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## LOCAL INVERSION OF A CLASS OF PIECEWISE REGULAR MAPS

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**ABSTRACT.** This paper provides sufficient conditions for any map  $L$ , that is strongly piecewise linear relatively to a decomposition of  $\mathbb{R}^k$  in admissible cones, to be invertible. Namely, via a degree theory argument, we show that when there are at most four convex pieces (or three pieces with at most a non convex one), the map is invertible. Examples show that the result cannot be plainly extended to a greater number of pieces. Our result is obtained by studying the structure of strongly piecewise linear maps. We then extend the results to the  $PC^1$  case.

**1. Introduction.** This paper is devoted to the investigation of the local invertibility of a class of  $PC^1$  maps that enjoy a simple nondifferentiability structure. We point out that  $PC^1$  or, more generally,  $PC^r$  maps naturally appear in many contexts, see e.g., [16, Ch. 1], in particular in applied dynamical systems see, e.g., [3] and the references therein.

Our research is carried out by a preliminary analysis of the global invertibility of a type of continuous piecewise linear maps that we call strongly piecewise linear maps (see Definition 2.2). Indeed, in this paper, we continue the study of local invertibility of  $PC^1$  maps that we began in [13] in the low dimensional case, motivated by invertibility problems arising in Optimal Control Theory, see [11, 12, 14]. Here, whenever possible, we extend the results obtained in [13] to the higher dimensional case throughout a deeper investigation of the structure of strongly piecewise linear maps. Our main result concerning strongly piecewise linear maps, Theorem 3.11, implies that when there are at most four convex pieces, the map is globally invertible with a strongly piecewise linear inverse. The same holds when there are three pieces and at most one is not convex. We provide some counterexamples to show that

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the assumptions of Theorem 3.11 are sharp: Example 3.12 describes a pie-sliced map with four pieces one of which is not convex, while Example 3.13 shows a non invertible map with five convex pieces.

The approach followed in this paper relies mainly on [10] through an estimation of the degree of the map one wishes to invert. This degree, in two dimensions, was estimated in [13] by means of the winding number of the image of the one-dimensional sphere  $S^1$ . In higher dimension we follow a similar technique using a hypersurface integral formulation described in [15].

It is worth mentioning that already in Example 4.3 of [13] we partially addressed the invertibility problem of strongly piecewise linear maps with an additional special structure that, in the present paper, we call *pie-sliced*, see Definition 2.3. It would be tempting to try to get invertibility results by combining sufficient local conditions for invertibility like Theorem 2.3.2 in [16] with lower dimensional results as Theorem 4.2 of [13]. Such an approach would bear fruit in the case when the map to be inverted has (at least locally) a pie-sliced structure. However, Example 3.5 shows that there are strongly piecewise linear maps which are not pie-sliced. Indeed, the pie-slicing is not merely a property of the spatial decomposition but it also depends on the map structure, as shown in Example 2.4. Theorem 3.1 shows that non pie-sliced strongly piecewise linear maps in  $\mathbb{R}^3$  are subject to a certain rigidity, namely there are no such maps having exactly three genuine pieces.

In the last section of the paper we focus on local invertibility of  $PC^1$  maps. To do so, we take advantage of Theorem 3.11 and the results of [7, 10], which use the notion of Bouligand derivative, to extend the local invertibility results for  $PC^1$  maps of [13]. Notice that in the literature there are sufficient local invertibility results for more general nonsmooth maps based on more advanced notions of differential, for instance the results of [4] based on Clarke's generalized gradient. Bouligand derivative, however, seems to be particularly well suited for  $PC^1$  maps where the linearized cones have nonempty interior. In fact, in [13] we provided an example showing that our invertibility result cannot be deduced from Clarke's inverse function theorem. It should not be a surprise that it may be convenient to invoke different invertibility theorems according to the particular situation at hand. For example in the papers [11, 12, 14] we had to resort both to the results in [13] and to Clarke's inversion theorem depending on the structure of the piecewise linearization of the map that had to be inverted.

Finally, we point out that our usage of degree theory and of the notions of Bouligand derivative are completely transparent to the end-user of our invertibility result, since their application is completely hidden inside the proofs.

## 2. Preliminaries.

**2.1. Some notions of nonsmooth analysis.** In this Section we give some basic definitions from nonsmooth analysis. For the sake of readability we adapt such definitions to the framework where we are going to use them.

Following [7], a continuous function  $f: U \subseteq \mathbb{R}^k \rightarrow \mathbb{R}^m$  is a *continuous selection of  $C^1$  functions* if there exists a finite number of  $C^1$  functions  $f_1, \dots, f_\ell$ , of  $U$  into  $\mathbb{R}^m$  such that the *active index set*  $\mathcal{I} := \{i: f(x) = f_i(x)\}$  is nonempty for each  $x \in U$ . The functions  $f_i$ 's are called *selection functions* of  $f$ . The function  $f$  is called a  *$PC^1$  function* if at every point  $x \in U$  there exists a neighborhood  $V$  such that the restriction of  $f$  to  $V$  is a continuous selection of  $C^1$  functions.

A function  $f: \mathbb{R}^k \rightarrow \mathbb{R}^m$  is said to be *piecewise linear* if it is a continuous selection of linear functions.

**Definition 2.1.** A cone  $C \subseteq \mathbb{R}^k$  with vertex at the origin is a positively homogeneous set, i.e. if  $v \in C$  then  $\alpha v \in C$  for all  $\alpha \geq 0$ .

For our purposes we need some more specialized definitions

- A *polyhedral cone* is a cone  $C \subseteq \mathbb{R}^k$  with nonempty connected interior and vertex at the origin which is the intersection of a finite number of closed half-spaces whose boundary contains the origin.
- Let  $\pi_1 \neq \pi_2 \subset \mathbb{R}^k$  be two half hyper-planes with common boundary  $\partial\pi_1 = \partial\pi_2$  containing the origin. Thus  $\mathbb{R}^k \setminus (\pi_1 \cup \pi_2)$  is an open set with two connected components  $A_1$  and  $A_2$ . We call each connected component an *open wedge* of  $\mathbb{R}^k$ . The closure of an open wedge of  $\mathbb{R}^k$  is called a *wedge* of  $\mathbb{R}^k$ .
- An *admissible cone*  $C \subseteq \mathbb{R}^k$  is a cone with nonempty interior and vertex at the origin which is given by the intersection of a finite number of wedges of  $\mathbb{R}^k$ , hence it is closed.

Notice that a polyhedral cone is necessarily convex whereas an admissible one might not be so.

Linearity and the open mapping theorem easily yield that if  $L: \mathbb{R}^k \rightarrow \mathbb{R}^k$  is an invertible linear mapping and  $C$  is a polyhedral cone, then so is  $L(C)$ . As one can easily check, this property no longer holds if  $L$  is not invertible.

**Definition 2.2.** A *decomposition in admissible cones* of  $\mathbb{R}^k$  is a finite collection  $C_1, \dots, C_n$  of closed admissible cones with pairwise disjoint interiors and such that  $\mathbb{R}^k = \cup_{i=1}^n C_i$ . Notice that  $C_i \cap C_j = \partial C_i \cap \partial C_j$  for any  $1 \leq i < j \leq n$ .

A *strongly piecewise linear map (at the origin of  $\mathbb{R}^k$ )* is a continuous function  $G: \mathbb{R}^k \rightarrow \mathbb{R}^k$  such that there exist a decomposition  $C_1, \dots, C_n$  of  $\mathbb{R}^k$  in admissible cones, and linear maps  $L_1, \dots, L_n$  with

$$G(x) = L_i x, \quad \text{for } x \in C_i.$$

Finally we say that  $G$  is *nondegenerate* if  $\text{sign}(\det L_i)$  is constant and nonzero for all  $i = 1, \dots, n$ .

