Pointwise Calderon-Zygmund gradient estimates for the p-Laplace system

POINTWISE CALDERÓN-ZYGMUND GRADIENT ESTIMATES FOR THE p-LAPLACE SYSTEM

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Abstract

Pointwise estimates for the gradient of solutions to the p-Laplace system with right-hand side in divergence form are established. Their formulation involves the sharp maximal operator, whose properties enable us to develop a nonlinear counterpart of the classical Calderón-Zygmund theory for the Laplacian. As a consequence, a flexible, comprehensive approach to gradient bounds for the p-Laplace system for a broad class of norms is derived. The relevant gradient bounds are just reduced to norm inequalities for a classical operator of harmonic analysis. In particular, new gradient estimates are exhibited which augment the available literature in the elliptic regularity theory.

Résumé

Des inegalités en chaque point pour le gradient de la solution du système du p-Laplacien lorsque le second membre est en forme de divergence sont établies ici. Dans leur formulation apparait l'opérateur maximal, don't les propriétés nous permettent d'établir une contrepartie non linéaire de la théorie classique de Calderon-Zygmund pour le Laplacien. En conséquence, une approche flexible et générale pour obtenir des majorations du gradient pour le système du p-Laplacien dans une large classe de normes est obtenue. Les majorations du gradient sont réduites à des majorations de normes pour un opérateur classique d'analyse harmonique. En particulier, de nouvelles majorations du gradient sont obtenues et complètent la littérapture en régularité elliptique.

1. Introduction and main results

The present paper deals with the p-Laplace elliptic system

$$-\mathrm{div}(|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}) = -\mathrm{div}\mathbf{F} \qquad \text{in } \Omega. \tag{1.1}$$

Here, Ω is an open set in $\mathbb{R}^n$, with $n \geq 2$, the exponent $p \in (1,\infty)$, the function $\mathbf{F} : \Omega \to \mathbb{R}^{N\times n}$, with $N \geq 1$, is assigned, and $\mathbf{u} : \Omega \to \mathbb{R}^N$ is the unknown. The notation $\mathbb{R}^{N\times n}$ stands for the space of $N \times n$ matrices.

We are concerned with gradient estimates for local weak solutions $\mathbf{u}$ to this system. Our purpose is to establish pointwise bounds for $\nabla \mathbf{u}$, or more precisely, for $|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}$, in terms of $\mathbf{F}$, which, in a sense, linearize the problem. The quantities $|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}$ and $\mathbf{F}$ are suitably linked via sharp maximal operators. This provides us with a powerful tool for a unified regularity theory of the gradient, which is turned into a merely harmonic-analytic framework.

Mathematics Subject Classifications: 35J60, 35B65.
Keywords: Nonlinear elliptic systems, Gradient regularity, Sharp maximal function, Rearrangement-invariant spaces, Campanato spaces.
This study was partly funded by: Research project of MIUR (Italian Ministry of Education, University and Research) Prin 2012 "Elliptic and parabolic partial differential equations: geometric aspects, related inequalities, and applications" (grant number 2012TC7588); GNAMPA of the Italian INdAM (National Institute of High Mathematics) (grant number not available); Program PRVOUK P47 at the Department of Analysis at Charles University, Prague (grant number not available); Academy of Finland.
The authors declare that they have no conflict of interest.

A brief digression to the linear setting, corresponding to the choice $p = 2$, may help to grasp the spirit and novelty of our contribution. In this case, system (1.1) reduces to

$$-\mathrm{div}(\nabla \mathbf{u}) = -\mathrm{div}\mathbf{F}\,, \tag{1.2}$$

namely,

$$-\Delta \mathbf{u} = -\mathrm{div}\mathbf{F}\,. \tag{1.3}$$

The classical Calderón-Zygmund theory offers an exhaustive picture for gradient bounds in this framework. In particular, it implies that the divergence operator can "almost" be canceled in (1.2). Indeed, assume, for simplicity, that $\Omega = \mathbb{R}^n$. A standard representation formula, in terms of Riesz transforms, tells us that, if $\mathbf{u}$ is the solution to (1.3) under suitable assumptions – for instance $\mathbf{F} \in L^2(\mathbb{R}^n)$ and $\nabla \mathbf{u} \in L^2(\mathbb{R}^n)$ – then

$$\nabla \mathbf{u} = T(\mathbf{F}), \tag{1.4}$$

where the mapping $\mathbf{F} \mapsto T(\mathbf{F})$ is a Calderón-Zygmund singular integral operator. The operator T is known to be bounded in any non-borderline function space. As a consequence, bounds for the norm of $\nabla \mathbf{u}$ in any such space via the same norm of $\mathbf{F}$ immediately follows from (1.4). A typical instance amounts to an estimate in Lebesgue spaces, which reads

$$\|\nabla \mathbf{u}\|_{L^r(\mathbb{R}^n)} \le C\|\mathbf{F}\|_{L^r(\mathbb{R}^n)} \tag{1.5}$$

for every $r \in (1, \infty)$. Here, and in what follows, C denotes a constant whose dependence will be specified whenever needed. Observe that, by contrast, (1.5) fails if either $r = 1$, or $r = \infty$. Replacements for (1.5) in these endpoint cases are known. For example, the space of functions of bounded mean oscillation, denoted by $\mathrm{BMO}(\mathbb{R}^n)$, is a well known substitute for $L^\infty(\mathbb{R}^n)$. Actually, one has that

$$\|\nabla \mathbf{u}\|_{\mathrm{BMO}(\mathbb{R}^n)} \le C\|\mathbf{F}\|_{\mathrm{BMO}(\mathbb{R}^n)}.$$

Also, classically

$$\|\nabla \mathbf{u}\|_{C^\alpha(\mathbb{R}^n)} \le C\|\mathbf{F}\|_{C^\alpha(\mathbb{R}^n)},$$

for $\alpha \in (0, 1)$, where $\|\cdot\|_{C^\alpha(\mathbb{R}^n)}$ denotes the Hölder seminorm.

Let us now turn to the nonlinear case, corresponding to $p \neq 2$. The beginning of a systematic study of the so-called nonlinear Calderón-Zygmund theory, associated with (1.1), can be traced back to [32]. In particular, in that paper it is shown that, if $N = 1$, $p \ge 2$ and $r \in [p, \infty)$, then

$$\|\nabla \mathbf{u}\|_{L^r(\mathbb{R}^n)} \le C\|\mathbf{F}\|^{\frac{1}{p-1}}_{L^{\frac{r}{p-1}}(\mathbb{R}^n)}, \tag{1.6}$$

or, equivalently,

$$\||\nabla \mathbf{u}|^{p-1}\|_{L^q(\mathbb{R}^n)} \le C\|\mathbf{F}\|_{L^q(\mathbb{R}^n)} \tag{1.7}$$

for $q \in [p', \infty)$. Estimate (1.7) was extended to every $N \ge 1$ and $p > 1$ in [19], a contribution which is also devoted to the the inequality

$$\|\nabla \mathbf{u}\|_{\mathrm{BMO}(\mathbb{R}^n)} \le C\|\mathbf{F}\|^{\frac{1}{p-1}}_{\mathrm{BMO}(\mathbb{R}^n)},$$

but only for $p \ge 2$. On the other hand, the recent paper [21] contains the inequality

$$\||\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}\|_{\mathrm{BMO}(\mathbb{R}^n)} \le C\|\mathbf{F}\|_{\mathrm{BMO}(\mathbb{R}^n)} \tag{1.8}$$

for every $N \ge 1$ and $p > 1$. In particular, this suggests that gradient bounds for solutions to (1.1) are suitably formulated in terms of the nonlinear expression $|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}$.
As far as Hölder regularity of the gradient of solutions to (1.1) is concerned, the scalar case ($N = 1$) was settled in [57]. The same result for systems ($N \ge 1$), for $p \ge 2$, goes back to the paper [56], in the homogeneous case when $\mathbf{F} = 0$. Systems involving differential operators depending only on the length of the gradient are hence usually called with Uhlenbeck

structure. The contribution [56] was extended to the situation when $1 < p < 2$ in [1] and [13]. In particular, the latter paper includes the case of non-vanishing smooth $\mathbf{F}$. As is well known, regularity of solutions to nonlinear elliptic systems is a critical issue, and exhibits special features compared to the case of a single equation. This has been demonstrated via several counterexamples, including those from [33, 52, 31, 58, 59].

The study of pointwise elliptic gradient regularity has received an impulse from the papers [51] and [28], where Havin-Maz'ya-Wolff nonlinear potentials have been shown to yield precise estimates for the gradient of local solutions to nonlinear p-Laplacian type equations, but with right-hand side in non-divergence form. Enhancements and extensions of these results, involving classical Riesz potentials instead of nonlinear potentials, are the object of a series of papers starting from [40, 39]. Systems are considered in [27, 41, 42]. These estimates for the gradient of solutions were preceded by parallel results for the solutions themselves obtained in [35]. Estimates in rearrangement invariant form for the gradient of solutions to boundary value problems are established in [4, 3, 18].
Besides the papers mentioned above, which are focused on pointwise results, in recent years gradient regularity has been the object of a number of contributions on elliptic equations and systems with different peculiarities. For instance, results on elliptic problems involving differential operators affected by weak regularity properties can be found in [25, 26, 7, 36, 37]. The papers [9, 15, 23, 29, 45, 46, 47, 49] are concerned with operators governed by general growth conditions. Recent global gradient estimates for boundary value problems, under minimal regularity assumptions on the boundary of the ground domain, are the object of [2, 5, 10, 16, 17, 53, 43, 44].

Our main results show that, as in the linear case $\nabla \mathbf{u}$ and $\mathbf{F}$ are linked through the (singular integral) linear operator T appearing in (1.4), which acts "almost diagonally" between function spaces, likewise $|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}$ and $\mathbf{F}$ are related via a sublinear operator in the nonlinear setting. The operator which now comes into play is the sharp maximal operator, and has to be applied both to $|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}$ and $\mathbf{F}$. The crucial use of this operator in the analysis of gradient regularity for equations and systems of the form (1.1) is rooted in the papers [32] and [19].

Recall that the sharp maximal operator $M^\sharp$ is defined as

$$M^\sharp \mathbf{f}(x) = \sup_{B \ni x} \fint_B |\mathbf{f} - \langle \mathbf{f} \rangle_B| dy \qquad \text{for } x \in \mathbb{R}^n, \tag{1.9}$$

for any locally integrable function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$. Here, $m \in \mathbb{N}$, B denotes a ball in $\mathbb{R}^n$, $|B|$ stands for its Lebesgue measure, $\fint_B = \frac{1}{|B|}\int$, and $\langle \mathbf{f} \rangle_B = \fint_B \mathbf{f}(y)dy$. More generally, if $q \geq 1$, the operator $M^{\sharp,q}$ is given by

$$M^{\sharp,q} \mathbf{f}(x) = \sup_{B \ni x} \left(\fint_B |\mathbf{f} - \langle \mathbf{f} \rangle_B|^q dy \right)^{\frac{1}{q}} \qquad \text{for } x \in \mathbb{R}^n,. \tag{1.10}$$

Hence, $M^{\sharp,1} = M^\sharp$. These operators are known to be bounded in customary, non-borderline, function spaces endowed with a norm which is locally stronger than $L^q(\mathbb{R}^n)$. For instance, $M^{\sharp,q}$ is bounded in $L^r(\mathbb{R}^n)$ for every $r \in (q, \infty]$. The operator $M^{\sharp,q}$, as well as other operators to be considered below, will also be applied to matrix-valued functions, with a completely analogous definition. In fact, matrices will be identified with vectors, with an appropriate number of components, whenever they are elements of the target space of functions.

We denote by $W^{1,p}(\Omega)$, $W^{1,p}_{\rm loc}(\Omega)$ and $W^{1,p}_0(\Omega)$ the usual Sobolev spaces of weakly differentiable functions. Moreover, we shall make use of the homogeneous Sobolev space

$$V^{1,p}(\Omega) = \{\mathbf{u} : \mathbf{u} \text{ is a weakly differentiable function in } \Omega, \text{ and } |\nabla \mathbf{u}| \in L^p(\Omega)\}.$$

Here, and in similar occurrences below, we do not indicate the target space in the notation of function spaces. What are the elements of the target space in question (real numbers, vectors, matrices) will be clear from the context. Usually, real-valued functions will be denoted in standard-face, and vector-valued or matrix-valued functions in bold-face.

A basic version of our pointwise estimates is stated in the following theorem, where, in particular, we deal with the case when $\Omega = \mathbb{R}^n$ in (1.1).

Theorem 1.1. [Pointwise estimate in $\mathbb{R}^n$] *Let $n \geq 2$, $N \geq 1$ and $p \in (1, \infty)$. Assume that $\mathbf{F} \in L^{p'}_{\mathrm{loc}}(\mathbb{R}^n)$. Let $\mathbf{u} \in V^{1,p}(\mathbb{R}^n)$ be a local weak solution to system (1.1), with $\Omega = \mathbb{R}^n$. Then there exists a constant $c = c(n, N, p)$ such that*

$$M^\sharp\big(|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}\big)(x) \leq c M^{\sharp,p'}\mathbf{F}(x) \quad \text{for a.e. } x \in \mathbb{R}^n. \tag{1.11}$$

Theorem 1.1 enables one to transfer the problem of bounds for so called Banach function norms of $\nabla \mathbf{u}$ in terms of $\mathbf{F}$ to boundedness properties of $M^{\sharp,p'}$, and reverse boundedness properties of $M^\sharp$ between function spaces endowed with norms of this kind. In particular, the results for Lebesgue norms recalled above can easily be recovered via classical properties of the sharp maximal operator. More interestingly, new estimates also follow from Theorem 1.1, including gradient bounds in Lorentz and Orlicz norms. These can be derived as special instances of a general approach, developed in Section 4 below, for gradient regularity in norms depending only on its size.

