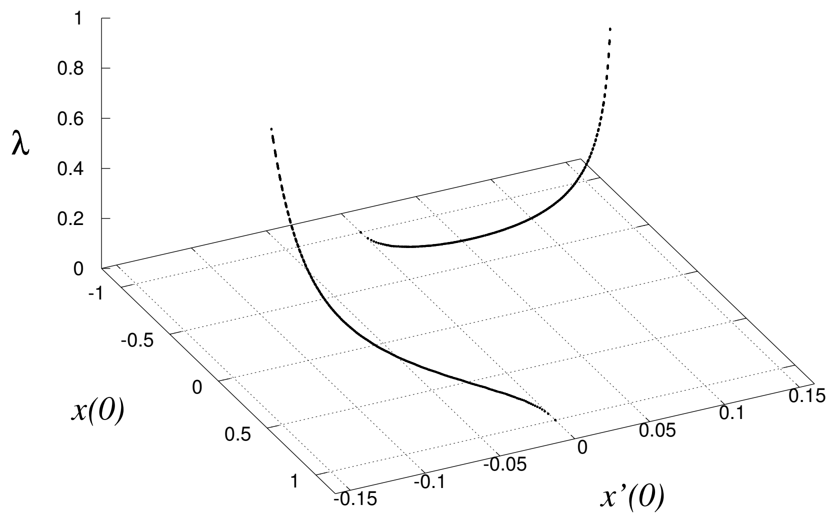


Figure 4. Initial conditions in the box $[-1.2, 1.2] \times [-0.16, 0.16] \times [0, 1]$ of 1-periodic solutions of equation $x''(t) = -\frac{(x^2-1)|x^2-1|}{1+x^4} + 10\lambda(x\sin(2\pi t) - |x|)$.

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