As a particular case of decomposition in admissible cones, we consider the so-called (by analogy to the 3-dimensional case) *pie-slice decompositions*.

**Definition 2.3** (Pie-sliced decomposition). A decomposition of  $\mathbb{R}^k$  in admissible cones  $C_1, C_2, \dots, C_n$  is called a pie-slice decomposition if the intersection of all the cones contains a  $(k-2)$ -dimensional space  $V$  through the origin. Furthermore, let  $G: \mathbb{R}^k \rightarrow \mathbb{R}^k$  be a strongly piecewise linear map with respect to this decomposition with

$$G(x) = L_i x, \quad x \in C_i, \quad i = 1, \dots, n.$$

We say that  $G$  *respects this pie-slicing* (or, informally, that  $G$  is pie-sliced) if there exists a subspace  $W$  of  $\mathbb{R}^k$  such that  $\mathbb{R}^k = W \oplus V$  and,

- (A)  $W$  is invariant under the action of  $L_1, \dots, L_n$ , i.e.,  $L_1(W) = \dots = L_n(W) = W$ ,
- (B)  $L_1|_V = \dots = L_n|_V$ .

**Remark 1.** When  $k = 3$  a simple characterization of not pie-sliced decompositions can be given:  $C_1, \dots, C_n$  is a not pie-sliced decomposition of  $\mathbb{R}^3$  if and only if  $\bigcap_{i=1}^n C_i = \{0\}$ .

Using Theorem 4.2 of [13] we get an invertibility result for pie-sliced maps.

**Proposition 1.** *Assume  $C_1, C_2, \dots, C_n$  is a pie-sliced decomposition of  $\mathbb{R}^k$ , with  $n \leq 4$  and let  $V$  be the common  $(k-2)$ -dimensional linear space. If  $n = 4$  also assume that the cones are convex. Let  $G: \mathbb{R}^k \rightarrow \mathbb{R}^k$  be a nondegenerate strongly piecewise linear map with respect to this decomposition with*

$$G(x) = L_i x, \quad x \in C_i, \quad i = 1, \dots, n.$$

*Assume also that  $G$  respects this pie-slicing. Then  $G$  is invertible with strongly piecewise linear inverse.*

*Proof.* Since  $G$  respects the pie-slicing  $C_1, \dots, C_n$  then there exist subspaces  $V$  and  $W$  as in Definition 2.3. Let  $w_1, w_2, v_3, \dots, v_k$  be a basis of  $\mathbb{R}^k$  such that  $W = \text{span}\{w_1, w_2\}$  and  $V = \text{span}\{v_3, \dots, v_k\}$ . By assumptions (A) and (B) in Definition 2.3, with respect to this basis we can write  $L_i$  in block-matrix form according to the following template:

$$L_i \simeq \left( \begin{array}{c|c} A_i & * \\ \hline 0 & B \end{array} \right), \quad (1)$$

where  $A_i$ , for  $i = 1, \dots, n$ , and  $B$  are square matrices expressing the actions of the maps  $L_i$  on  $W$  and  $V$ , respectively. Notice that the  $A_i$ 's are  $2 \times 2$  matrices and  $B$  is a  $(k-2) \times (k-2)$  matrix that does not depend on  $i$ . The matrix representing  $L_i$  is block-upper-triangular and, since all the maps  $L_i$  are nonsingular, the matrices  $A_i$ 's and  $B$  are nonsingular.

Clearly, the restriction  $G|_W$  is invertible by Theorem 4.2 of [13]. Similarly, since  $G(x) = L_1 x = \dots = L_n x$ , for any point  $x \in V$ , and the  $L_i$ 's are isomorphisms,  $G$  is invertible on  $V$ . Given any vector  $y \in \mathbb{R}^k$ , we can obtain  $G^{-1}(y)$  by the following argument. Write  $y = z + x$  where  $z \in W$  and  $x \in V$ . Let  $i$  be such that  $y \in C_i$ , then

$$G^{-1}(y) = L_i^{-1}(y) = L_i^{-1}(x) + L_i^{-1}(z) = L_1^{-1}(x) + (G|_W)^{-1}(z). \quad \square$$

Notice that if  $G$  is a strongly piecewise linear map as in Definition 2.2 above, then  $G$  is positively homogeneous.

It is very easy to construct examples of strongly piecewise linear maps respecting an inherent pie-sliced decomposition. Such maps, however, do not represent the whole range of possibilities. The following example shows a strongly piecewise linear map that does *not* respect its underlying pie-sliced decomposition.

**Example 2.4.** Let

$$C_1 = \{(x, y, z) \in \mathbb{R}^3 : x \geq 0, y \geq 0\}, \quad C_3 = \{(x, y, z) \in \mathbb{R}^3 : y \leq x, y \leq 0\}, \\ C_2 = \mathbb{R}^3 \setminus (C_1 \cup C_3).$$

$C_1, C_2, C_3$  is a pie-sliced decomposition of  $\mathbb{R}^3$ , with  $V = \{(x, y, z) \in \mathbb{R}^3 : x = y = 0\}$ . Set

$$L_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

One can check directly that, with this choice of  $L_1, \dots, L_3$ ,  $G$  as in Definition 2.2 is continuous. By inspection of the proof of Proposition 1, in particular by Equation (1), one sees that  $G$  does not respect the pie-slicing  $\{C_1, C_2, C_3\}$  of  $\mathbb{R}^3$ . It is very easy to prove that the only invariant plane under the action of  $L_1$  is the plane  $z = 0$ . However, this plane is not invariant for the map  $G$ , as illustrated by figure 1 which shows that the image of the plane  $z = 0$  under  $G$  is not even planar.

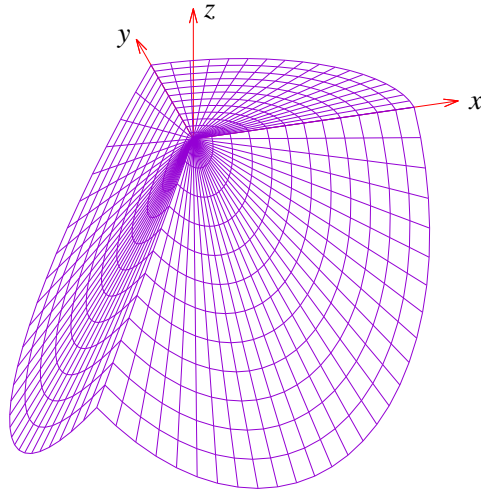


FIGURE 1. The image of the unit disk centered at the origin in the plane  $z = 0$  under the map  $G$  of Example 2.4.

Example 3.2 in the next section provides further examples in this direction. In the first part of this paper we deal with global invertibility of nondegenerate strongly piecewise linear maps. The following facts from [13] will be used in the sequel, see also [16].

**Lemma 2.5** ([13, Lemma 2.2]). *Let  $G: \mathbb{R}^k \rightarrow \mathbb{R}^k$  be a strongly piecewise linear map as in Definition 2.2, and let  $U$  be an open neighborhood of  $0 \in \mathbb{R}^k$ . Assume that the restriction  $G|_U: U \rightarrow G(U)$  is invertible with continuous inverse, then  $G$  is globally invertible and its inverse is a strongly piecewise linear map as well.*

**Proposition 2** ([13, Lemma 2.1]). *Let  $G$  be a strongly piecewise linear map from  $\mathbb{R}^k$  into itself. If  $G$  is invertible, then it is nondegenerate.*

It is not difficult to see that the converse of the latter statement is not true (see for instance Examples 4.1 and 4.2 in [13]). Our main concern will be finding simple sufficient conditions for the invertibility. Before dealing with this problem, however, we need some preliminaries.