Remark 1.2. It will be clear from our proof of Theorem 1.1 that the operator $M^\sharp$ can be replaced with $M^{\sharp,\min\{p',2\}}$ on the left-hand side of inequality (1.11). Thus, the slightly stronger inequality

$$M^{\sharp,\min\{p',2\}}\big(|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}\big)(x) \leq c M^{\sharp,p'}\mathbf{F}(x) \tag{1.12}$$

actually holds for a.e. $x \in \mathbb{R}^n$. However, in all our applications, (1.11) and (1.12) turn out to lead to the same conclusions.

Although quite general, inequality (1.11) can still be enhanced to a form which is also well suited for gradient bounds in norms possibly depending on oscillations. The resulting inequality can be given a local form, which applies to solutions to system (1.1) in any open set Ω. A localized and weighted sharp maximal operator comes into play, which is defined as follows. Let $q \in [1, \infty)$, and, given $R > 0$, let $\omega : (0, R) \to (0, \infty)$ be a function. Define, for $\mathbf{f} \in L^q_{\mathrm{loc}}(\Omega)$,

$$M^{\sharp,q}_{\omega,R}\mathbf{f}(x) = \sup_{\substack{B_r \ni x \\ r < R}} \frac{1}{\omega(r)} \Big(\fint_{B_r} |\mathbf{f} - \langle \mathbf{f} \rangle_{B_r}|^q dy \Big)^{\frac{1}{q}} \quad \text{for } x \in B_R, \tag{1.13}$$

for every $x \in \Omega$ such that $\mathrm{dist}(x, \mathbb{R}^n \setminus \Omega) > R$. Here, B_r denotes any ball of radius $r > 0$. When needed, we shall use the notation $B_r(x)$ for a ball of radius r, centered at the point $x \in \mathbb{R}^n$. The simplified notation

$$M^\sharp_{\omega,R} = M^{\sharp,1}_{\omega,R}$$

will be employed for $q = 1$. If $\Omega = \mathbb{R}^n$, then the right-hand side of (1.13) is well defined also for $R = \infty$. In this case, we set

$$M^{\sharp,q}_{\omega} = M^{\sharp,q}_{\omega,\infty}.$$

In view of our purposes, the additional property that the function $\omega(r)r^{-\beta}$ be almost decreasing in $(0, R)$, for a suitable $\beta > 0$, will be needed. This amounts to requiring that

$$\omega(r) \leq c_\omega \rho^{-\beta} \omega(r\rho) \qquad \text{for } r \in (0, R) \text{ and } \rho \in (0, 1), \tag{1.14}$$

for some constant c_ω.

Theorem 1.3. [Pointwise local oscillation estimate] *Let $n \geq 2$, $N \geq 1$, $R > 0$ and $p \in (1,\infty)$. Let Ω be an open set in $\mathbb{R}^n$ and let $\mathbf{F} \in L^{p'}_{\rm loc}(\Omega)$. Let $\mathbf{u} \in W^{1,p}_{\rm loc}(\Omega)$ be a local weak solution to system (1.1). Then there exists a constant $\beta = \beta(n,p,N) > 0$ such that, if $\omega : (0,R) \to (0,\infty)$ is any function with the property that $\omega(r)r^{-\beta}$ is almost decreasing in $(0,R)$ in the sense of (1.14), then*

$$
\begin{aligned}
(1.15)\quad & M^{\sharp}_{\omega,R}\big(|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}\big)(x) \\
& \leq c M^{\sharp,p'}_{\omega,R}\mathbf{F}(x) + \frac{c}{\omega(R)}\bigg(\fint_{B_{2R}} \big||\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u} - \langle |\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}\rangle_{B_{2R}}\big|^{p'}\,dy\bigg)^{\frac{1}{p'}} \quad \text{for a.e. } x \in B_R,
\end{aligned}
$$

for some constant $c = c(n,N,p,c_\omega)$, and for every concentric balls $B_R \subset B_{2R} \subset \Omega$. Here, c_ω denotes the constant appearing in (1.14). In particular, the conclusion holds for any $\beta \in \big(0, \min\{1, \frac{2\alpha}{p'}\}\big)$, where $\alpha = \alpha(n,N,p)$ is the Hölder exponent appearing in a gradient estimate for the solutions to the p-harmonic system (see Theorem 2.6, Section 2 below).

Remark 1.4. As in (1.11), the operator $M^{\sharp}_{\omega,R}$ can be replaced with $M^{\sharp,\min\{p',2\}}_{\omega,R}$ on the left-hand side of (1.15).

Remark 1.5. In particular, if $\Omega = \mathbb{R}^n$, and assumption (1.14) holds with $R = \infty$, then inequality (1.15) implies that

$$
(1.16)\qquad M^{\sharp}_{\omega}\big(|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}\big)(x) \leq c M^{\sharp,p'}_{\omega}\mathbf{F}(x) \quad \text{for a.e. } x \in \mathbb{R}^n,
$$

provided that $\mathbf{u} \in V^{1,p}(\mathbb{R}^n)$.

As a consequence of Theorem 1.3, the Hölder regularity of the gradient of solutions to the p-Laplace system is easily recovered. However, more general regularity properties can be deduced. Inequalities between semi-norms of $\nabla \mathbf{u}$ and $\mathbf{F}$ in (generalized) Campanato spaces, associated with a function ω as above, stem from inequality (1.15). In particular, they tell us that information on the modulus of continuity of $\nabla \mathbf{u}$ in terms of that of $\mathbf{F}$ can still be derived, if the modulus of continuity ω of $\mathbf{F}$, although not of power type, yet satisfies the Dini type condition

$$
\int_0 \frac{\omega(r)}{r}\,dr < \infty.
$$

Such a condition is sharp, even in the simplest linear case when $p = 2$, as shown by Example 5.5, Section 5. These consequences of Theorem 1.3 are presented in Section 5, which also includes a discussion of BMO and VMO gradient regularity.

Pointwise gradient bounds for solutions to system (1.1), which are maximal operator free, also follow from the methods of this paper. Interestingly, they involve an unconventional Havin-Maz'ya-Wolff type nonlinear potential of the right-hand side of (1.1), defined in terms of its integral oscillations on balls.

Theorem 1.6. [Potential estimate] *Let $n \geq 2$, $N \geq 1$ and $p \in (1,\infty)$. Let Ω be an open set in $\mathbb{R}^n$, and let $\mathbf{F} \in L^{p'}_{\rm loc}(\Omega)$. Let $\mathbf{u} \in W^{1,p}_{\rm loc}(\Omega)$ be a local weak solution to system (1.1). Then there exists a constant $c = c(n,N,p)$ such that*

$$
(1.17)\qquad |\nabla \mathbf{u}(x)|^{p-1} \leq c\int_0^R \bigg(\fint_{B_r(x)} |\mathbf{F} - \langle \mathbf{F}\rangle_{B_r(x)}|^{p'}\,\mathrm{d}y\bigg)^{\frac{1}{p'}}\frac{dr}{r} + c\fint_{B_R(x)} |\nabla \mathbf{u}|^{p-1}\,\mathrm{d}y
$$

for a.e. $x \in \Omega$, and every $R > 0$ such that $B_R(x) \subset \Omega$. Moreover, a point $x \in \Omega$ is a Lebesgue point of $|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}$ whenever the right-hand side of (1.17) is finite for some $R > 0$.

From Theorem 1.6 one immediately infers, for instance, that $|\nabla \mathbf{u}|$ is locally bounded in Ω, provided that $\mathbf{F}$ has a modulus of continuity ω satisfying the Dini type condition displayed above. Example 5.5 again demonstrates the sharpness of the relevant condition with this regard.

Remark 1.7. If $\Omega = \mathbb{R}^n$, and a solution $\mathbf{u}$ to system (1.1) belongs to $V^{1,p}(\mathbb{R}^n)$, then letting R tend to infinity in inequality (1.17) tells us that

$$|\nabla \mathbf{u}(x)|^{p-1} \leq c \int_0^\infty \bigg(\fint_{B_r(x)} |\mathbf{F} - \langle \mathbf{F} \rangle_{B_r(x)}|^{p'} \, \mathrm{d}y \bigg)^{\frac{1}{p'}} \frac{dr}{r} \quad \text{for a.e. } x \in \mathbb{R}^n. \tag{1.18}$$

The approach to our pointwise inequalities for the gradient of solutions to system (1.1) relies upon precise decay estimates on balls for suitable nonlinear expressions of the gradient itself. These estimates are obtained through comparisons with the gradient of solutions to auxiliary systems, whose choice critically depends on whether $p \in (1, 2)$ or $p \in [2, \infty)$, and on whether the integral, on the relevant balls, of an appropriate function of the gradient is "small" or "large", compared with the integral oscillation of the function in question on the same balls. Merging the resulting comparison estimates requires a fine tuning of various parameters which come into play. Most of the intermediate steps, that eventually lead to our final results, call for the replacement of the p-power function with a smoothed (still convex) function near 0, called "shifted p-power function" in what follows, which again depends on a parameter. Of course, a crucial feature of the relevant intermediate steps is that the involved constants are independent of this parameter.

2. Decay estimates

The present section is devoted to decay estimates for the oscillation on balls of the gradient of local weak solutions to system (1.1). A function $\mathbf{u} \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is called a local weak solution to (1.1) if

$$\int_{\Omega'} |\nabla \mathbf{u}|^{p-2} \nabla \mathbf{u} \cdot \nabla \boldsymbol{\varphi} \, dx = \int_{\Omega'} \mathbf{F} \cdot \nabla \boldsymbol{\varphi} \, dx \tag{2.1}$$

for every function $\boldsymbol{\varphi} \in W^{1,p}_0(\Omega')$, and every open set $\Omega' \subset\subset \Omega$. Here, the dot " $\cdot$ " stands for scalar product.
The relevant estimates constitute the core of our proofs, and involve the function $\mathbf{A} : \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$ given by

$$\mathbf{A}(\mathbf{P}) = |\mathbf{P}|^{p-2}\mathbf{P} \qquad \text{for } \mathbf{P} \in \mathbb{R}^{N\times n}$$

and $\mathbf{V} : \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$ given by

$$\mathbf{V}(\mathbf{P}) = |\mathbf{P}|^{\frac{p-2}{2}}\mathbf{P} \qquad \text{for } \mathbf{P} \in \mathbb{R}^{N\times n}.$$

Our final goal here is the following result.

Proposition 2.1. *Let $p \in (1, \infty)$, and let $\mathbf{u}$ be a local weak solution to (1.1). Assume that $\delta \in (0, 1)$. Then there exist constants $\theta = \theta(n, p, N, \delta) \in (0, 1)$ and $c_\delta = c_\delta(n, p, N, \delta) > 0$ such that*

$$\begin{aligned} &\bigg(\fint_{\theta B} |\mathbf{A}(\nabla \mathbf{u}) - \langle \mathbf{A}(\nabla \mathbf{u}) \rangle_{\theta B}|^{\min\{2,p'\}} \, \mathrm{d}x \bigg)^{\frac{1}{\min\{2,p'\}}} \\ &\qquad \leq \delta \bigg(\fint_B |\mathbf{A}(\nabla \mathbf{u}) - \langle \mathbf{A}(\nabla \mathbf{u}) \rangle_B|^{\min\{2,p'\}} \, \mathrm{d}x \bigg)^{\frac{1}{\min\{2,p'\}}} + c_\delta \bigg(\fint_B |\mathbf{F} - \mathbf{F}_0|^{p'} \, \mathrm{d}x \bigg)^{\frac{1}{p'}} \end{aligned} \tag{2.2}$$

for every ball $B \subset \Omega$. In particular, inequality (2.2) *holds with*

$$\theta = c\delta^{\varsigma} \tag{2.3}$$

for small δ, where ς is any number larger than $\max\{1, \frac{p'}{2\alpha}\}$, *$\alpha = \alpha(n, N, p)$ is the Hölder exponent appearing in a gradient estimate for the solutions to the p-harmonic system (see Theorem 2.6 below), and $c = c(n, N, p, \varsigma)$.*

The proof of Proposition 2.1 is accomplished through several steps, to which the following subsections are devoted.

2.1. **Preliminary estimates.** Several inequalities will be conveniently formulated in terms of a "shifted" p-power function $\varphi_{p,a} : [0, \infty) \to [0, \infty)$, introduced in [20], and defined for $a \geq 0$ and $p \in (1, \infty)$ as

$$\varphi_{p,a}(t) = (a + t)^{p-2} t^2 \qquad \text{for } t \geq 0.$$

Clearly, $\varphi_{p,0}(t) = t^p$. The function $\varphi_{p,a}$ is nonnegative and convex, and vanishes at 0; it is hence a Young function. Consequently, for every $\delta > 0$, there exists a constant $c = c(\delta, p)$ such that

$$ts \leq \delta\, \varphi_{p,a}(t) + c\, \varphi_{p',a^{p-1}}(s) \qquad \text{for } t, s \geq 0. \tag{2.4}$$

Basic algebraic relations among the functions $\mathbf{A}$, $\mathbf{V}$, and $\varphi_{p,a}$ are summarized hereafter. They are the content of [20, Lemmas 3, 21, and 26] and [22, Appendix]. Throughout, we denote by c or C a generic constant, which may change from line to line, which depends on specified quantities. Moreover, given two nonnegative functions f and g, we write $f \preceq g$ to denote that there exists a positive constant c such that $f \leq cg$. The notation $f \approx g$ means that $f \preceq g \preceq f$.

Let $p \in (1, \infty)$. Then

$$\begin{aligned}\big(\mathbf{A}(\mathbf{P}) - \mathbf{A}(\mathbf{Q})\big) \cdot \big(\mathbf{P} - \mathbf{Q}\big) &\approx \big|\mathbf{V}(\mathbf{P}) - \mathbf{V}(\mathbf{Q})\big|^2 \approx (|\mathbf{Q}| + |\mathbf{P}|)^{p-2} |\mathbf{Q} - \mathbf{P}|^2 \\ &\approx \varphi_{p,|\mathbf{Q}|}\big(|\mathbf{P} - \mathbf{Q}|\big) \approx \varphi_{p',|\mathbf{Q}|^{p-1}}\big(|\mathbf{A}(\mathbf{P}) - \mathbf{A}(\mathbf{Q})|\big),\end{aligned} \tag{2.5}$$

for $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{N \times n}$, up to equivalence constants depending only on n, N, p. Moreover,

$$\mathbf{A}(\mathbf{Q}) \cdot \mathbf{Q} = |\mathbf{V}(\mathbf{Q})|^2 \approx \varphi_p(|\mathbf{Q}|), \tag{2.6}$$

and

$$|\mathbf{A}(\mathbf{P}) - \mathbf{A}(\mathbf{Q})| \approx \big(\varphi_{p,|\mathbf{Q}|}\big)'\big(|\mathbf{P} - \mathbf{Q}|\big) \approx (|\mathbf{Q}| + |\mathbf{P}|)^{p-2} |\mathbf{Q} - \mathbf{P}|, \tag{2.7}$$

for $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{N \times n}$, up to equivalence constants depending only on n, N, p.

For every $\gamma \in (0, 1]$, the "shift change" formula

$$\varphi_{p',|\mathbf{P}|^{p-1}}(t) \leq c\, \gamma^{1-\max\{p,2\}} \varphi_{p',|\mathbf{Q}|^{p-1}}(t) + \gamma |\mathbf{V}(\mathbf{P}) - \mathbf{V}(\mathbf{Q})|^2. \tag{2.8}$$

holds for some constant $c = c(n, N, p)$, and for every $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{N \times n}$.

Let us now recall some inequalities, in integral form, for merely measurable functions, to be repeatedly used in our proofs. In what follows, we denote by m any number in $\mathbb{N}$. To begin with, it is classical, and easily verified, that

$$\|\mathbf{f} - \langle \mathbf{f} \rangle_E\|_{L^2(E)} = \min_{\mathbf{c} \in \mathbb{R}^m} \|\mathbf{f} - \mathbf{c}\|_{L^2(E)}, \tag{2.9}$$

for any measurable set E in $\mathbb{R}^n$, and and every function $\mathbf{f} : E \to \mathbb{R}^m$ such that $\mathbf{f} \in L^2(E)$. If $q \in [1, \infty]$, then

$$\|\mathbf{f} - \langle \mathbf{f} \rangle_E\|_{L^q(E)} \leq 2 \min_{\mathbf{c} \in \mathbb{R}^m} \|\mathbf{f} - \mathbf{c}\|_{L^q(E)}, \tag{2.10}$$

for every $f \in L^q(E)$.

The following result is less standard, and concerns the equivalence of certain integral averages on balls.

Lemma 2.2. [21, Lemma 6.2] *Let $p \in (1,\infty)$, and let B be a ball in $\mathbb{R}^n$. Given any function $\mathbf{g} : \Omega \to \mathbb{R}^m$ such that $\mathbf{g} \in L^p(B)$, denote by $\mathbf{g}_{\mathbf{A}}$ the vector in $\mathbb{R}^m$ obeying $\mathbf{A}(\mathbf{g}_{\mathbf{A}}) = \langle \mathbf{A}(\mathbf{g})\rangle_B$. Then*

$$\fint_B |\mathbf{V}(\mathbf{g}) - \langle \mathbf{V}(\mathbf{g})\rangle_B|^2\, dx \approx \fint_B |\mathbf{V}(\mathbf{g}) - \mathbf{V}(\langle \mathbf{g}\rangle_B)|^2\, dx \approx \fint_B |\mathbf{V}(\mathbf{g}) - \mathbf{V}(\mathbf{g}_{\mathbf{A}})|^2\, dx, \tag{2.11}$$

up to equivalence constants independent of B and $\mathbf{g}$.

The next lemma encodes self-improving properties of reverse Hölder inequalities for shifted functions. In what follows, given a ball B in $\mathbb{R}^n$ and a positive number θ, we denote by θB the ball, with the same center as B, whose radius is θ times the radius of B.

Lemma 2.3. [21, Corollary 3.4] *Let Ω be an open subset of $\mathbb{R}^n$. Let $p \in (1,\infty)$, $a \in [0,\infty)$, and let $\mathbf{g}, \mathbf{h} : \Omega \to \mathbb{R}^m$ be such that $\mathbf{g} \in L^p_{\mathrm{loc}}(\Omega)$ and $\mathbf{h} \in L^1_{\mathrm{loc}}(\Omega)$. Assume that there exist constants $\sigma \in (0,1)$ and $c_0 > 0$ such that*

$$\fint_B \varphi_{p,a}(|\mathbf{g}|)\, dx \le c_0 \bigg(\fint_{2B} \varphi_{p,a}(|\mathbf{g}|)^\sigma\, dx\bigg)^{\frac{1}{\sigma}} + \fint_{2B} |\mathbf{h}|\, dx,$$

for every ball B such that $2B \subset \Omega$. Then there exists a constant $c_1 = c_1(c_0, p, n, \sigma)$ such that

$$\fint_B \varphi_{p,a}(|\mathbf{g}|)\, dx \le c_1\, \varphi_{p,a}\bigg(\fint_{2B} |\mathbf{g}|\, dx\bigg) + c_1 \fint_{2B} |\mathbf{h}|\, dx$$

for every ball B such that $2B \subset \Omega$.

We begin our discussion of system (1.1) by recalling a reverse Hölder type inequality for the excess functional $\fint_B |\mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(\mathbf{P})|^2\, \mathrm{d}x$.

Lemma 2.4. [21, Lemma 3.2] *Let $p \in (1,\infty)$, and let $\mathbf{u}$ be a local weak solution to (1.1). Then there exist constants $\sigma = \sigma(n,N,p) \in (0,1)$ and $c = c(n,N,p) > 0$ such that*

$$\begin{aligned}\fint_B |\mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(\mathbf{P})|^2\, \mathrm{d}x &\le c \bigg(\fint_{2B} |\mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(\mathbf{P})|^{2\sigma}\, \mathrm{d}x\bigg)^{\frac{1}{\sigma}} \\ &\quad + c \fint_{2B} \varphi_{p',|\mathbf{P}|^{p-1}}(|\mathbf{F} - \mathbf{F}_0|)\, \mathrm{d}x\end{aligned}$$

for every $\mathbf{P}, \mathbf{F}_0 \in \mathbb{R}^{N\times n}$, and every ball B such that $2B \subset \Omega$.

Lemmas 2.3 and 2.4 enable us to transfer information from the excess functional involving $\mathbf{V}$, to an excess functional involving $\mathbf{A}$.

Corollary 2.5. *Let $p \in (1,\infty)$, and let $\mathbf{u}$ be a local weak solution to (1.1). Then there exists a constant $c = c(n,N,p)$ such that*

$$\begin{aligned}\fint_B |\mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(\mathbf{P})|^2\, \mathrm{d}x &\le c\, \varphi_{p',|\mathbf{P}|^{p-1}}\bigg(\fint_{2B} |\mathbf{A}(\nabla \mathbf{u}) - \mathbf{A}(\mathbf{P})|\, \mathrm{d}x\bigg) \\ &\quad + c \fint_{2B} \varphi_{p',|\mathbf{P}|^{p-1}}(|\mathbf{F} - \mathbf{F}_0|)\, \mathrm{d}x\end{aligned} \tag{2.12}$$

for every $\mathbf{P}, \mathbf{F}_0 \in \mathbb{R}^{N\times n}$, and every ball B such that $2B \subset \Omega$.

Proof. By (2.5), we have

$$|\mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(\mathbf{P})|^2 \approx \varphi_{p',|\mathbf{P}|^{p-1}}(|\mathbf{A}(\nabla \mathbf{u}) - \mathbf{A}(\mathbf{P})|).$$

Combining this relation with Lemmas 2.4 and 2.3 yields (2.12). □

Crucial use will be made in what follows of decay estimates for p-harmonic maps, namely for solutions to (1.1) with $\mathbf{F} = 0$. In particular, the unique solution $\mathbf{v} \in W^{1,p}(B)$ to the Dirichlet problem

$$\begin{cases} -\mathrm{div}\mathbf{A}(\nabla \mathbf{v}) = 0 & \text{in } B, \\ \mathbf{v} = \mathbf{u} & \text{on } \partial B \end{cases} \tag{2.13}$$

will come into play. Here, B is a ball in Ω, and $\mathbf{u}$ is a local weak solution to system (1.1). As usual, the boundary condition in (2.13) has to be understood in the sense that $\mathbf{u} - \mathbf{v} \in W_0^{1,p}(B)$.
The relevant decay estimates are collected in the following statement.

Theorem 2.6. **[Decay estimate for p-harmonic maps]** *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $p \in (1, \infty)$. Assume that $\mathbf{v}$ is a p-harmonic function in Ω. Then there exist positive constants $\alpha = \alpha(n, N, p)$ and $c = c(n, N, p)$ such that*

$$\sup_{z,y \in \theta B} |\mathbf{V}(\nabla \mathbf{v})(z) - \mathbf{V}(\nabla \mathbf{v})(y)|^2 \le c\, \theta^{2\alpha} \fint_B |\mathbf{V}(\nabla \mathbf{v}) - \langle \mathbf{V}(\nabla \mathbf{v})\rangle_B|^2 \, dx \tag{2.14}$$

for every $\theta \in (0, \frac{1}{2}]$, and every ball $B \subset \Omega$. Moreover, for every $\kappa < \min\{1, \frac{2\alpha}{p'}\}$, there exists a constant $c = c(n, N, p, \kappa)$ such that

$$\sup_{z,y \in \theta B} |\mathbf{A}(\nabla \mathbf{v})(z) - \mathbf{A}(\nabla \mathbf{v})(y)| \le c\, \theta^{\kappa} \fint_B |\mathbf{A}(\nabla \mathbf{v}) - \langle \mathbf{A}(\nabla \mathbf{v})\rangle_B| \, dx \tag{2.15}$$

for every $\theta \in (0, \frac{1}{2}]$, and every ball $B \subset \Omega$.

A version of (2.14) with a left-hand-side in integral form is a special case of [23, Theorem 6.4]; an integral form of (2.15) can be found in [21, Remark 5.6]. The present version follows from Campanato's characterization of Hölder spaces [54, Lemma A.2].

A preliminary relation between the decay of a solution to system (1.1), and that of the corresponding solution to the Dirichlet problem (2.13) is established in the following lemma.

Lemma 2.7. *Let $p \in (1, \infty)$. Let $\mathbf{u}$ be a local weak solution to system (1.1), and let $\mathbf{v}$ be the solution to the Dirichlet problem (2.13). Then, for every $\beta > 0$, there exists a constant $c_\beta = c(n, N, p, \beta)$ such that*

$$\begin{aligned} \fint_B |\mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(\nabla \mathbf{v})|^2 \, \mathrm{d}x \le\; & \beta\, \varphi_{p', |\langle \mathbf{A}(\nabla \mathbf{u})\rangle_{2B}|}\bigg(\fint_{2B} |\mathbf{A}(\nabla \mathbf{u}) - \langle \mathbf{A}(\nabla \mathbf{u})\rangle_{2B}| \, \mathrm{d}x \bigg) \\ & + c_\beta \fint_{2B} \varphi_{p', |\langle \mathbf{A}(\nabla \mathbf{u})\rangle_{2B}|}(|\mathbf{F} - \mathbf{F}_0|) \, \mathrm{d}x \end{aligned} \tag{2.16}$$

for every $\mathbf{F}_0 \in \mathbb{R}^{N \times n}$, and every ball B such that $2B \subset \Omega$.

Proof. Choosing $\mathbf{u} - \mathbf{v}$ as a test function in (2.1) and making use of equation (2.5) yield

$$\begin{aligned} \fint_B |\mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(\nabla \mathbf{v})|^2 \, \mathrm{d}x &\le c \fint_B \big(\mathbf{A}(\nabla \mathbf{u}) - \mathbf{A}(\nabla \mathbf{v})\big) \cdot \nabla(\mathbf{u} - \mathbf{v}) \, \mathrm{d}x \\ &= c \fint_B \big(\mathbf{F} - \mathbf{F}_0\big) \cdot \nabla\big(\mathbf{u} - \mathbf{v}\big) \, \mathrm{d}x. \end{aligned}$$

Hence, via an application of inequality (2.4), with $a = |\nabla \mathbf{u}|$, and equation (2.5) again, one obtains that

(2.17)
$$\begin{aligned}\fint_B |\mathbf{V}(\nabla\mathbf{u}) - \mathbf{V}(\nabla\mathbf{v})|^2\,\mathrm{d}x &\le \delta\fint_B \varphi_{p,|\nabla\mathbf{u}|}(|\nabla(\mathbf{u}-\mathbf{v})|)\,\mathrm{d}x + c_\delta\fint_B \varphi_{p',|\nabla\mathbf{u}|^{p-1}}(|\mathbf{F}-\mathbf{F}_0|)\,\mathrm{d}x\\ &\le c\delta\fint_B |\mathbf{V}(\nabla\mathbf{u}) - \mathbf{V}(\nabla\mathbf{v})|^2\,\mathrm{d}x + c_\delta\fint_B \varphi_{p',|\nabla\mathbf{u}|^{p-1}}(|\mathbf{F}-\mathbf{F}_0|)\,\mathrm{d}x\end{aligned}$$

for every $\delta > 0$. Let $\mathbf{Q} \in \mathbb{R}^{N\times n}$ be such that

$$\mathbf{A}(\mathbf{Q}) = \langle \mathbf{A}(\nabla\mathbf{u})\rangle_{2B}. \tag{2.18}$$

Owing to (2.8), applied with $\mathbf{P} = \nabla\mathbf{u}$ and $\mathbf{Q}$ as in (2.18), we deduce from (2.17) that

$$\begin{aligned}\fint_B |\mathbf{V}(\nabla\mathbf{u}) - \mathbf{V}(\nabla\mathbf{v})|^2\,\mathrm{d}x &\le c\fint_B \varphi_{p',|\nabla\mathbf{u}|^{p-1}}(|\mathbf{F}-\mathbf{F}_0|)\,\mathrm{d}x\\ &\le \gamma\fint_B |\mathbf{V}(\nabla\mathbf{u}) - \mathbf{V}(\mathbf{Q})|^2\,\mathrm{d}x + c_\gamma\fint_B \varphi_{p',|\mathbf{A}(\mathbf{Q})|}(|\mathbf{F}-\mathbf{F}_0|)\,\mathrm{d}x,\end{aligned} \tag{2.19}$$

for every $\gamma > 0$, and a corresponding suitable constant c_γ. On the other hand, by Corollary 2.5 and the equality $|A(Q)| = |Q|^{p-1}$,

$$\begin{aligned}\fint_B |\mathbf{V}(\nabla\mathbf{u}) - \mathbf{V}(\mathbf{Q})|^2\,\mathrm{d}x &\le c\,\varphi_{p',|\mathbf{A}(\mathbf{Q})|}\bigg(\fint_{2B} |\mathbf{A}(\nabla\mathbf{u}) - \mathbf{A}(\mathbf{Q})|\,\mathrm{d}x\bigg)\\ &\quad + c\fint_{2B} \varphi_{p',|\mathbf{A}(\mathbf{Q})|}(|\mathbf{F}-\mathbf{F}_0|)\,\mathrm{d}x.\end{aligned} \tag{2.20}$$

Inequality (2.16) follows from (2.18), (2.19) and (2.20), via a suitable choice of γ. □

We are ready to prove a first decay estimate for $\mathbf{u}$.