A classical notion which we need is that of Bouligand derivative. Let  $U \subseteq \mathbb{R}^s$  be open and let  $f: U \rightarrow \mathbb{R}^m$  be locally Lipschitz. We say that  $f$  is *Bouligand differentiable* at  $x_0 \in U$  if there exists a positively homogeneous function,  $f'(x_0, \cdot): \mathbb{R}^s \rightarrow \mathbb{R}^m$  with the property that

$$\lim_{x \rightarrow x_0} \frac{\|f(x) - f(x_0) - f'(x_0, x - x_0)\|}{\|x - x_0\|} = 0. \quad (2)$$

This uniquely determined function  $f'(x_0, \cdot)$  is called the *Bouligand derivative* of  $f$  at  $x_0$  (see Examples 5.1 and 5.2 [13]). An important fact proved by Kuntz/Scholtes [7] is the following:

**Proposition 3** (Prop. 2.1 in [7]). *Let  $U \subseteq \mathbb{R}^s$  be an open set. Any  $PC^1$  function  $f: U \rightarrow \mathbb{R}^m$  is locally Lipschitz and, at every  $x_0 \in U$ , it has a piecewise linear Bouligand derivative  $f'(x_0, \cdot)$  which is a continuous selection of the Fréchet derivatives of the selection functions of  $f$  at  $x_0$ .*

Following [10] we consider a generalization of the notion of Jacobian matrix  $\nabla f(x)$  of a function  $f: \mathbb{R}^k \rightarrow \mathbb{R}^k$  at a Fréchet differentiability point  $x$ . Let  $f: \mathbb{R}^k \rightarrow \mathbb{R}^k$  be locally Lipschitz at  $x_0$ . We define  $\text{Jac}(f, x_0)$  as the set of limit points of sequences  $\{\nabla f(x_j)\}$  where  $\{x_j\}$  is a sequence converging to  $x_0$  and such that  $f$  is Fréchet differentiable at  $x_j$  with Jacobian  $\nabla f(x_j)$ . One can see, as a consequence of Rademacher's Theorem, that  $\text{Jac}(f, x_0)$  is nonempty, see [10]. Moreover the convex hull of  $\text{Jac}(f, x_0)$  is equal to the Clarke generalized Jacobian  $\partial f(x_0)$  of  $f$  at  $x_0$ , see [4] or the book [5].

Let  $f: U \subseteq \mathbb{R}^k \rightarrow \mathbb{R}^k$  be a  $PC^1$  function (with selection functions  $f_i$ ). The relation between the Bouligand derivative and the above generalized notion of Jacobian is clarified by the following formula [10, Lemma 2]:

$$\text{Jac}(f'(x_0, \cdot), 0) \subseteq \text{Jac}(f, x_0) = \{\nabla f_i(x_0) : i \in \bar{\mathcal{I}}(x_0)\}, \quad (3)$$

where  $\bar{\mathcal{I}}(x_0) = \{i : x_0 \in \text{cl int}\{x \in U : i \in \mathcal{I}(x)\}\}$ , see e.g. [7]. Notice that by Proposition 3 the map  $f'(x_0, \cdot)$  is continuous and piecewise linear, hence it is locally Lipschitz. Thus  $\text{Jac}(f'(x_0, \cdot), 0)$  is well defined.

An important notion that we shall use in the remainder of the paper is that of Brouwer degree of a map. Major references for this topic are, for instance, Milnor [9], Deimling [6] and Lloyd [8]; see also [1] for a quick introduction. Let  $\Omega \subseteq \mathbb{R}^k$  be an open set and let  $f: \Omega \rightarrow \mathbb{R}^k$ . We say that  $f$  is *proper* if the preimage of any compact set is compact. A triple  $(f, U, p)$ , with  $p \in \mathbb{R}^k$  and  $f$  a proper map defined in some neighborhood of the closure of the open set  $U \subseteq \mathbb{R}^k$ , is said to be *admissible* if  $f^{-1}(p) \cap U$  is compact. Given an admissible triple  $(f, U, p)$ , it is defined an integer  $\text{deg}(f, U, p)$ , called the *degree of  $f$  in  $U$  respect to  $p$* . There are several possible approaches to this notion, the one pursued in the books cited above regards  $\text{deg}(f, U, p)$  as a sort of algebraic count of the elements of  $f^{-1}(p)$  that are contained in  $U$ . However, in this paper, following [2] and [15] it will be convenient to use an integral computational formula. For the sake of simplicity we consider only the case  $k \geq 3$ , since it is the only one we are concerned with. For the admissible triple  $(f, U, p)$ , if  $U$  is the interior of a manifold with piecewise smooth boundary, one has

$$\text{deg}(f, U, p) = \int_{\partial U} (f|_{\partial U})^* \varphi_{F_p}, \quad (4)$$

where “ $*$ ” denotes the pullback,  $\partial U$  is the boundary of  $U$  oriented by its exterior normal and  $\varphi_{F_p}$  is the  $k-1$  differential form given by

$$\varphi_{F_p}(x) := \sum_{i=1}^k (-1)^{i+1} \frac{\partial F_p}{\partial x_i}(x) dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_k.$$

In this formula the hat over  $dx_i$  denotes cancellation and  $F_p: \mathbb{R}^k \setminus \{p\} \rightarrow \mathbb{R}$  is defined by

$$F_p(x) := \frac{1}{k(2-k)\omega_k} |x-p|^{2-k}, \quad (5)$$

where  $\omega_k$  denotes the volume of the unit ball in  $\mathbb{R}^k$ . When  $p=0$ , the map  $F_0$  is the fundamental harmonic function in  $\mathbb{R}^k$  and we shall denote it just as  $F$ .

Observe that if  $f: \mathbb{R}^k \rightarrow \mathbb{R}^k$  is proper then  $\text{deg}(f, \mathbb{R}^k, p)$  is well-defined for any  $p \in \mathbb{R}^k$ . Moreover, one can prove that it is actually independent of the choice of  $p$ . In this case we shall simply write  $\text{deg}(f)$  instead of the more cumbersome  $\text{deg}(f, \mathbb{R}^k, p)$ .

The following two results of [10] play a crucial role in the following. Here, we slightly reformulate them to match our notation.

**Theorem 2.6** (Thm. 4 of [10]). *Let  $f: U \subseteq \mathbb{R}^k \rightarrow \mathbb{R}^k$  be a  $PC^1$  function. Then  $f$  is a Lipschitz local homeomorphism at  $x_0 \in U$  if and only if  $\text{Jac}(f, x_0)$  consists of matrices whose determinants have the same nonzero sign and, for a sufficiently small neighborhood  $U_0$  of  $x_0$ ,  $\deg(f, U_0, y_0)$ ,  $y_0 := f(x_0)$ , is well-defined and has value  $\pm 1$ .*

**Theorem 2.7** (Thm. 5 of [10]). *Let  $f: U \subseteq \mathbb{R}^k \rightarrow \mathbb{R}^k$  be a  $PC^1$  function, and let  $x_0 \in U$ . Assume that*

$$\text{Jac}(f, x_0) = \text{Jac}(f'(x_0, \cdot), 0),$$

*then the following statements are equivalent:*

1.  $f$  is a Lipschitz local homeomorphism at  $x_0 \in U$ ;
2.  $f'(x_0, \cdot)$  is bijective;
3.  $f'(x_0, \cdot)$  is a Lipschitz (global) homeomorphism.