Lemma 2.8. *Let $p \in (1,\infty)$, and let $\mathbf{u}$ be a local weak solution to* (1.1)*. Let α be the exponent appearing in Theorem 2.6. Assume that $\theta \in (0, \frac{1}{2})$. Then there exist constants $c = c(n,N,p)$ and $c_\theta = c_\theta(n,N,p,\theta)$ such that*

$$\begin{aligned}\fint_{\theta B} |\mathbf{V}(\nabla\mathbf{u}) - \langle\mathbf{V}(\nabla\mathbf{u})\rangle_{\theta B}|^2\,\mathrm{d}x &\le c\,\theta^{2\alpha}\varphi_{p',|\langle\mathbf{A}(\nabla\mathbf{u})\rangle_{2B}|}\bigg(\fint_{2B} |\mathbf{A}(\nabla\mathbf{u}) - \langle\mathbf{A}(\nabla\mathbf{u})\rangle_{2B}|\,\mathrm{d}x\bigg)\\ &\quad + c_\theta\fint_{2B} \varphi_{p',|\langle\mathbf{A}(\nabla\mathbf{u})\rangle_{2B}|}(|\mathbf{F}-\mathbf{F}_0|)\,\mathrm{d}x\end{aligned} \tag{2.21}$$

for every ball B such that $2B \subset \Omega$.

Proof. From inequality (2.14) and property (2.9) we deduce that

$$\begin{aligned}&\fint_{\theta B} |\mathbf{V}(\nabla\mathbf{u}) - \langle\mathbf{V}(\nabla\mathbf{u})\rangle_{\theta B}|^2\,\mathrm{d}x\\ &\quad\le c\fint_{\theta B} |\mathbf{V}(\nabla\mathbf{v}) - \langle\mathbf{V}(\nabla\mathbf{v})\rangle_{\theta B}|^2\,\mathrm{d}x + c\fint_{\theta B} |\mathbf{V}(\nabla\mathbf{u}) - \mathbf{V}(\nabla\mathbf{v})|^2\,\mathrm{d}x\\ &\quad\le c\,\theta^{2\alpha}\fint_{B} |\mathbf{V}(\nabla\mathbf{v}) - \langle\mathbf{V}(\nabla\mathbf{v})\rangle_{B}|^2\,\mathrm{d}x + c\theta^{-n}\fint_{B} |\mathbf{V}(\nabla\mathbf{u}) - \mathbf{V}(\nabla\mathbf{v})|^2\,\mathrm{d}x\end{aligned} \tag{2.22}$$

$$\leq c\,\theta^{2\alpha} \fint_B |\mathbf{V}(\nabla \mathbf{u}) - \langle \mathbf{V}(\nabla \mathbf{u})\rangle_B|^2\,\mathrm{d}x + c\,\theta^{-n} \fint_B |\mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(\nabla \mathbf{v})|^2\,\mathrm{d}x.$$

$$\leq c\,\theta^{2\alpha} \fint_B |\mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(\mathbf{Q})|^2\,\mathrm{d}x + c\,\theta^{-n} \fint_B |\mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(\nabla \mathbf{v})|^2\,\mathrm{d}x,$$

where $\mathbf{Q}$ is defined in (2.18). Hence, inequality (2.21) follows, via Corollary 2.5 (applied with $\mathbf{P} = \mathbf{Q}$), and Lemma 2.7 (applied with $\beta = \theta^{n+2\alpha}$). □

From here on, our decay estimate on a given ball $B \subset \Omega$ takes a different form depending on whether the quantity $\fint_{2B} |\nabla \mathbf{u}|^p\,\mathrm{d}x$ is "small" or "large" compared to $\fint_{2B} |\mathbf{V}(\nabla \mathbf{u}) - \langle \mathbf{V}(\nabla \mathbf{u})\rangle_{2B}|^2\,\mathrm{d}x$. We shall refer to the former situation as the "degenerate case", and to the latter as the "non-degenerate case".

2.2. **The degenerate case.** Throughout this subsection, we assume that $\mathbf{u}$ is a local weak solution to system (1.1) such that

$$\fint_{2B} |\mathbf{V}(\nabla \mathbf{u})|^2\,\mathrm{d}x \leq \frac{1}{\varepsilon} \fint_{2B} |\mathbf{V}(\nabla \mathbf{u}) - \langle \mathbf{V}(\nabla \mathbf{u})\rangle_{2B}|^2\,\mathrm{d}x \tag{2.23}$$

for some fixed ball B such that $4B \subset \Omega$, and some fixed number $\varepsilon > 0$.
We begin with the following lemma.

Lemma 2.9. *Let $p \in (1, \infty)$. Let $\mathbf{u}$ be a local weak solution to* (1.1) *satisfying* (2.23) *for some ball B and some $\varepsilon > 0$. Let $\mathbf{P} \in \mathbb{R}^{N\times n}$ be such that $\mathbf{A}(\mathbf{P}) = \langle \mathbf{A}(\nabla \mathbf{u})\rangle_{2B}$. Then there exists a constant $c = c(n, N, p)$ such that*

(2.24)

$$\fint_{2B} |\mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(\mathbf{P})|^2\,\mathrm{d}x + \varphi_{p',|\mathbf{A}(\mathbf{P})|}\Big(\fint_{2B} |\mathbf{A}(\nabla \mathbf{u}) - \mathbf{A}(\mathbf{P})|\,\mathrm{d}x\Big) + \fint_{2B} \varphi_{p',|\mathbf{A}(\mathbf{P})|}(|\mathbf{F} - \mathbf{F}_0|)\,\mathrm{d}x$$
$$\leq c\varepsilon^{1-\max\{2,p\}} \Big(\fint_{4B} |\mathbf{A}(\nabla \mathbf{u}) - \mathbf{A}(\mathbf{P}_0)|\,\mathrm{d}x\Big)^{p'} + c\varepsilon^{1-\max\{2,p\}} \fint_{4B} |\mathbf{F} - \mathbf{F}_0|^{p'}\,\mathrm{d}x$$

for every $\mathbf{F}_0, \mathbf{P}_0 \in \mathbb{R}^{N\times n}$.

Proof. Set

$$m = \max\{2, p\}. \tag{2.25}$$

By Corollary 2.5, applied with B replaced by $2B$, and by inequality (2.8), with $\mathbf{Q} = 0$ and $\gamma = \varepsilon c'$, where c' is a positive constant to be chosen later, one has that

(2.26)

$$\fint_{2B} |\mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(\mathbf{P})|^2\,\mathrm{d}x \leq \varphi_{p',|\mathbf{A}(\mathbf{P})|}\Big(\fint_{2B} |\mathbf{A}(\nabla \mathbf{u}) - \mathbf{A}(\mathbf{P})|\,\mathrm{d}x\Big) + \fint_{2B} \varphi_{p',|\mathbf{A}(\mathbf{P})|}(|\mathbf{F} - \mathbf{F}_0|)\,\mathrm{d}x$$
$$\leq c\,\varphi_{p',|\mathbf{P}|^{p-1}}\Big(\fint_{4B} |\mathbf{A}(\nabla \mathbf{u}) - \mathbf{A}(\mathbf{P})|\,\mathrm{d}x\Big) + c \fint_{4B} \varphi_{p',|\mathbf{P}|^{p-1}}(|\mathbf{F} - \mathbf{F}_0|)\,\mathrm{d}x$$
$$\leq c\varepsilon^{1-m} \Big(\fint_{4B} |\mathbf{A}(\nabla \mathbf{u}) - \mathbf{A}(\mathbf{P})|\,\mathrm{d}x\Big)^{p'} + cc'\varepsilon|\mathbf{V}(\mathbf{P})|^2$$
$$+ c\varepsilon^{1-m} \fint_{4B} |\mathbf{F} - \mathbf{F}_0|^{p'}\,\mathrm{d}x + cc'\varepsilon|\mathbf{V}(\mathbf{P})|^2.\,\mathrm{d}x$$

Since, by (2.23) and (2.9),

(2.27)
$$|\mathbf{V}(\mathbf{P})|^2 \leq \fint_{2B} |\mathbf{V}(\nabla \mathbf{u})|^2\, dx \leq \frac{1}{\varepsilon} \fint_{2B} |\mathbf{V}(\nabla \mathbf{u}) - \langle \mathbf{V}(\nabla \mathbf{u})\rangle_{2B}|^2\, \mathrm{d}x \leq \frac{1}{\varepsilon} \fint_{2B} |\mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(\mathbf{P})|^2\, \mathrm{d}x,$$

choosing $c' = \frac{1}{4c}$ in (2.26) ensures that

$$\begin{aligned}
&\fint_{2B} |\mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(\mathbf{P})|^2\, \mathrm{d}x + \varphi_{p',|\mathbf{A}(\mathbf{P})|}\Big(\fint_{2B} |\mathbf{A}(\nabla \mathbf{u}) - \mathbf{A}(\mathbf{P})|\, \mathrm{d}x\Big) + \fint_{2B} \varphi_{p',|\mathbf{A}(\mathbf{P})|}(|\mathbf{F} - \mathbf{F}_0|)\, \mathrm{d}x \\
&\quad \leq c\varepsilon^{1-m} \Big(\fint_{4B} |\mathbf{A}(\nabla \mathbf{u}) - \mathbf{A}(\mathbf{P})|\, \mathrm{d}x\Big)^{p'} + c\varepsilon^{1-m} \fint_{4B} |\mathbf{F} - \mathbf{F}_0|^{p'}\, \mathrm{d}x \\
&\quad \leq c\varepsilon^{1-m} \Big(\fint_{4B} |\mathbf{A}(\nabla \mathbf{u}) - \mathbf{A}(\mathbf{P}_0)|\, \mathrm{d}x\Big)^{p'} + c\varepsilon^{1-m} |\langle \mathbf{A}(\nabla \mathbf{u})\rangle_{2B} - \mathbf{A}(\mathbf{P}_0)|^{p'} \\
&\quad + c\varepsilon^{1-m} \fint_{4B} |\mathbf{F} - \mathbf{F}_0|^{p'}\, \mathrm{d}x \\
&\quad \leq c\varepsilon^{1-m} \Big(\fint_{4B} |\mathbf{A}(\nabla \mathbf{u}) - \mathbf{A}(\mathbf{P}_0)|\, \mathrm{d}x\Big)^{p'} + c\varepsilon^{1-m} \fint_{4B} |\mathbf{F} - \mathbf{F}_0|^{p'}\, \mathrm{d}x.
\end{aligned}$$

This establishes inequality (2.24). □

The next lemma provides us with an estimate for a distance between the gradient of a solution $\mathbf{u}$ to (1.1), and the gradient of the solution $\mathbf{v}$ to the associated Dirichlet problem (2.13) in a form suitable for our purposes.

Lemma 2.10. *Let $p \in (1, \infty)$. Let $\mathbf{u}$ be a local weak solution to* (1.1) *satisfying* (2.23) *for some ball B and some $\varepsilon > 0$. Let $\mathbf{v}$ be the solution to the Dirichlet problem* (2.13)*. Then for every $\delta > 0$, there exists a constant $c = c(n, N, p, \varepsilon, \delta)$ such that*

(2.28)
$$\fint_B |\mathbf{A}(\nabla \mathbf{u}) - \mathbf{A}(\nabla \mathbf{v})|^{p'}\, \mathrm{d}x \leq \delta \Big(\fint_{4B} |\mathbf{A}(\nabla \mathbf{u})) - \langle \mathbf{A}(\nabla \mathbf{u})\rangle_{4B}|\, dx\Big)^{p'} + c \fint_{4B} |\mathbf{F} - \mathbf{F}_0|^{p'}\, \mathrm{d}x$$

for every $\mathbf{F}_0 \in \mathbb{R}^{N\times n}$.

Proof. Fix $\gamma > 0$, and define m as in (2.25). Then the following chain holds:

(2.29)
$$\begin{aligned}
&\fint_B |\mathbf{A}(\nabla \mathbf{u}) - \mathbf{A}(\nabla \mathbf{v})|^{p'}\, \mathrm{d}x \leq c\gamma^{1-m} \fint_B \varphi_{p',|\nabla \mathbf{u}|^{p-1}}\big(\mathbf{A}(\nabla \mathbf{u}) - \mathbf{A}(\nabla \mathbf{v})\big)\, \mathrm{d}x + \gamma \fint_B |\mathbf{V}(\nabla \mathbf{u})|^2\, \mathrm{d}x \\
&\quad \leq c\gamma^{1-m} \fint_B |\mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(\nabla \mathbf{v})|^2\, \mathrm{d}x + \gamma \fint_B |\mathbf{V}(\nabla \mathbf{u})|^2\, \mathrm{d}x \\
&\quad \leq c\gamma^{1-m} \fint_B |\mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(\nabla \mathbf{v})|^2\, \mathrm{d}x + \frac{\gamma}{\varepsilon} \fint_{2B} |\mathbf{V}(\nabla \mathbf{u}) - \langle \mathbf{V}(\nabla \mathbf{u})\rangle_{2B}|^2\, \mathrm{d}x\,,
\end{aligned}$$

where the first inequality follows from (2.8) applied with $\mathbf{P} = 0$, $\mathbf{Q} = \nabla \mathbf{u}$, $t = \mathbf{A}(\nabla \mathbf{u}) - \mathbf{A}(\nabla \mathbf{v})$, the second inequality from (2.5) applied with $\mathbf{P} = \nabla \mathbf{v}$, $\mathbf{Q} = \nabla \mathbf{u}$, and the last one from (2.23).