*Moreover, if any of (1)–(3) holds, then  $f$  is a local  $PC^1$  homeomorphism at  $x_0$ .*

We conclude this subsection recalling the classical notion of Bouligand tangent cone. Let  $C \subseteq \mathbb{R}^k$  be a nonempty closed subset. Given  $x \in C$ , the *Bouligand tangent cone to  $C$  at  $x$*  is the set:

$$\{v \in \mathbb{R}^k : \exists \alpha_j \rightarrow 0^+, \exists v_j \rightarrow v \text{ s.t. } x + \alpha_j v_j \in C\}.$$

We also recall Proposition 3.1 from [13]:

**Proposition 4.** *Let  $A$  and  $B$  be linear automorphisms of  $\mathbb{R}^k$ . Assume that for some  $v \in \mathbb{R}^k \setminus \{0\}$ ,  $A$  and  $B$  coincide on the hyperplane  $\{v\}^\perp$ . Then, the map  $\mathcal{L}_{AB}$  defined by  $x \mapsto Ax$  if  $\langle v, x \rangle \geq 0$ , and by  $x \mapsto Bx$  if  $\langle v, x \rangle \leq 0$ , is a homeomorphism if and only if  $\det(A) \cdot \det(B) > 0$ .*

**3. A restriction on the structure of piecewise linear mappings.** In this section we show that the structure of a piecewise linear mapping in dimension greater than or equal to three is subject to some restriction. In particular in the following theorem we show that the space decomposition underlying a *nontrivial* continuous piecewise linear map, is either pie-sliced, or it must have at least four pieces.

**Theorem 3.1.** *Let  $C_1, C_2, C_3$  be a not pie-sliced decomposition of  $\mathbb{R}^3$  in admissible cones. Let  $L_1, L_2, L_3$  and  $G$  be as in Definition 2.2.*

*Then, either one of the cones is a half space and  $G$  is the continuous selection of two linear mappings only or  $L_1 = L_2 = L_3$  so that  $G: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  is a linear mapping.*

**Remark 2.** In particular every strongly piecewise linear mapping associated to a three pieces non-pie-sliced decomposition is either linear or has only two genuine selection functions. On the other hand there exist three pieces pie-sliced decompositions, as shown in Example 3.2. However the pie-sliced case for maps reduces to a two-dimensional problem which was considered in Proposition 1. In Example 3.5 we show that there exist piecewise linear maps which have four genuine selection functions where the associated decomposition of  $\mathbb{R}^3$  is non-pie-sliced. We recall that the case of a continuous selection of only two maps in  $\mathbb{R}^k$  has already been treated in [13], see also Proposition 4.

Before proving the theorem we describe the result with one example and provide two technical lemmata.

**Example 3.2.** Let  $\alpha, \beta \in \mathbb{R} \setminus \{0\}$ , and set

$$C_1 = \{(x, y, z) \in \mathbb{R}^3 : x \geq 0, y \geq 0\},$$

$$C_2 = \left\{ (x, y, z) \in \mathbb{R}^3 : x \leq 0, y \geq \frac{\beta}{\alpha}x \right\}, \quad C_3 = \mathbb{R}^3 \setminus \text{int}(C_1 \cup C_2).$$

as illustrated in Figure 2.

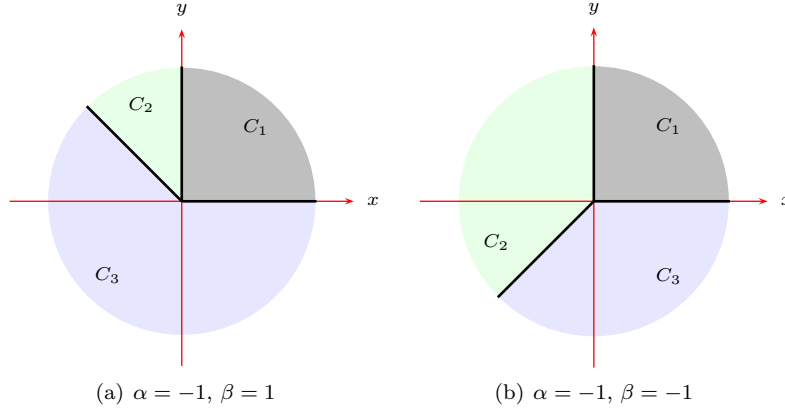


FIGURE 2. The three cones in Example 3.2 for different choices of  $\alpha$  and  $\beta$ . The  $z$  axis is not shown because it is assumed perpendicular to the page.

Let

$$L_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad L_2 = \begin{pmatrix} s_1 & 0 & 0 \\ s_2 & 1 & 0 \\ s_3 & 0 & 1 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 1 & (s_1 - 1)\frac{\alpha}{\beta} & 0 \\ 0 & 1 + s_2\frac{\alpha}{\beta} & 0 \\ 0 & s_3\frac{\alpha}{\beta} & 1 \end{pmatrix}.$$

For any choice of  $(s_1, s_2, s_3) \in \mathbb{R}^3$  such that  $s_1 > 0, 1 + s_2\frac{\alpha}{\beta} > 0$ , we have a strongly piecewise linear mapping  $G$ , as in Definition 2.2, of  $\mathbb{R}^3$  onto itself. Such choice is nontrivial, in the sense that it is not a linear map, if  $(s_1, s_2, s_3) \neq (1, 0, 0)$ . Notice that when  $s_3 = 0$  the map  $G$  respects the pie-slicing  $\{C_1, C_2, C_3\}$  and it is non-pie-sliced otherwise. Figure 3 shows the image of the unit sphere centered at the origin under  $G$  with different combinations of parameters; in particular, in the case of Figure 3(a),  $G$  is not pie-sliced (although the underlying space decomposition is pie-sliced) and, on the contrary, in the case of Figure 3(b),  $G$  is pie-sliced.

**Lemma 3.3.** *Let  $G: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be a strongly piecewise linear map associated to a decomposition in admissible cones  $C_1, \dots, C_n$  of  $\mathbb{R}^3$ . Assume there exist  $i, j$  such that  $C_i, C_j$  share three linearly independent vectors. Then  $L_i = L_j$ .*

*Proof.* Let  $v_1, v_2, v_3$  be linearly independent vectors belonging to  $C_i \cap C_j$ . The claim follows immediately from the simple fact that if  $A$  and  $B$  are two  $3 \times 3$  matrices, such that  $Av_s = Bv_s$  for  $s = 1, 2, 3$ , then  $A = B$ .  $\square$

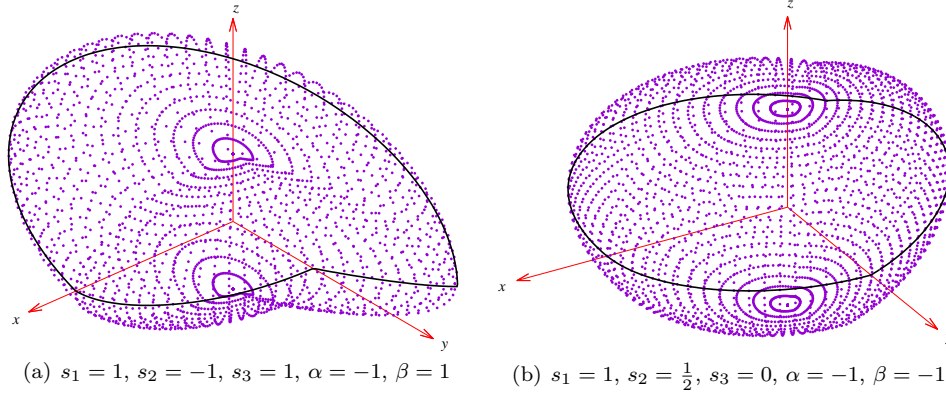


FIGURE 3. Image of the unit sphere under  $\mathcal{G}$  as in Example 3.2 for different choices of the parameters  $\alpha, \beta, s_1, s_2, s_3$ . The dark continuous line represents the image of the circle of radius 1 centered at the origin.