Let $\mathbf{P}$ be such that $\mathbf{A}(\mathbf{P}) = \langle \mathbf{A}(\nabla \mathbf{u})\rangle_{2B}$. Given $\beta > 0$, by Lemma 2.7 and inequality (2.9) one has that

(2.30)
$$\begin{aligned} &c\gamma^{1-m} \fint_B |\mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(\nabla \mathbf{v})|^2 \, dx + \frac{\gamma}{\varepsilon} \fint_{2B} |\mathbf{V}(\nabla \mathbf{u}) - \langle \mathbf{V}(\nabla \mathbf{u})\rangle_{2B}|^2 \, dx \\ &\quad \le c\gamma^{1-m} \beta \, \varphi_{p',|\mathbf{A}(\mathbf{P})|}\Big(\fint_{2B} |\mathbf{A}(\nabla \mathbf{u}) - \mathbf{A}(\mathbf{P})| \, dx \Big) + c\gamma^{1-m} c_\beta \fint_{2B} \varphi_{p',|\mathbf{A}(\mathbf{P})|}(|\mathbf{F} - \mathbf{F}_0|) \, dx \\ &\qquad + \frac{c\gamma}{\varepsilon} \fint_{2B} |\mathbf{V}(\nabla \mathbf{u}) - \mathbf{A}(\mathbf{P})|^2 \, dx \, . \end{aligned}$$

On the other hand, inequality (2.8), applied with $\mathbf{P} = \mathbf{P}$, $\mathbf{Q} = 0$ and $t = |\mathbf{F} - \mathbf{F}_0|$, and inequality (2.27) tell us that, if $\tau > 0$, then

$$\begin{aligned} \fint_{2B} \varphi_{p',|\mathbf{A}(\mathbf{P})|}(|\mathbf{F} - \mathbf{F}_0|) \, dx &\le c\tau^{1-m} \fint_{2B} |\mathbf{F} - \mathbf{F}_0|^{p'} \, dx + \tau |\mathbf{V}(\mathbf{P})|^2 \, dx \\ &\le c\tau^{1-m} \fint_{2B} |\mathbf{F} - \mathbf{F}_0|^{p'} \, dx + \frac{\tau}{\varepsilon} \fint_{2B} |\mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(\mathbf{P})|^2 \, dx. \end{aligned} \tag{2.31}$$

Combining inequalities (2.29)–(2.31), and then making use of Lemma 2.9, yield

$$\begin{aligned} \fint_B |\mathbf{A}(\nabla \mathbf{u}) - \mathbf{A}(\nabla \mathbf{v})|^{p'} \, dx &\le c\gamma^{1-m} \beta \, \varphi_{p',|\mathbf{A}(\mathbf{P})|}\Big(\fint_{2B} |\mathbf{A}(\nabla \mathbf{u}) - \mathbf{A}(\mathbf{P})| \, dx \Big) \\ &\quad \frac{c}{\varepsilon} \big(\gamma + \gamma^{1-m} c_\beta \tau\big) \fint_{2B} |\mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(\mathbf{P})|^2 \, dx + c\gamma^{1-m} c_\beta \tau^{1-m} \fint_{2B} |\mathbf{F} - \mathbf{F}_0|^{p'} \, dx \\ &\le c\gamma^{1-m} \beta \varepsilon^{1-m} \Big(\fint_{4B} |\mathbf{A}(\nabla \mathbf{u}) - \langle \mathbf{A}(\nabla \mathbf{u})\rangle_{4B}| \, dx \Big)^{p'} + c\gamma^{1-m} \beta \varepsilon^{1-m} \fint_{4B} |\mathbf{F} - \mathbf{F}_0|^{p'} \, dx \\ &\quad + \frac{c}{\varepsilon} \big(\gamma + \gamma^{1-m} c_\beta \tau\big) \varepsilon^{1-m} \Big(\fint_{4B} |\mathbf{A}(\nabla \mathbf{u}) - \langle \mathbf{A}(\nabla \mathbf{u})\rangle_{4B}| \, dx \Big)^{p'} \\ &\quad + \frac{c}{\varepsilon} \big(\gamma + \gamma^{1-m} c_\beta \tau\big) \varepsilon^{1-m} \fint_{2B} |\mathbf{F} - \mathbf{F}_0|^{p'} \, dx + c\gamma^{1-m} c_\beta \tau^{1-m} \fint_{2B} |\mathbf{F} - \mathbf{F}_0|^{p'} \, dx. \end{aligned} \tag{2.32}$$

Choosing first γ, then β and finally τ sufficiently small in (2.32) yields (2.28). $\square$

Proposition 2.11. *Let $p \in (1, \infty)$. Let $\mathbf{u}$ be a local weak solution to (1.1) satisfying (2.23) for some ball B and some $\varepsilon > 0$, and let $\mathbf{v}$ be the solution to the Dirichlet problem (2.13). Let α be the exponent appearing in Theorem 2.6. Then, for every $\theta \in (0, 1)$ and every $\kappa < \min\{1, \frac{2\alpha}{p'}\}$, there exist constants $c = c(n, N, p, \kappa)$, and $c' = c'(n, N, p, \varepsilon, \theta, \kappa)$ such that*

$$\begin{aligned} &\fint_{\theta B} |\mathbf{A}(\nabla \mathbf{u}) - \langle \mathbf{A}(\nabla \mathbf{u})\rangle_{\theta B}|^{p'} \, dx \\ &\qquad \le c\theta^{\kappa p'} \Big(\fint_{4B} |\mathbf{A}(\nabla \mathbf{u}) - \langle \mathbf{A}(\nabla \mathbf{u})\rangle_{4B}| \, dx \Big)^{p'} + c' \fint_{4B} |\mathbf{F} - \mathbf{F}_0|^{p'} \, dx \end{aligned} \tag{2.33}$$

for every $\mathbf{F}_0 \in \mathbb{R}^{N \times n}$.

Proof. By property (2.10),

$$\fint_{\theta B} |\mathbf{A}(\nabla \mathbf{u}) - \langle \mathbf{A}(\nabla \mathbf{u})\rangle_{\theta B}|^{p'}\,\mathrm{d}x \le c \fint_{\theta B} |\mathbf{A}(\nabla \mathbf{v}) - \langle \mathbf{A}(\nabla \mathbf{v})\rangle_{\theta B}|^{p'}\,\mathrm{d}x + c\theta^{-n} \fint_{B} |\mathbf{A}(\nabla \mathbf{u}) - \mathbf{A}(\nabla \mathbf{v})|^{p'}\,\mathrm{d}x = I_1 + I_2. \tag{2.34}$$

We exploit inequality (2.15), property (2.10), and Lemma 2.10 to obtain the following chain of inequalities:

$$\begin{aligned} I_1 &\le c\theta^{\kappa p'} \bigg(\fint_{B} |\mathbf{A}(\nabla \mathbf{v}) - \langle \mathbf{A}(\nabla \mathbf{v})\rangle_{B}|\,\mathrm{d}x \bigg)^{p'} \\ &\le \widetilde{c}\theta^{\kappa p'} \fint_{B} |\mathbf{A}(\nabla \mathbf{v}) - \mathbf{A}(\nabla \mathbf{u})|^{p'}\,\mathrm{d}x + \widetilde{c}\theta^{2\alpha} \bigg(\fint_{B} |\mathbf{A}(\nabla \mathbf{u}) - \langle \mathbf{A}(\nabla \mathbf{u})\rangle_{B}|\,\mathrm{d}x \bigg)^{p'} \\ &\le \widetilde{c}\theta^{\kappa p'} \bigg(\fint_{4B} |\mathbf{A}(\nabla \mathbf{u}) - \langle \mathbf{A}(\nabla \mathbf{u})\rangle_{4B}|\,\mathrm{d}x \bigg)^{p'} + c' \fint_{4B} |\mathbf{F} - \mathbf{F}_0|^{p'}\,\mathrm{d}x, \end{aligned} \tag{2.35}$$

where $\widetilde{c} = \widetilde{c}(n, p, N, \kappa)$ and $c' = c'(n, p, N, \varepsilon, \kappa)$. One can further make use of Lemma 2.10 to infer that

$$I_2 \le c\delta\theta^{-n} \bigg(\fint_{4B} |\mathbf{A}(\nabla \mathbf{u}) - \langle \mathbf{A}(\nabla \mathbf{u})\rangle_{4B}|\,\mathrm{d}x \bigg)^{p'} + c_\delta\theta^{-n} \fint_{4B} |\mathbf{F} - \mathbf{F}_0|^{p'}\,\mathrm{d}x \tag{2.36}$$

for any $\delta > 0$, and for some constant $c_\delta = c_\delta(n, N, p, \varepsilon, \delta)$. On choosing $\delta = \theta^{n+\kappa p'}$, inequalities (2.34)–(2.36) yield the result. □

2.3. **The non-degenerate case.** In the present subsection, we assume that $\mathbf{u}$ is a local weak solution to (1.1) such that

$$\fint_{2B} |\mathbf{V}(\nabla \mathbf{u}) - \langle \mathbf{V}(\nabla \mathbf{u})\rangle_{2B}|^{2}\,\mathrm{d}x < \varepsilon \fint_{2B} |\mathbf{V}(\nabla \mathbf{u})|^{2}\,\mathrm{d}x. \tag{2.37}$$

for some fixed reference ball B such that $4B \subset \Omega$, and some number $\varepsilon \in (0, \frac{1}{4})$. Since in the degenerate case appropriate estimates are possible for any ε, we can consider it to be a free parameter at this stage. In particular, our choice of ε in (2.37) will depend on the parameter θ in Proposition 2.1.

Let us start by introducing a few notations to be used in what follows. We shall have to deal with balls whose centers differ from the reference ball B in (2.37). We call $\sigma B(z)$ the ball, centered at the point z, whose radius is σ times the radius of B. Given $z \in \mathbb{R}^n$ and $\sigma > 0$, we denote by $\mathbf{A}_{\sigma,z}, \mathbf{V}_{\sigma,z}, \mathbf{U}_{\sigma,z}$ the matrices from $\mathbb{R}^{N\times n}$ satisfying

$$\mathbf{A}(\mathbf{A}_{\sigma,z}) = \langle \mathbf{A}(\nabla \mathbf{u})\rangle_{\sigma B(z)}, \quad \mathbf{V}(\mathbf{V}_{\sigma,z}) = \langle \mathbf{V}(\nabla \mathbf{u})\rangle_{\sigma B(z)}, \quad \mathbf{U}_{\sigma,z} = \langle \nabla \mathbf{u}\rangle_{\sigma B(z)}, \tag{2.38}$$

respectively. Whenever z is the center of the reference ball B, it will be omitted, and we just write

$$\mathbf{A}(\mathbf{A}_{\sigma}) = \langle \mathbf{A}(\nabla \mathbf{u})\rangle_{\sigma B}, \quad \mathbf{V}(\mathbf{V}_{\sigma}) = \langle \mathbf{V}(\nabla \mathbf{u})\rangle_{\sigma B}, \quad \mathbf{U}_{\sigma} = \langle \nabla \mathbf{u}\rangle_{\sigma B}. \tag{2.39}$$

The next lemma tells us that, under condition (2.37), the expressions defined above are equivalent, up to multiplicative constants proportional to ε.

Lemma 2.12. *Let $p \in (1, \infty)$. Let $\mathbf{u}$ be a local weak solution to* (1.1) *satisfying* (2.37) *for some ball B and some $\varepsilon \in (0, \frac{1}{4})$. Then there exists a constant $c = c(n, N, p)$ such that, if*

$$\varepsilon c \le \sigma^{5n}\,, \tag{2.40}$$

then

$$\max\{|\mathbf{U}_{\theta,z}|,|\mathbf{A}_{\theta,z}|\}\leq 2|\mathbf{V}_2|\leq 4\min\{|\mathbf{U}_{\theta,z}|,|\mathbf{A}_{\theta,z}|\}, \tag{2.41}$$

and

$$\max\{|\mathbf{U}_2|,|\mathbf{A}_2|\}\leq 2|\mathbf{V}_{\theta,z}|\leq 4\min\{|\mathbf{U}_2|,|\mathbf{A}_2|\}, \tag{2.42}$$

for every $\theta\in[\sigma,2]$ *and* $z\in B$ *satisfying* $\theta B(z)\subset 2B$*. Moreover,* $\mathbf{U}_2\neq 0$*, and*

$$|\mathbf{A}_{\theta,z}-\mathbf{U}_2|+|\mathbf{V}_{\theta,z}-\mathbf{U}_2|+|\mathbf{U}_{\theta,z}-\mathbf{U}_2|\leq \tfrac{1}{8}\sigma^{2n}|\mathbf{U}_2| \tag{2.43}$$

for every $\theta\in[\sigma,2]$.