**Lemma 3.4.** *Let  $C_1, C_2, C_3$  be a not pie-sliced decomposition of  $\mathbb{R}^3$  in admissible cones. Then there exist indices  $\alpha, \beta, 1 \leq \alpha < \beta \leq 3$  such that  $\partial C_\alpha \cap \partial C_\beta$  contains three linearly independent vectors.*

*Proof.* Assume by contradiction that  $\partial C_i \cap \partial C_j, 1 \leq i < j \leq 3$ , never contains three linearly independent vectors. Then there exist three different planes  $\pi_1, \pi_2, \pi_3$  through the origin such that

$$C_1 \cap C_2 \subset \pi_3, \quad C_2 \cap C_3 \subset \pi_1, \quad C_1 \cap C_3 \subset \pi_2.$$

Let us now prove that  $\pi_1 \cap \pi_2 \cap \pi_3 = \{0\}$ . Clearly, since  $C_1, C_2, C_3$  is a decomposition of  $\mathbb{R}^3$  in admissible cones, this intersection cannot be a two-dimensional space. Assume, by contradiction, that  $\pi_1 \cap \pi_2 \cap \pi_3$  is a one-dimensional space  $r$ . The planes  $\pi_1, \pi_2, \pi_3$  define six (minimal) convex wedges  $W_1, \dots, W_6$ . Observe that if  $\text{int}(C_1) \cap W_{\bar{j}} \neq \emptyset$  for some  $\bar{j} \in \{1, \dots, 6\}$ , then  $\text{int}(W_{\bar{j}}) \subseteq \text{int}(C_1)$ . Else, there would exist  $x \in \text{int}(C_1) \cap \text{int}(W_{\bar{j}})$  and  $y \in \text{int}(W_{\bar{j}}) \cap (\mathbb{R}^3 \setminus C_1)$  so that  $y \in \text{int}(C_2) \cup \text{int}(C_3)$ . Thus the segment joining  $x$  and  $y$  is contained in  $\text{int}(W_{\bar{j}})$  and intersects  $\partial(C_1) \cap \partial(C_2) = C_1 \cap C_2$  or  $\partial(C_1) \cap \partial(C_3) = C_1 \cap C_3$ . A contradiction since it does not intersect  $\pi_1 \cup \pi_2 \cup \pi_3$ . This contradiction proves that each of the cones  $C_1, C_2, C_3$  is given by the union of some of the  $W_j$ 's. Hence, since  $\mathbb{R}^3 = C_1 \cup C_2 \cup C_3$ , we have  $C_1 \cap C_2 \cap C_3 \supseteq r$ , a contradiction since the decomposition is not pie-sliced.

Up to a linear change of coordinates we can assume that

$$\pi_s = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_s = 0\}$$

so that each admissible cone  $C_i, i = 1, 2, 3$  is union of a finite number of adjacent octants. It is easy to see that in any such a combination, the assumption  $C_1 \cap C_2 \cap C_3 = \{0\}$  is violated. A contradiction.  $\square$

*Proof of Theorem 3.1.* Let  $C_\alpha, C_\beta$  be as in Lemma 3.4. Up to a permutation of the indices we can assume  $\alpha = 1$  and  $\beta = 2$  so that  $L_1 = L_2$  by Lemma 3.3. Let  $\hat{C} = C_1 \cup C_2$ . Either  $\partial \hat{C} \cap \partial C_3$  is a plane or it contains three linearly independent

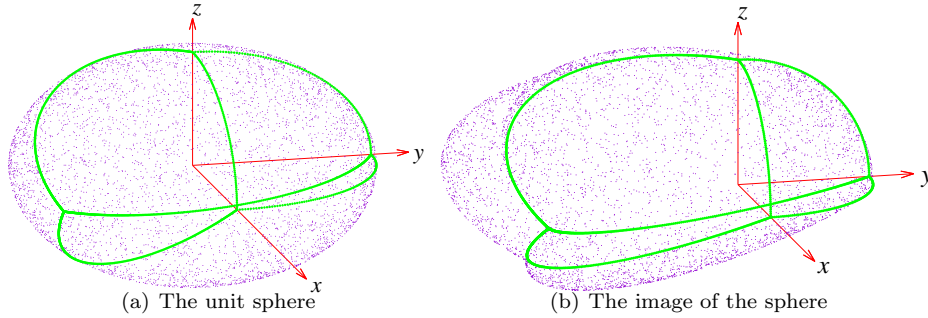


FIGURE 4. The image of the unit sphere centered at the origin under the map  $G$  of Example 3.5. The green lines on the left are the intersections of the cones  $C_1, \dots, C_4$  with the unit sphere, those on the right are their images.

vectors. In the former case the assertion follows from Lemma 4. In the latter case the assertion follows from Lemma 3.4.  $\square$

The following example shows the existence of piecewise linear maps associated to not pie-sliced decompositions consisting of at least four pieces.

**Example 3.5.** Let  $p_1, \dots, p_4$  be the following points in  $\mathbb{R}^3$ :

$$p_1 = (1, 0, 0), \quad p_2 = (0, 1, 0), \quad p_3 = (0, 0, 1), \quad p_4 = (-1, -1, -1),$$

And consider the following cones:

$$C_i := \left\{ (x, y, z) \in \mathbb{R}^3 : \exists (\alpha_1, \dots, \alpha_4) \in \mathbb{R}_+^4 \text{ s.t. } (x, y, z) = \sum_{j=1, j \neq i}^4 \alpha_j p_j \right\},$$

each one consisting of the set of half lines through one face of the simplex with vertices in the points  $p_1, \dots, p_4$  (i.e., their convex envelope). Put

$$L_1 = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$L_3 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad L_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(See Figure 4.) It is not difficult to prove that the piecewise function  $\mathbb{R}^3 \rightarrow \mathbb{R}^3$  defined by

$$G := L_i x \text{ for } x \in C_i,$$

is strongly piecewise linear. Thus  $G$  is an example of a continuous piecewise linear function which is not the trivial extension of a planar continuous strongly piecewise linear function.

As a consequence of our main result, Theorem 3.10 below, we have that the map  $G$  of Example 3.5 is invertible.

The map  $G$  features another structure (or lack of it): In a neighborhood of any one of the points  $p_1, p_2$  and  $p_4$  it is not locally reducible (by a translation) to a

strongly piecewise linear map that respects a pie-slicing of the space. This follows by an argument as in the first part of the proof of Proposition 1 see, in particular, formula (1), where  $V$  is the line  $x = y = 0$  and  $W$  is the plane  $z = 0$ . This is not the case with  $p_3$ .

**3.1. Computation of the degree.** We start with an easy but fundamental technical lemma.

**Lemma 3.6.** *Let  $c_1, \dots, c_k \in \mathbb{R}$ . For any  $n \in \{1, \dots, k\}$  let  $A_n = (a_{ij}^{(n)})_{i,j=1,\dots,k}$  be the  $k \times k$  matrix such that*

$$a_{ij}^{(n)} = \begin{cases} \delta_{ij} & j \neq n, \\ c_i & j = n. \end{cases}$$

Then  $\det A_n = c_n$  for any  $n = 1, \dots, k$ .

*Proof.* In each matrix  $A_n$  each column  $j$  but the  $n$ -th column is given by the  $j$ -th element of the canonical basis of  $\mathbb{R}^k$ . Thus, evaluating  $\det A_n$  along the  $n$ -th row of  $A_n$  we get  $\det A_n = c_n \det I_{k-1} = c_n$ , where  $I_{k-1}$  denotes the identity matrix on  $\mathbb{R}^{k-1}$ .  $\square$

Let  $S \subset \mathbb{R}^k$  be a  $(k-1)$ -dimensional parametrised manifold defined in an open set  $W \subset \mathbb{R}^{k-1}$  and let  $\varphi$  be a  $(k-1)$ -form on  $S$ . We write  $\varphi$  in coordinates as

$$\varphi(x) = \sum_{i=1}^k \varphi_i(x) dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_k, \quad (6)$$

where, we recall, the hat “ $\widehat{\phantom{x}}$ ” over  $dx_i$  denotes cancellation. If  $S := \{x(u) : u \in W\}$  we get

$$\int_S \varphi = \int_W \sum_{i=1}^k \varphi_i(x(u)) \det \left[ \frac{\partial (x_1, \dots, \widehat{x_i}, \dots, x_k)}{\partial (u_1, \dots, u_{k-1})} \right] (u_1, \dots, u_{k-1}) du_1 \dots du_{k-1} \quad (7)$$

where  $S$  is given the orientation induced by the parametrization. Assume  $S$  is contained in a hyperplane passing through the origin. Up to a change of coordinates we can assume the equations of  $S$  are given by

$$S: \begin{cases} x_i(u) = u_i, & i = 1, \dots, k-1, \\ x_k(u) = \sum_{i=1}^{k-1} \alpha_i u_i, \end{cases} \quad (8)$$

i.e. we can assume  $S$  is contained in the hyperplane  $x_k = \sum_{i=1}^{k-1} \alpha_i x_i$ .

**Lemma 3.7.** *Let  $x: W \rightarrow \mathbb{R}^k$  be as in (8). Then*

$$\det \left[ \frac{\partial (x_1, \dots, \widehat{x_i}, \dots, x_k)}{\partial (u_1, \dots, u_{k-1})} \right] (u_1, \dots, u_{k-1}) = \begin{cases} (-1)^{k+i-1} \alpha_i, & i = 1, \dots, k-1 \\ 1, & i = k. \end{cases}$$

*Proof.* For each  $i = 1, \dots, k$  let

$$B_i := \frac{\partial (x_1, \dots, \widehat{x_i}, \dots, x_k)}{\partial (u_1, \dots, u_{k-1})} (u_1, \dots, u_{k-1}).$$

Clearly  $B_k = I_{k-1}$ , the identity matrix on  $\mathbb{R}^{k-1}$ . If  $i = 1, \dots, k-1$ , in the  $k$ -th column of  $B_i$ , all the entries but the last one are null, with  $(B_i)_{k-1,i} = \alpha_i$  and, by Lemma 3.6,  $\det B_i = (-1)^{k+i-1} \alpha_i$ .  $\square$

Thus, we can now give an explicit expression for the integral in (7):

$$\int_S \varphi = \int_W \sum_{i=1}^{k-1} (-1)^{k+i-1} \alpha_i \varphi_i(x(u)) \, du_1 \dots du_{k-1} + \int_W \varphi_k(x(u)) \, du_1 \dots du_{k-1}.$$

We now go back to the fundamental harmonic function of  $\mathbb{R}^k$ ,  $k \geq 3$ , defined in (5)

$$F(x) = F_0(x) = \frac{1}{k(2-k)\omega_k} |x|^{2-k}, \quad x \in \mathbb{R}^k \setminus \{0\},$$

and we show that  $\deg(F) = 1$ . In order to prove this equality we take advantage of the results in [15] which reduce the computation of the degree to the integration of the following  $(k-1)$ -form on an hypersurface of  $\mathbb{R}^k$ :

$$\varphi_F(x) := \sum_{i=1}^k (-1)^{i+1} \frac{\partial F}{\partial x_i}(x) \, dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_k. \quad (9)$$

Recalling that we stipulated to write  $\deg(f)$  instead of  $\deg(f, \mathbb{R}^k, 0)$  whenever  $f: \mathbb{R}^k \rightarrow \mathbb{R}^k$  is a proper map, from (4) we obtain

$$\deg(f) = \int_{\mathcal{S}^{k-1}} (f|_{\mathcal{S}^{k-1}})^* \varphi_F = \int_{f(\mathcal{S}^{k-1})} \varphi_F, \quad (10)$$

where  $\mathcal{S}^{k-1}$  is oriented by its exterior normal and  $f(\mathcal{S}^{k-1})$  has the orientation induced by  $f$ . See also Definition 3.2 and Proposition 3.3 in [15].

**Lemma 3.8.** *With  $\varphi_F$  and  $S$  as above  $\int_S \varphi_F = 0$ .*

*Proof.* As  $\frac{\partial F}{\partial x_i}(x) = \frac{1}{k\omega_k} |x|^{-k} x_i$ , we get

$$\begin{aligned} \int_S \varphi_F &= \frac{1}{k\omega_k} \int_U \sum_{i=1}^{k-1} (-1)^{k+i-1} \alpha_i (-1)^{i+1} |x(u)|^{-k} u_i \, du_1 \dots du_{k-1} + \\ &\quad + \frac{1}{k\omega_k} \int_U (-1)^{k+1} |x(u)|^{-k} x_k(u) \, du_1 \dots du_{k-1} = 0 \end{aligned}$$

since  $x_k(u) = \sum_{i=1}^{k-1} \alpha_i u_i$ .  $\square$

**Lemma 3.9.** *Let  $\varphi_F$  be the  $(k-1)$ -form defined in (9). Then  $d\varphi_F = 0$ .*

*Proof.*

$$\begin{aligned} d\varphi_F &= \frac{1}{k(2-k)\omega_k} \sum_{i=1}^k \left\{ (-1)^{i+1} \left( \sum_{j=1}^k \frac{\partial^2 F}{\partial x_i \partial x_j} dx_j \right) \wedge dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_k \right\} \\ &= \frac{1}{k(2-k)\omega_k} \sum_{i=1}^k \left\{ (-1)^{i+1} (-1)^{i-1} \frac{\partial^2 F}{\partial x_i^2}(x) dx_1 \wedge \dots \wedge dx_k \right\} \\ &= \frac{1}{k(2-k)\omega_k} \Delta F(x) dx_1 \wedge \dots \wedge dx_k = 0 \end{aligned}$$

since  $F$  is the fundamental harmonic function of  $\mathbb{R}^k$ .  $\square$

Let us now consider a polyhedral cone  $C$ , let  $L: \mathbb{R}^k \rightarrow \mathbb{R}^k$  be a linear nonsingular map and let  $\Gamma := L(\mathcal{S}^{k-1} \cap C)$ . We seek an estimate for the integral of  $\varphi_F$  on  $\Gamma$ . For our purposes it is sufficient to confine ourselves to the case when  $L$  is orientation-preserving, that is when  $\det L > 0$ . Orienting  $\mathcal{S}^{k-1}$  by its exterior normal as above, we get an orientation for  $\mathcal{S}^{k-1} \cap C$ , so that  $\Gamma$  receives the orientation induced by  $L$ .