Proof. By condition (2.37),

$$\begin{aligned}|\mathbf{V}(\mathbf{V}_2)|^2\leq&\fint_{2B}|\mathbf{V}(\nabla\mathbf{u})|^2\,\mathrm{d}x\\ \leq&2\fint_{2B}|\mathbf{V}(\nabla\mathbf{u})-\mathbf{V}(\mathbf{V}_2)|^2\,\mathrm{d}x+2|\mathbf{V}(\mathbf{V}_2)|^2<2\varepsilon\fint_{2B}|\mathbf{V}(\nabla\mathbf{u})|^2\,\mathrm{d}x+2|\mathbf{V}(\mathbf{V}_2)|^2.\end{aligned}$$

Hence,

$$|\mathbf{V}(\mathbf{V}_2)|^2\leq\fint_{2B}|\mathbf{V}(\nabla\mathbf{u})|^2\,\mathrm{d}x<4|\mathbf{V}(\mathbf{V}_2)|^2, \tag{2.44}$$

since we are assuming that $\varepsilon\leq\frac14$. Hence, in particular, $|\mathbf{V}(\mathbf{V}_2)|>0$.
Next, denote by $\mathbf{T}_{\theta,z}$ any of the quantities $\mathbf{A}_{\theta,z}$, $\mathbf{V}_{\theta,z}$, $\mathbf{U}_{\theta,z}$ for $\theta\in[\sigma,2]$. Then the following chain holds:

$$\begin{aligned}|\mathbf{V}(\mathbf{V}_2)-\mathbf{V}(\mathbf{T}_{\theta,z})|^2\leq&2\fint_{\theta B}|\mathbf{V}(\nabla\mathbf{u})-\mathbf{V}(\mathbf{T}_{\theta,z})|^2\,\mathrm{d}x+2\fint_{\theta B}|\mathbf{V}(\nabla\mathbf{u})-\mathbf{V}(\mathbf{V}_2)|^2\,\mathrm{d}x\\ \leq&c\fint_{\theta B}|\mathbf{V}(\nabla\mathbf{u})-\mathbf{V}(\mathbf{V}_\theta)|^2\,\mathrm{d}x+c\theta^{-n}\fint_{2B}|\mathbf{V}(\nabla\mathbf{u})-\mathbf{V}(\mathbf{V}_2)|^2\,\mathrm{d}x\\ \leq&c\theta^{-n}\fint_{2B}|\mathbf{V}(\nabla\mathbf{u})-\mathbf{V}(\mathbf{V}_2)|^2\,\mathrm{d}x\\ \leq&c\varepsilon\theta^{-n}\fint_{2B}|\mathbf{V}(\nabla\mathbf{u})|^2\,\mathrm{d}x\leq\hat{c}\varepsilon\theta^{-n}|\mathbf{V}(\mathbf{V}_2)|^2,\end{aligned} \tag{2.45}$$

where the second inequality is a consequence of Lemma 2.2, the third of (2.10), the fourth of (2.37), and the last one of (2.44). Hence, via the triangle inequality, and (2.40) with sufficiently large c, we obtain that

$$|\mathbf{T}_{\theta,z}|^p=|\mathbf{V}(\mathbf{T}_{\theta,z})|^2\leq\big(1+\tfrac12\big)|\mathbf{V}(\mathbf{V}_2)|^2$$

and

$$|\mathbf{V}(\mathbf{V}_2)|^2\leq|\mathbf{T}_{\theta,z}|^p+\frac12|\mathbf{V}(\mathbf{V}_2)|^2.$$

Therefore

$$|\mathbf{T}_{\theta,z}|\leq 2|\mathbf{V}_2|\leq 4|\mathbf{T}_{\theta,z}|. \tag{2.46}$$

Equation (2.41) is thus established.
As far as (2.43) is concerned, observe that, by (2.5) and (2.45),

$$|\mathbf{V}_2|^{p-2}|\mathbf{V}_2-\mathbf{T}_{\theta,z}|^2\leq c|\mathbf{V}(\mathbf{V}_2)-\mathbf{V}(\mathbf{T}_{\theta,z})|^2\leq\varepsilon c'\sigma^{-n}|\mathbf{V}_2|^p. \tag{2.47}$$

Coupling equations (2.47) and (2.41) tells us that

$$|\mathbf{V}_2-\mathbf{T}_{\theta,z}|\leq\sqrt{\varepsilon c\sigma^{-n}}|\mathbf{V}_2|\leq 2\sqrt{\varepsilon c\sigma^{-n}}|\mathbf{U}_2|\,.$$

Hence,

$$|\mathbf{V}_2 - \mathbf{T}_{\theta,z}| \le 2\sqrt{\varepsilon c' \sigma^{-n}}|\mathbf{U}_2|\,, \tag{2.48}$$

$$|\mathbf{V}_2 - \mathbf{U}_2| \le 2\sqrt{\varepsilon c' \sigma^{-n}}|\mathbf{U}_2|\,. \tag{2.49}$$

Inequality (2.43) follows from (2.48) and (2.49), via the triangle inequality, on choosing c in (2.40) larger than a suitable absolute constant times $\max\{c', \hat{c}\}$.
Finally, equation (2.42) can be derived from (2.41) and (2.43). □

Lemma 2.13. *Let $p \in (1,\infty)$. Assume that $\mathbf{u}$ is a local weak solution to* (1.1) *satisfying* (2.37) *for some ball B and some $\varepsilon \in (0, \frac{1}{4})$. Assume, in addition, that there exists a constant $\overline{c}$ such that*

$$|\langle \mathbf{A}(\nabla \mathbf{u})\rangle_{2B}|^{p'} \le \frac{\overline{c}^{p'}}{\varepsilon} \fint_{2B} |\mathbf{F} - \mathbf{F}_0|^{p'}\,\mathrm{d}x \tag{2.50}$$

for every $\mathbf{F}_0 \in \mathbb{R}^{N\times n}$. Then there exists a constant $c = c(n, N, p)$ such that

$$\fint_{\sigma B} |\mathbf{A}(\nabla \mathbf{u}) - \langle \mathbf{A}(\nabla \mathbf{u})\rangle_{\sigma B}|^{p'}\,\mathrm{d}x \le c\frac{\overline{c}^{p'}}{\varepsilon\sigma^n} \fint_{2B} |\mathbf{F} - \mathbf{F}_0|^{p'}\,\mathrm{d}x \tag{2.51}$$

for every σ satisfying (2.40).

Proof. One has that

$$\begin{aligned}\fint_{\sigma B} |\mathbf{A}(\nabla \mathbf{u}) - \mathbf{A}(\mathbf{A}_\sigma)|^{p'}\,\mathrm{d}x \le& c\sigma^{-n}\fint_{2B} |\mathbf{A}(\nabla \mathbf{u}) - \mathbf{A}(\mathbf{A}_2)|^{p'}\,\mathrm{d}x \le c\sigma^{-n}\Big(\fint_{2B} |\mathbf{V}(\nabla \mathbf{u})|^2\,\mathrm{d}x + |\mathbf{A}_2|^p\Big)\\ \le& c\sigma^{-n}\big(|\mathbf{V}(\mathbf{V}_2)|^2 + |\mathbf{A}_2|^p\big)\end{aligned}$$

where the first inequality is due to (2.10), the third to (2.44) and the last one to Lemma 2.12. Hence, the result follows via (2.50). □

In view of (2.44) and (2.41), assumption (2.37) implies that

$$\fint_{2B} |\mathbf{V}(\nabla \mathbf{u}) - \langle \mathbf{V}(\nabla \mathbf{u})\rangle_{2B}|^2\,\mathrm{d}x < c\varepsilon|\langle \mathbf{A}(\nabla \mathbf{u})\rangle_{2B}|^{p'} \tag{2.52}$$

for some constant $c = c(n, N, p)$, provided that (2.40) holds. Moreover, Lemma 2.13 enables us to argue under the additional condition that

$$\overline{c}^{p'} \fint_{2B} |\mathbf{F} - \mathbf{F}_0|^{p'}\,\mathrm{d}x \le \varepsilon|\langle \mathbf{A}(\nabla \mathbf{u})\rangle_{2B}|^{p'} \tag{2.53}$$

for some given constant $\overline{c}$. Equations (2.52) and (2.53) amount to requiring that both $\mathbf{V}(\nabla \mathbf{u})$ and $\mathbf{F}$ are small compared to the averages of $\mathbf{A}(\nabla \mathbf{u})$. Next, given $\sigma > 0$, we choose

$$\overline{c} = c_\sigma \tag{2.54}$$

in (2.53), where c_σ is the constant appearing in Lemma 2.8 for $\theta = \sigma$. Also, we introduce the notation:

$$E = \fint_{2B} |\mathbf{A}(\nabla \mathbf{u}) - \langle \mathbf{A}(\nabla \mathbf{u})\rangle_{2B}|\,\mathrm{d}x + \Big(\sigma^{-2\alpha} c_\sigma \fint_{2B} |\mathbf{F} - \mathbf{F}_0|^{p'}\,\mathrm{d}x\Big)^{1/p'}. \tag{2.55}$$

Our key decay estimate in the non-degenerate case, for $p \in (1, 2)$, is contained in the following proposition.

Proposition 2.14. *Assume that $p \in (1,2)$. Let $\mathbf{u}$ be a local weak solution to (1.1) satisfying (2.37) and (2.53) for some ball B, some $\varepsilon \in (0, \frac{1}{4})$, and $\overline{c} = c_\sigma$ for some σ as in (2.40). Here, c_σ denotes the constant appearing in Lemma 2.8, with $\theta = \sigma$. Then there exist constants $c = c(n,N,p)$ and $\alpha = \alpha(n,p,N) \in (0,1)$ such that*

$$\fint_{\theta B} |\mathbf{A}(\nabla \mathbf{u}) - \langle \mathbf{A}(\nabla \mathbf{u})\rangle_{\theta B}|^2 \,\mathrm{d}x \le c\,\theta^{\frac{4\alpha}{p'}} \bigg[\Big(\fint_{2B} |\mathbf{A}(\nabla \mathbf{u}) - \langle \mathbf{A}(\nabla \mathbf{u})\rangle_{2B}| \,\mathrm{d}x \Big)^2 + \Big(\sigma^{-2\alpha} c_\sigma \fint_{2B} |\mathbf{F} - \mathbf{F}_0|^{p'} \,\mathrm{d}x \Big)^{\frac{2}{p'}} \bigg] \tag{2.56}$$

for every $\theta \in [\sigma, \frac{1}{4})$ and $z \in B$.

Proof. One has that

$$\begin{aligned} \fint_{2B} |\mathbf{A}(\nabla \mathbf{u}) - \langle \mathbf{A}(\nabla \mathbf{u})\rangle_{2B}| \,\mathrm{d}x &\le \Big(\fint_{2B} |\mathbf{A}(\nabla \mathbf{u}) - \langle \mathbf{A}(\nabla \mathbf{u})\rangle_{2B}|^{p'} \,\mathrm{d}x \Big)^{\frac{1}{p'}} \\ &\le \Big(\fint_{2B} \varphi_{p',|\langle \mathbf{A}(\nabla \mathbf{u})\rangle_{2B}|}\Big(|\mathbf{A}(\nabla \mathbf{u}) - \langle \mathbf{A}(\nabla \mathbf{u})\rangle_{2B}|\Big) \,\mathrm{d}x \Big)^{\frac{1}{p'}} \\ &\le c\Big(\fint_{2B} |\mathbf{V}(\nabla \mathbf{u}) - \langle \mathbf{V}(\nabla \mathbf{u})\rangle_{2B}|^2 \,\mathrm{d}x \Big)^{\frac{1}{p'}} \le c\varepsilon^{\frac{1}{p'}} |\langle \mathbf{A}(\nabla \mathbf{u})\rangle_{2B}|, \end{aligned} \tag{2.57}$$

where the second inequality holds since $t^{p'} \le \varphi_{p',a}(t)$ when $p' \ge 2$, $a > 0$, $t \ge 0$, the third inequality holds by (2.5) and Lemma 2.2, and the last inequality is a consequence of (2.52). Coupling (2.57) with (2.53), and taking into account the fact that $\varepsilon < 1$, imply that

$$E \le c\varepsilon^{\frac{1}{p'}} |\langle \mathbf{A}(\nabla \mathbf{u})\rangle_{2B}|, \tag{2.58}$$

where E is defined by (2.55). Now, observe that

$$\varphi_{p',|\mathbf{A}(\mathbf{A}_2)|}(t) \le ct^{p'} + c|\mathbf{A}(\mathbf{A}_2)|^{p'-2} t^2 \quad \text{for all} \quad t \ge 0 \tag{2.59}$$

inasmuch as $p' \ge 2$. Therefore, owing to (2.58),

$$\begin{aligned} \fint_{2B} \varphi_{p',|\mathbf{A}(\mathbf{A}_2)|}(|\mathbf{F} - \mathbf{F}_0|) \,\mathrm{d}x &\le c|\mathbf{A}(\mathbf{A}_2)|^{p'-2} \fint_{2B} |\mathbf{F} - \mathbf{F}_0|^2 \,\mathrm{d}x + c \fint_{2B} |\mathbf{F} - \mathbf{F}_0|^{p'} \,\mathrm{d}x \\ &\le c|\mathbf{A}(\mathbf{A}_2)|^{p'-2} \Big(\fint_{2B} |\mathbf{F} - \mathbf{F}_0|^{p'} \Big)^{\frac{2}{p'}} + c\sigma^{2\alpha} E^{p'} \le c\theta^{\frac{4\alpha}{p'}} |\mathbf{A}(\mathbf{A}_2)|^{p'-2} E^2. \end{aligned} \tag{2.60}$$

An application of Lemma 2.8, with B replaced by $\frac{1}{2}B(z)$, tells us that, if $\theta \in [\sigma, \frac{1}{4})$, then

$$\begin{aligned} \fint_{\theta B(z)} |\mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(\mathbf{U}_{\theta,z})|^2 \,\mathrm{d}x \le{}& c\,\theta^{2\alpha} \varphi_{p',|\mathbf{A}(\mathbf{A}_{1,z}|}\Big(\fint_{B(z)} |\mathbf{A}(\nabla \mathbf{u}) - \langle \mathbf{A}(\nabla \mathbf{u})\rangle_{B(z)}| \,\mathrm{d}x \Big) \\ &+ c_\sigma \fint_{B(z)} \varphi_{p',|\mathbf{A}(\mathbf{A}_{1.z})|}(|\mathbf{F} - \mathbf{F}_0|) \,\mathrm{d}x \end{aligned} \tag{2.61}$$

for any $z \in B$. Since $B(z) \subset 2B$, by Lemma 2.12 one has that $|\mathbf{A}_{1,z}| \le 2|\mathbf{A}_2| \le 4|\mathbf{A}_{1,z}|$. Hence, by (2.61),

$$\fint_{\theta B(z)} |\mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(\mathbf{U}_{\theta,z})|^2 \,\mathrm{d}x \le c\,\theta^{2\alpha} \varphi_{p',|\mathbf{A}(\mathbf{A}_2)|}\Big(\fint_{2B} |\mathbf{A}(\nabla \mathbf{u}) - \langle \mathbf{A}(\nabla \mathbf{u})\rangle_{2B}| \,\mathrm{d}x \Big) \tag{2.62}$$

$$+ c_\sigma \fint_{2B} \varphi_{p',|\mathbf{A}(\mathbf{A}_2)|}(|\mathbf{F}-\mathbf{F}_0|)\,\mathrm{d}x.$$