Analogously to Lemma 4.2 in [13] and using the fact that  $\det L > 0$  we obtain that  $L(C)$  is still a polyhedral cone. Let  $\rho$  be small enough so that  $\rho\mathcal{S}^{k-1}$  does not intersect  $\Gamma$ . Let  $R \subset \mathbb{R}^k$  be the bounded region of  $\mathbb{R}^k$  delimited by the smooth surfaces  $\Gamma$ ,  $\rho\mathcal{S}^{k-1} \cap L(C)$  and  $L(\partial C)$  (so that the boundary of  $R$  is piecewise regular). By Lemma 3.9  $\int_R d\varphi_F = 0$ . Applying Stokes' theorem to  $R$  and taking into account that, by Lemma 3.8,  $\int_S \varphi_F = 0$  for any  $S \subset L(\partial C)$ , we obtain

$$0 = \int_R d\varphi_F = \int_{\partial R} \varphi_F = - \int_{\rho\mathcal{S}^{k-1} \cap L(C)} \varphi_F + \int_{\Gamma} \varphi_F$$

where, we recall,  $\partial R$  is oriented by its exterior normal and  $\rho\mathcal{S}^{k-1} \cap L(C)$  receives the orientation of  $\mathcal{S}^{k-1}$ , which explains the minus sign in the first addendum of the right hand side of the equation. Thus we get

$$\int_{\rho\mathcal{S}^{k-1} \cap L(C)} \varphi_F = \int_{\Gamma} \varphi_F. \quad (11)$$

Notice that, as  $\rho\mathcal{S}^{k-1} \cap L(C)$  has nonempty interior relatively to  $\mathcal{S}^{k-1}$ ,

$$\int_{\rho\mathcal{S}^{k-1} \cap L(C)} \varphi_F > 0. \quad (12)$$

Moreover, if  $x: u \in U \subset \mathbb{R}^{k-1} \mapsto x(u) \in \mathbb{R}^k$  is a parametrization for  $\mathcal{S}^{k-1} \cap L(C)$ , then  $\rho x: u \in U \subset \mathbb{R}^{k-1} \mapsto \rho x(u) \in \mathbb{R}^k$  is a parametrization for  $\rho\mathcal{S}^{k-1} \cap L(C)$ . Thus, by (11), being  $L$  orientation preserving, we obtain

$$\begin{aligned} \int_{\Gamma} \varphi_F &= \int_{\rho\mathcal{S}^{k-1} \cap L(C)} \varphi_F = \\ &= \int_U \sum_{i=1}^k (-1)^{i+1} \rho^{1-k} \frac{x_i}{|x|^k} \det \frac{\partial(\rho x_1, \dots, \widehat{\rho x_i}, \dots, \rho x_k)}{\partial(u_1, \dots, u_{k-1})} du_1 \dots du_{k-1} = \\ &= \int_U \sum_{i=1}^k (-1)^{i+1} \frac{x_i}{|x|^k} \det \frac{\partial(x_1, \dots, \widehat{x_i}, \dots, x_k)}{\partial(u_1, \dots, u_{k-1})} du_1 \dots du_{k-1} = \int_{\mathcal{S}^{k-1} \cap L(C)} \varphi_F. \end{aligned} \quad (13)$$

With our notation, according to formula (4) we have

$$1 = \deg(\mathbb{I}_k) = \int_{\mathcal{S}^{k-1}} \varphi_F, \quad (14)$$

$\mathbb{I}_k$  being the identity map on  $\mathbb{R}^k$ . Thus, the radial symmetry of  $F$  shows that for any given hemisphere  $H \subseteq \mathcal{S}^{k-1}$ , one has  $\int_H \varphi_F = \frac{1}{2}$ , where  $H$  receives its orientation from  $\mathcal{S}^{k-1}$ .

Since  $L(C)$  is convex (being a polyhedral cone) we have that  $\mathcal{S}^{k-1} \cap L(C)$  is strictly contained in some hemisphere  $\mathcal{H} \subseteq \mathcal{S}^{k-1}$ . Thus,

$$\int_{\Gamma} \varphi_F = \int_{\mathcal{S}^{k-1} \cap L(C)} \varphi_F < \int_{\mathcal{H}} \varphi_F = \frac{1}{2}.$$

Finally, taking also (12) and (13) into account, we find the estimate we were looking for:

$$0 < \int_{\Gamma} \varphi_F < \frac{1}{2}. \quad (15)$$

**Remark 3.** Inequality (15) could be also deduced from a geometrical argument. Observe that  $k\omega_k\varphi_F$  is the area element of  $\mathcal{S}^{k-1}$ . Thus, by (13),

$$k\omega_k \int_{\Gamma} \varphi_F = \int_{\mathcal{S}^{k-1} \cap L(C)} k\omega_k \varphi_F.$$

The integral on the right hand side is the solid angle described by  $L(C)$  and thus,  $\int_{\Gamma} \varphi_F$  represents the fraction of solid angle described by  $L(C)$  (recall that  $L$  is orientation-preserving). The left and right inequalities in (15) follow, respectively, from the facts that  $L(C)$  has nonempty interior and that  $\mathcal{S}^{k-1} \cap L(C)$  is contained in a hemisphere.

**Remark 4.** Notice that if  $C$  is not a polyhedral cone but only an admissible cone, then it may not be convex. Nevertheless  $L(C)$  cannot be equal to the whole space  $\mathbb{R}^k$ . Hence, for an orientation preserving  $L$  as above, we get the strict inequality

$$0 < \int_{\Gamma} \varphi_F < 1. \quad (16)$$

**Theorem 3.10.** *Let  $n = 2, 3, 4$  and let  $C_1, \dots, C_n \subset \mathbb{R}^k$  be a finite family of polyhedral cones. Assume also that the interiors of  $\text{int}(C_1), \dots, \text{int}(C_n)$  are pairwise disjoint and  $\cup_{i=1}^n C_i = \mathbb{R}^k$ . Let  $G$  be a strongly piecewise linear map as in definition 2.2 such that  $\text{sgn}(\det L_i) = 1$ . Then  $\deg(G) = 1$ .*

*Proof.* By formula (10),  $\deg(G)$  can be computed via the formula

$$\deg(G) = \int_{\mathcal{S}^{k-1}} G|_{\mathcal{S}^{k-1}}^* \varphi_F = \int_{G(\mathcal{S}^{k-1})} \varphi_F,$$

where, we recall,  $G(\mathcal{S}^{k-1})$  has the orientation induced by  $G$ . Applying (15) to each addendum we get

$$0 < \int_{G(\mathcal{S}^{k-1})} \varphi_F = \sum_{i=1}^n \int_{L_i(C_i \cap \mathcal{S}^{k-1})} \varphi_F < \sum_{i=1}^n \frac{1}{2} < 2,$$

so that  $0 < \deg(G) < 2$ . As  $\deg(G)$  is an integer, the claim is proven.  $\square$

As a consequence of Theorem 3.10 we can now prove the main result of this paper:

**Theorem 3.11.** *Let  $G: \mathbb{R}^k \rightarrow \mathbb{R}^k$  be as in Definition 2.2 and non-degenerate. If one of the following conditions holds:*

1.  $n \in \{1, 2, 3\}$  and at most one cone is not convex;
2.  $n = 4$  and all the admissible cones are convex;

*then  $G$  has a strongly piecewise linear inverse.*

*Proof.* The case  $n = 1$  is trivial since  $G$  is linear. If  $n = 2$  it suffices to apply Proposition 4.

Let us now consider the cases  $n = 3, 4$ . Let us consider first the case when all the maps  $L_i$  that appear in the definition of  $G$  have positive determinants. If all the

cones are strictly convex then  $\deg(G) = 1$  by Theorem 3.10 and the claim follows from Theorem 2.6.

If, in the case  $n = 3$ , one cone is not convex then, as in the proof of Theorem 3.10 we get

$$0 < \deg(G) = \int_{G(S^{k-1})} \varphi_F = \sum_{i=1}^3 \int_{L_i(C_i \cap S^{k-1})} \varphi_F < \frac{1}{2} + \frac{1}{2} + 1 = 2,$$

where  $L_i$  and  $C_i$  are as in Theorem 3.10. Hence  $\deg(G) = 1$  and the claim follows from Theorem 2.6.