Starting from (2.62), and making use of (2.9), (2.59) and (2.60) tell us that

$$\fint_{\theta B(z)} |\mathbf{V}(\nabla \mathbf{u}) - \langle \mathbf{V}(\nabla \mathbf{u})\rangle_{\theta B(z)}|^2\,\mathrm{d}x \leq c\,\theta^{\frac{4\alpha}{p'}}|\mathbf{A}(\mathbf{A}_2)|^{p'-2}E^2\,.$$

Thus,

$$\begin{aligned}\fint_{\theta B} |\mathbf{A}(\nabla \mathbf{u}) - \mathbf{A}(\mathbf{A}_\theta)|^2\,\mathrm{d}x &\leq \fint_{\theta B} |\mathbf{A}(\nabla \mathbf{u}) - \mathbf{A}(\mathbf{V}_\theta)|^2\,\mathrm{d}x \\ &\leq c\fint_{\theta B} (|\nabla \mathbf{u}| + |\mathbf{V}_\theta|)^{2(p-2)}|\nabla \mathbf{u} - \mathbf{V}_\theta|^2\,\mathrm{d}x \\ &\leq c|\mathbf{V}_\theta|^{p-2}\fint_{\theta B} (|\nabla \mathbf{u}| + |\mathbf{V}_\theta|)^{(p-2)}|\nabla \mathbf{u} - \mathbf{V}_\theta|^2\,\mathrm{d}x \\ &\leq c|\mathbf{V}_\theta|^{p-2}\fint_{\theta B} |\mathbf{V}(\nabla \mathbf{u}) - \langle \mathbf{V}(\nabla \mathbf{u})\rangle_{\theta B}|^2\,\mathrm{d}x \\ &\leq c\,\theta^{\frac{4\alpha}{p'}}|\mathbf{A}_2|^{p-2}|\mathbf{A}(\mathbf{A}_2)|^{p'-2}E^2 \\ &\leq c\,\theta^{\frac{4\alpha}{p'}}E^2,\end{aligned} \tag{2.63}$$

where the first inequality holds owing to (2.10), the second to (2.7), the third to the fact that $p-2<0$, the fourth to (2.5), the fifth to Lemma 2.12, and the last one to the fact that $p-2+(p'-2)(p-1)=0$. This establishes (2.56). □

The remaining part of the present section is devoted to a counterpart of inequality (2.56) in the case when $p \in [2,\infty)$, which requires some further steps.
A first decay conclusion in the spirit of (2.56), but with $\mathbf{A}(\nabla \mathbf{u})$ replaced by $\mathbf{V}(\nabla \mathbf{u})$, reads as follows.

Lemma 2.15. *Assume that $p \in [2,\infty)$. Let $\mathbf{u}$ be a local weak solution to* (1.1) *satisfying* (2.37) *and* (2.53) *for some ball B, some $\varepsilon \in (0,\frac14)$, and $\overline{c} = c_\sigma$ for some σ as in* (2.40). *Here, c_σ denotes the constant appearing in Lemma 2.8, with $\theta = \sigma$. Then there exist constants $c = c(n,N,p)$ and $\alpha = \alpha(n,p,N) \in (0,1)$ such that*

$$\begin{aligned}&\fint_{\theta B(z)} |\mathbf{V}(\nabla \mathbf{u}) - \langle \mathbf{V}(\nabla \mathbf{u})\rangle_{\theta B(z)}|^2\,\mathrm{d}x \\ &\quad\leq c\,\theta^{2\alpha}\bigg[\bigg(\fint_{2B} |\mathbf{A}(\nabla \mathbf{u}) - \langle \mathbf{A}(\nabla \mathbf{u})\rangle_{2B}|\,\mathrm{d}x\bigg)^{p'} + \sigma^{-2\alpha}c_\sigma\fint_{2B} |\mathbf{F}-\mathbf{F}_0|^{p'}\,\mathrm{d}x\bigg]\end{aligned} \tag{2.64}$$

for every $\theta \in [\sigma,\frac14)$ and $z \in B$.

Proof. An inspection of the proof of inequality (2.62) reveals that it holds, in fact, also for $p \in [2,\infty)$. On the other hand, for these values of p, we have that $\varphi_{p',|\mathbf{A}(\mathbf{A}_2)|}(t) \leq t^{p'}$ for $t \geq 0$. Inequality (2.64) thus follows from (2.62). □

The key idea which enables us to turn the decay estimate for $\mathbf{V}(\nabla \mathbf{u})$ contained in Lemma 2.15 into the desired estimate for $\mathbf{A}(\nabla \mathbf{u})$ is to exploit a linearization argument. Specifically, denote by $D\mathbf{A} : \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n} \times \mathbb{R}^{N\times n}$ the differential of the map $\mathbf{A}$, and let $\mathbf{z}$ be the solution to the linear Dirichlet problem

$$\begin{cases} \operatorname{div}(D\mathbf{A}(\mathbf{A}_2)\nabla \mathbf{z}) = 0 & \text{in} \quad B, \\ \mathbf{z} = \mathbf{u} & \text{on} \quad \partial B, \end{cases} \tag{2.65}$$

where $\mathbf{A}_2$ is the (constant) matrix defined as in (2.39). We shall compare the gradient of the solution $\mathbf{u}$ to the original system (1.1) with the gradient of $\mathbf{z}$.
To begin with, recall that, since $\mathbf{z}$ solves a linear uniformly elliptic system with constant coefficients, the standard linear theory provides us with the decay estimate

$$\sup_{x,x' \in \theta B} |\nabla \mathbf{z}(x) - \nabla \mathbf{z}(x')| \leq \theta c \fint_B |\nabla \mathbf{z} - \langle \nabla \mathbf{z} \rangle_B| \, \mathrm{d}y \tag{2.66}$$

for any $\theta \in (0, 1)$, for some constant $c = c(n, p, N)$.
Next, observe that, on defining the function $\mathbf{H} : \mathbb{R}^{N\times n} \times \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$ as

$$\mathbf{H}(\mathbf{P}, \mathbf{Q}) = \mathbf{A}(\mathbf{P}) - \mathbf{A}(\mathbf{Q}) - D\mathbf{A}(\mathbf{Q})(\mathbf{P} - \mathbf{Q}) \quad \text{for } (\mathbf{P}, \mathbf{Q}) \in \mathbb{R}^{N\times n} \times \mathbb{R}^{N\times n}, \tag{2.67}$$

system (1.1) can be rewritten as

$$\operatorname{div}(D\mathbf{A}(\mathbf{A}_2)\nabla(\mathbf{u} - \mathbf{z})) = -\operatorname{div}\left(\mathbf{H}(\nabla \mathbf{u}, \mathbf{A}_2)\right) + \operatorname{div}\left(\mathbf{F} - \mathbf{F}_0\right) . \tag{2.68}$$

The following classical result (see e.g. [24, Lemma 2]) applies to systems of the form (2.67).

Theorem 2.16. [Calderón–Zygmund] *Let B be a ball in $\mathbb{R}^n$. Let $\mathbf{B} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ be such that $\lambda|\boldsymbol{\xi}|^2 \leq \langle \mathbf{B}\boldsymbol{\xi}, \boldsymbol{\xi} \rangle \leq \Lambda|\boldsymbol{\xi}|^2$ for some constants $\Lambda \geq \lambda > 0$, and for every $\boldsymbol{\xi} \in \mathbb{R}^n$. Assume that $r \in (1, \infty)$, and let $\mathbf{G} \in L^r(B, \mathbb{R}^{N\times n})$. Let $\mathbf{w}$ be the unique solution to the system*

$$\begin{cases} -\operatorname{div}(\mathbf{B}\nabla \mathbf{w}) = -\operatorname{div}\mathbf{G} & \text{in } B \\ \mathbf{w} = 0 & \text{on } \partial B. \end{cases}$$

Then, there exists a constant $c = c(n, r, \frac{\Lambda}{\lambda})$ such that

$$\lambda \|\nabla \mathbf{w}\|_{L^r(B)} \leq c \|\mathbf{G}\|_{L^r(B)}.$$

Since the matrix $\mathbf{B}$, given by $\mathbf{B} = D\mathbf{A}(\mathbf{A}_2)$, satisfies the assumptions of Theorem 2.16 with $\lambda = \lambda(n, N, p)$ and $\Lambda = \Lambda(n, N, p)$, an application of this theorem with $\mathbf{w} = \mathbf{u} - \mathbf{z}$ and $r = p'$ yields the following result.

Lemma 2.17. *Assume that $p \in (1, \infty)$, and let $\mathbf{u}$ be a local weak solution to (1.1). Let B be a ball such that $B \subset \Omega$, and let $\mathbf{z}$ be the solution to the Dirichlet problem (2.65). Then there exists a constant $c = c(n, N, p)$ such that*

$$|\mathbf{A}_2|^{(p-2)p'} \fint_B |\nabla \mathbf{u} - \nabla \mathbf{z}|^{p'} \, \mathrm{d}x \leq c \fint_B |\mathbf{F} - \mathbf{F}_0|^{p'} \, \mathrm{d}x + c \fint_B |\mathbf{H}(\nabla \mathbf{u}, \mathbf{A}_2)|^{p'} \, \mathrm{d}x. \tag{2.69}$$

The error term containing $\mathbf{H}$ in (2.69) will be estimated with the aid of an algebraic inequality which is the object of the next lemma.

Lemma 2.18. *Let $p \in [2, \infty)$, and let $\mathbf{H}$ be the function defined by (2.67). Then, there exists a constant $c = c(n, N, p)$ such that*

$$\mathbf{H}(\mathbf{P}, \mathbf{Q}) \leq c\delta \left(\frac{|\mathbf{P} - \mathbf{Q}|}{\delta} \right)^{p-1} \chi_{\{|\mathbf{P}-\mathbf{Q}| \geq \delta|\mathbf{Q}|\}} + c\delta |\mathbf{A}(\mathbf{P}) - \mathbf{A}(\mathbf{Q})| \chi_{\{|\mathbf{P}-\mathbf{Q}| \leq \delta|\mathbf{Q}|\}}, \tag{2.70}$$

for every $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{N\times n}$, and $\delta \in (0, 1/2]$. Here, χ_G stands for the characteristic function of a set G.

Proof. The function $\mathbf{H}$ can be represented as

$$\begin{aligned}\mathbf{H}(\mathbf{P},\mathbf{Q}) &= \int_0^1 [D\mathbf{A}(\mathbf{Q}+t(\mathbf{P}-\mathbf{Q})) - D\mathbf{A}(\mathbf{Q})]\,\mathrm{d}t\,(\mathbf{P}-\mathbf{Q})\\ &= |\mathbf{Q}|^{p-1}\int_0^1 \Big[D\mathbf{A}(\widetilde{\mathbf{Q}}+t\widetilde{\mathbf{R}}) - D\mathbf{A}(\widetilde{\mathbf{Q}})\Big]\,\mathrm{d}t\,\widetilde{\mathbf{R}},\end{aligned}$$

where we have introduced the notations

$$\widetilde{\mathbf{Q}} = \frac{\mathbf{Q}}{|\mathbf{Q}|}\,, \qquad \widetilde{\mathbf{P}} = \frac{\mathbf{P}}{|\mathbf{Q}|}\,, \qquad \widetilde{\mathbf{R}} = \widetilde{\mathbf{P}} - \widetilde{\mathbf{Q}}\,.$$

Computations show that

$$|D^2\mathbf{A}(\mathbf{T})| \le c|\mathbf{T}|^{p-3} \tag{2.71}$$

for some constant $c = c(n,N,p)$ and for every $\mathbf{T}\in\mathbb{R}^{N\times n}\setminus\{0\}$. Owing to (2.71),

$$\left|\int_0^1 \Big[D\mathbf{A}(\widetilde{\mathbf{Q}}+t\widetilde{\mathbf{R}}) - D\mathbf{A}(\widetilde{\mathbf{Q}})\Big]\,\mathrm{d}t\right| \le c|\widetilde{\mathbf{R}}|\int_0^1\int_0^1 |\widetilde{\mathbf{Q}}+st\widetilde{\mathbf{R}}|^{p-3}\,\mathrm{d}s\,\mathrm{d}t$$

for some constant $c = c(n,N,p)$. Now, if $|\widetilde{\mathbf{R}}|\le\delta$, and hence, in particular, $|\widetilde{\mathbf{R}}|\le\frac12$, then

$$\int_0^1\int_0^1 |\widetilde{\mathbf{Q}}+st\widetilde{\mathbf{R}}|^{p-3}\,\mathrm{d}s\,\mathrm{d}t \le c.$$

Since our present assumption on $|\widetilde{\mathbf{R}}|$ is equivalent to $|\mathbf{P}-\mathbf{Q}|\le\delta|\mathbf{Q}|$, and hence, in particular, it implies that $|\mathbf{P}|+|\mathbf{Q}|\le 3|\mathbf{Q}|$, we have that

$$|\mathbf{H}(\mathbf{P},\mathbf{Q})| \le c\delta|\mathbf{Q}|^{p-2}|\mathbf{P}-\mathbf{Q}| \le c'\delta|\mathbf{A}(\mathbf{P})-\mathbf{A}(\mathbf{Q})|, \tag{2.72}$$

whence (2.70) follows. Note that, in the derivation of (2.72), we have also made use of (2.7). On the other hand, if $|\widetilde{\mathbf{R}}|>\delta$, then

$$\int_0^1\int_0^1 |\widetilde{\mathbf{Q}}+st\widetilde{\mathbf{R}}|^{p-3}\,\mathrm{d}s\,\mathrm{d}t \le c(1+|\widetilde{\mathbf{R}}|)^{p-3}.$$

Hence,

$$|\mathbf{H}(\mathbf{P},\mathbf{Q})| \le c(|\mathbf{Q}|+|\mathbf{P}-\mathbf{Q}|)^{p-3}|\mathbf{P}-\mathbf{Q}|^2\,,$$

and this inequality can be shown to imply (2.70) also in this case. □

Our last preparatory step is contained in the following lemma.