Consider now the case when  $\det L_i < 0$  for  $i = 1, \dots, n$ . Let  $r: \mathbb{R}^k \rightarrow \mathbb{R}^k$  be the reflection map given by  $(x_1, x_2, \dots, x_k) \mapsto (-x_1, x_2, \dots, x_k)$ . Let  $\mathcal{G}$  be the composition  $r \circ G$ . Clearly it is enough to prove that  $\mathcal{G}$  is invertible. One has that

$$\mathcal{G}(x) = \mathcal{L}_i x, \quad x \in C_i, \quad i = 1, \dots, n.$$

with  $\mathcal{L}_i = r \circ L_i$  for  $i = 1, \dots, n$ . Since  $\det \mathcal{L}_i = (\det r)(\det L_i) > 0$ , we have that  $\mathcal{G}$  is invertible by the first part of the proof.  $\square$

**Remark 5.** If  $k = n = 3$ , then the assumptions of Theorem 3.11 can be relaxed. Indeed by Theorem 3.1, then either the space decomposition is pie-sliced, hence there is at most one non convex piece, or the map  $G$  has at most two genuine pieces, so that the statement boils down to the case  $n \leq 2$ .

The following counterexamples show that the convexity assumption cannot be dropped for  $n = 4$  and that Theorem 3.11 cannot be extended to the case  $n \geq 5$ .

**Example 3.12.** In this example, invertibility fails because of lack of convexity of one sector. Let  $G := L_i x$  for  $x \in C_i$ , where

$$L_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 1 & 0 & 0 \\ 2\sqrt{3} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$L_3 = \begin{pmatrix} -2 & -\sqrt{3} & 0 \\ -\sqrt{3} & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad L_4 = \begin{pmatrix} 1 & 2\sqrt{3} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and the sectors are given, in cylindrical coordinates, by the triplets  $(\rho, \theta, \zeta)$  with arbitrary  $\rho$ 's and  $\zeta$ 's, and  $\theta$  chosen as in the following table:

$C_1$	$C_2$	$C_3$	$C_4$
$0 \leq \theta \leq \frac{\pi}{2}$	$\frac{\pi}{2} \leq \theta \leq \frac{2}{3}\pi$	$\frac{2}{3}\pi \leq \theta \leq \frac{11}{6}\pi$	$\frac{11}{6}\pi \leq \theta \leq 2\pi$

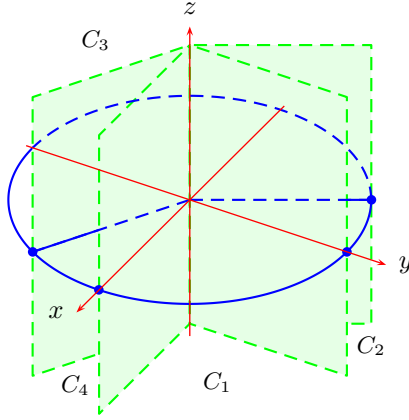
Figure 5 illustrates the decomposition of  $\mathbb{R}^3$  in the sectors  $C_1, \dots, C_4$ .

The above defined map  $L$  is not injective, hence not invertible. In fact, as shown in Example 4.2 in [13], the restriction of  $G$  to the plane  $z = 0$  is not injective.

The same technique used above to extend an example of [13] can be applied to construct a continuous piecewise linear function with selection index set consisting of five polyhedral cones, which is not invertible.

**Example 3.13.** Consider the nondegenerate continuous piecewise linear map  $G: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  defined by

$$G := L_i x \text{ for } x \in C_i,$$

FIGURE 5. The decomposition of  $\mathbb{R}^3$  of Example 3.12

where

$$L_1 = \begin{pmatrix} 1 & -\sqrt{2} & 0 \\ 0 & \sqrt{2}-1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad L_2 = \begin{pmatrix} -\sqrt{2} & 1-\sqrt{2} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$L_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1-\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad L_4 = \begin{pmatrix} \sqrt{2}-1 & 0 & 0 \\ -\sqrt{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad L_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the corresponding cones are given, in cylindrical coordinates, by the triplets  $(\rho, \theta, \zeta)$  with arbitrary  $\rho$ 's and  $\zeta$ 's, and  $\theta$  chosen as in the following table:

$C_1$	$C_2$	$C_3$	$C_4$	$C_5$
$0 \leq \theta \leq \frac{3}{8}\pi$	$\frac{3}{8}\pi \leq \theta \leq \frac{3}{4}\pi$	$\frac{3}{4}\pi \leq \theta \leq \frac{9}{8}\pi$	$\frac{9}{8}\pi \leq \theta \leq \frac{3}{2}\pi$	$\frac{3}{2}\pi \leq \theta \leq 2\pi$

As shown in Example 4.1 in [13], the restriction of  $G$  to the plane  $z = 0$  is not injective hence  $G$  cannot be invertible.

**4. Invertibility of some piecewise differentiable functions.** This section is devoted to a class of  $PC^1$  functions whose local invertibility can be established by the means of the theory developed in the previous section. A few results in this direction have been obtained in [13]. In particular, here, we are interested in a result in the same spirit of Theorem 3.11, namely, in the generalization to arbitrary dimension of [13, Corollary 5.3].

Here, for a Fréchet differentiable map  $\varphi$  at a point  $x_0$ , the (Fréchet) differential at  $x_0$  is denoted by  $d\varphi(x_0)$ . We base our discussion on the following Theorem from [13]:

**Theorem 4.1** ([13, Theorem 5.1]). *Let  $f$  be an  $\mathbb{R}^k$ -valued  $PC^1$  function in a neighborhood of  $x_0 \in \mathbb{R}^k$ . Assume that*

1. *All the determinants of all the elements of  $\text{Jac}(f, x_0)$  have the same sign;*
2. *The Bouligand differential of  $f$  at  $x_0$  is an invertible piecewise linear map.*

*Then  $f$  is locally invertible at  $x_0$ .*

The sought generalization is based on the following construction: Let  $f$  be an  $\mathbb{R}^k$ -valued  $PC^1$  function in a sufficiently small ball  $B(x_0, \rho) \subseteq \mathbb{R}^k$ , and let  $\mathcal{I}_0 = \{1, \dots, n\}$  be the active index set in  $B(x_0, \rho)$ . For each  $i \in \mathcal{I}_0$  define

$$S_i := \{x \in B(x_0, \rho) : f(x) = f_i(x)\}. \quad (17)$$

Let  $C_1, \dots, C_n$  be the tangent cones (in the sense of Bouligand) at  $x_0$  to the sectors  $S_1, \dots, S_n$ . Assume that the  $C_i$ 's are admissible cones and that

$$df_i(x_0)x = df_j(x_0)x \quad \text{for any } x \in C_i \cap C_j, i, j \in \{1, \dots, n\}, i \neq j.$$

Define

$$F(x) = df_i(x_0)x \quad x \in C_i, i = 1, \dots, n \quad (18)$$

so that  $F$  is a continuous piecewise linear map (compare [7]).

We are finally in a position to state and prove the following generalization of [13, Corollary 5.3] that was proven in dimension  $n = 2$ :

**Corollary 1.** *Let  $f$  and  $F$  be as above, with  $F$  nondegenerate at 0. We have that if either*

- $n \in \{1, 2, 3\}$ , and at most one admissible cone  $C_i$  associated to  $F$  is not convex or
- $n = 4$  and all the cones  $C_i$ 's are convex,

*then  $f$  is a Lipschitz homeomorphism in a sufficiently small neighborhood of  $x_0$ .*

**Remark 6.** If  $k = n = 3$ , then we can relax the assumptions as in Theorem 3.11.

*Proof.* Since  $F$  is nondegenerate then it is invertible by Theorem 3.11. Theorem 4.1, yields the assertion.  $\square$

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