Lemma 2.19. *Let $p\in[2,\infty)$, and let $\mathbf{u}$ be a local weak solution to* (1.1) *satisfying* (2.37) *for some ball B, and some $\varepsilon\in(0,\frac14)$. Assume that $\sigma\in(0,\frac14)$ is such that* (2.40) *holds. Let α be the exponent appearing in the statement of Theorem 2.6. Then there exist constants $c=c(n,p,N)$ and $\alpha=\alpha(n,p,N)$ such that*

$$\begin{aligned}&\int_{B\cap\{|\nabla\mathbf{u}-\mathbf{A}_2|\ge\sigma^{2n}|\mathbf{A}_2|\}} |\mathbf{V}(\nabla\mathbf{u})-\mathbf{V}(\mathbf{A}_2)|^2\,\mathrm{d}x\\ &\qquad\le c|B|\sigma^{2\alpha}\bigg[\bigg(\fint_{2B}|\mathbf{A}(\nabla\mathbf{u})-\langle\mathbf{A}(\nabla\mathbf{u})\rangle_{2B}|\,\mathrm{d}x\bigg)^{p'} + \sigma^{-2\alpha}c_\sigma\fint_{2B}|\mathbf{F}-\mathbf{F}_0|^{p'}\,\mathrm{d}x\bigg],\end{aligned} \tag{2.73}$$

where $\mathbf{A}_2$ is given by (2.39).

Proof. By Besicovich covering theorem, there exists a countable covering of the set $B\cap\{|\nabla\mathbf{u}-\mathbf{A}_2|\ge\sigma^{2n}|\mathbf{A}_2|\}$ by balls $\sigma B(z)$ whose number of overlaps is uniformly bounded by a constant depending only on n. On this set we have, by Lemma 2.12,

$$\sigma^{2n}|\mathbf{A}_2| \le |\nabla\mathbf{u}-\mathbf{A}_2| \le |\nabla\mathbf{u}-\mathbf{V}_{\sigma,z}| + |\mathbf{V}_{\sigma,z}-\mathbf{U}_2| + |\mathbf{U}_2-\mathbf{A}_2| \le |\nabla\mathbf{u}-\mathbf{V}_{\sigma,z}| + \frac{\sigma^{2n}}{4}|\mathbf{A}_2|$$

and hence $|\nabla\mathbf{u}-\mathbf{A}_2|\le c|\nabla\mathbf{u}-\mathbf{V}_{\sigma,z}|$. On the other hand, by Lemma 2.12, $|\mathbf{A}_2|\le 2|\mathbf{V}_{\sigma,z}|$. Thus, owing to (2.5) and (2.64),

$$\begin{aligned}
&\int_{\sigma B(z)\cap\{|\nabla\mathbf{u}-\mathbf{A}_2|\ge\sigma^{2n}|\mathbf{A}_2|\}}|\mathbf{V}(\nabla\mathbf{u})-\mathbf{V}(\mathbf{A}_2)|^2\,\mathrm{d}x\\
&\quad\le c\int_{\sigma B(z)\cap\{|\nabla\mathbf{u}-\mathbf{A}_2|\ge\sigma^{2n}|\mathbf{A}_2|\}}(|\nabla\mathbf{u}|+|\mathbf{A}_2|)^{p-2}|\nabla\mathbf{u}-\mathbf{A}_2|^2\,\mathrm{d}x\\
&\quad\le c\int_{\sigma B(z)\cap\{|\nabla\mathbf{u}-\mathbf{A}_2|\ge\sigma^{2n}|\mathbf{A}_2|\}}(|\nabla\mathbf{u}|+|\mathbf{V}_{\sigma,z}|)^{p-2}|\nabla\mathbf{u}-\mathbf{V}_{\sigma,z}|^2\,\mathrm{d}x\\
&\quad\le c\int_{\sigma B(z)}|\mathbf{V}(\nabla\mathbf{u})-\langle\mathbf{V}(\nabla\mathbf{u})\rangle_{\sigma B(z)}|^2\,\mathrm{d}x\le c|\sigma B(z)|\sigma^{2\alpha}E^{p'},
\end{aligned}$$

where E is defined as in (2.55). Hence, inequality (2.73) follows. □

We are now ready to prove the decay estimate in the case when $p\ge 2$.

Proposition 2.20. *Let $p\in[2,\infty)$, and let $\mathbf{u}$ be a local weak solution to (1.1). Let α be the exponent appearing in the statement of Theorem 2.6. Let $\theta\in(0,1)$. Then there exist constants $c=c(n,p,N)$, $\varepsilon_\theta=\varepsilon(n,p,N,\theta)\in(0,\frac14)$, and $c_\theta=c_\theta(n,p,N,\theta)$ such that if $\mathbf{u}$ satisfies (2.37) with $\varepsilon=\varepsilon_\theta$, then*

$$\begin{aligned}
&\Big(\fint_{\theta B}|\mathbf{A}(\nabla\mathbf{u})-\langle\mathbf{A}(\nabla\mathbf{u})\rangle_{\theta B}|^{p'}\,\mathrm{d}x\Big)^{\frac{1}{p'}}\\
&\quad\le c\theta^{\min\{\frac{2\alpha}{p'},1\}}\Big(\fint_{2B}|\mathbf{A}(\nabla\mathbf{u})-\langle\mathbf{A}(\nabla\mathbf{u})\rangle_{2B}|^{p'}\,\mathrm{d}x\Big)^{\frac{1}{p'}}+c_\theta\Big(\fint_{2B}|\mathbf{F}-\mathbf{F}_0|^{p'}\,\mathrm{d}x\Big)^{\frac{1}{p'}}
\end{aligned}\tag{2.74}$$

for every $\mathbf{F}_0\in\mathbb{R}^{N\times n}$.

Proof. From (2.10), (2.7) and (2.5) one can infer that

$$\begin{aligned}
\fint_{\theta B}|\mathbf{A}(\nabla\mathbf{u})-\mathbf{A}(\mathbf{A}_\theta)|^{p'}\,\mathrm{d}x&\le c\fint_{\theta B}|\mathbf{A}(\nabla\mathbf{u})-\mathbf{A}(\mathbf{U}_\theta)|^{p'}\,\mathrm{d}x\\
&\le c\fint_{\theta B}\Big(|\nabla\mathbf{u}-\mathbf{U}_\theta|+|\mathbf{U}_\theta|\Big)^{p'(p-2)}|\nabla\mathbf{u}-\mathbf{U}_\theta|^{p'}\\
&\le c\fint_{\theta B}|\mathbf{V}(\nabla\mathbf{u})-\mathbf{V}(\mathbf{U}_\theta)|^2\,\mathrm{d}x+c|\mathbf{U}_\theta|^{p'(p-2)}\fint_{\theta B}|\nabla\mathbf{u}-\mathbf{U}_\theta|^{p'}\,\mathrm{d}x.
\end{aligned}\tag{2.75}$$

By Lemma 2.15,

$$\fint_{\theta B}|\mathbf{V}(\nabla\mathbf{u})-\mathbf{V}(\mathbf{U}_\theta)|^2\,\mathrm{d}x\le c\theta^{2\alpha}E^{p'},\tag{2.76}$$

where E is defined by (2.55). Next, via a repeated use of inequality (2.10), the triangle inequality, Lemma 2.12, and inequality (2.66) one obtains the following chain:

(2.77)

$$\begin{aligned}
&|\mathbf{U}_\theta|^{p'(p-2)}\fint_{\theta B}|\nabla\mathbf{u}-\mathbf{U}_\theta|^{p'}\,\mathrm{d}x\le c\,|\mathbf{U}_\theta|^{p'(p-2)}\fint_{\theta B}|\nabla\mathbf{u}-\langle\nabla\mathbf{z}\rangle_{\theta B}|^{p'}\,\mathrm{d}x\\
&\le c\theta^{p'}|\mathbf{A}_2|^{p'(p-2)}\Big(\fint_{B}|\nabla\mathbf{z}-\langle\nabla\mathbf{z}\rangle_B|\,\mathrm{d}x\Big)^{p'}+c\theta^{-n}|\mathbf{A}_2|^{p'(p-2)}\fint_{B}|\nabla\mathbf{z}-\nabla\mathbf{u}|^{p'}\,\mathrm{d}x
\end{aligned}$$

$$\le c\theta^{p'}|\mathbf{A}_2|^{p'(p-2)}\bigg(\fint_B |\nabla\mathbf{u}-\langle\nabla\mathbf{u}\rangle_B|\,\mathrm{d}x\bigg)^{p'}+c(\theta^{-n}+\theta^{p'})|\mathbf{A}_2|^{p'(p-2)}\fint_B|\nabla\mathbf{z}-\nabla\mathbf{u}|^{p'}\,\mathrm{d}x$$

$$= I_1+I_2.$$

Owing to (2.10) and (2.7),

$$I_1 \le c\theta^{p'}\bigg(\fint_B|\mathbf{A}_2|^{p-2}|\nabla\mathbf{u}-\mathbf{A}_2|\,\mathrm{d}x\bigg)^{p'}\le c\theta^{p'}\bigg(\fint_B|\mathbf{A}(\nabla\mathbf{u})-\mathbf{A}(\mathbf{A}_2)|\,\mathrm{d}x\bigg)^{p'}\le c\theta^{p'}E^{p'}. \tag{2.78}$$

In order to estimate I_2, observe that, thanks to Lemma 2.18,

$$\int_B|\mathbf{H}(\nabla\mathbf{u},\mathbf{A}_2)|^{p'}\,\mathrm{d}x \le c\delta^{-(p-2)p'}\int_{B\cap\{|\nabla\mathbf{u}-\mathbf{A}_2|\ge\delta|\mathbf{A}_2|\}}|\nabla\mathbf{u}-\mathbf{A}_2|^p\,\mathrm{d}x + c\delta^{p'}\int_{2B}|\mathbf{A}(\nabla\mathbf{u})-\mathbf{A}(\mathbf{A}_2)|^{p'}\,\mathrm{d}x$$

for every $\delta\in(0,\frac12)$. This inequality, applied with $\delta=\sigma^{2\alpha/p}$, and Lemma 2.19 ensure that

$$\fint_B|\mathbf{H}(\nabla\mathbf{u},\mathbf{A}_2)|^{p'}\,\mathrm{d}x \le c\sigma^{\frac{2\alpha}{p-1}}E^{p'}+c\sigma^{\frac{2\alpha}{p-1}}\fint_{2B}|\mathbf{A}(\nabla\mathbf{u})-\mathbf{A}(\mathbf{A}_2)|^{p'}\,\mathrm{d}x. \tag{2.79}$$

Here, we have also exploited the fact that, by (2.5), $|\mathbf{P}-\mathbf{Q}|^p\le c|\mathbf{V}(\mathbf{P})-\mathbf{V}(\mathbf{Q})|^2$ for $\mathbf{P},\mathbf{Q}\in\mathbb{R}^{N\times n}$, since $p\ge 2$.
By (2.69) and (2.79),

$$\begin{aligned}|\mathbf{A}_2|^{p'(p-2)}\fint_B|\nabla\mathbf{z}-\nabla\mathbf{u}|^{p'}\,\mathrm{d}x &\le c\fint_B|\mathbf{F}-\mathbf{F}_0|^{p'}\,\mathrm{d}x+c\fint_B|\mathbf{H}(\nabla\mathbf{u},\mathbf{A}_2)|^{p'}\,\mathrm{d}x.\\ &\le c_\sigma\fint_B|\mathbf{F}-\mathbf{F}_0|^{p'}\,\mathrm{d}x+c\sigma^{\frac{2\alpha}{p-1}}\fint_{2B}|\mathbf{A}(\nabla\mathbf{u})-\mathbf{A}(\mathbf{A}_2)|^{p'}\,\mathrm{d}x.\end{aligned} \tag{2.80}$$

Choosing $\sigma=\theta^{2\alpha p}$ in (2.80) yields

$$I_2\le c\theta^{p'}\fint_{2B}|\mathbf{A}(\nabla\mathbf{u})-\mathbf{A}(\mathbf{A}_2)|^{p'}\,\mathrm{d}x+c_\theta\fint_B|\mathbf{F}-\mathbf{F}_0|^{p'}\,\mathrm{d}x\,, \tag{2.81}$$

where $c=c(n,p,N)$ and $c_\theta=c(n,p,N,\theta)$. We next fix $\varepsilon_\theta=\varepsilon(n,p,N,\theta)$ in such a way that (2.40) is satisfied. Combining inequalities (2.75)–(2.78) and (2.81) yields (2.74). □

2.4. **Proof of Proposition 2.1, concluded.** The decay estimates established above enable us to accomplish the proof of Proposition 2.1.

Proof of Proposition 2.1. We apply Proposition 2.11, Lemma 2.13, and either Proposition 2.14 or Proposition 2.20, according to whether $p\in(1,2)$ or $p\in[2,\infty)$, with $4B$ or $2B$ replaced by B in the integrals on the right-hand sides of the relevant decay estimates. Moreover, we denote by c_1, c_2 and c_3 the constants multiplying the integrals involving the expression $\mathbf{A}(\nabla\mathbf{u})$ on the right-hand sides of inequalities (2.33), (2.56), and (2.74), respectively.
Fix any $\kappa\in\big(0,\min\{1,\frac{2\alpha}{p'}\}\big)$ and any $\delta>0$. In the case when $p\in(1,2)$, we choose θ so small that

$$c_1\theta^\kappa+c_2\,\theta^{\frac{2\alpha}{p'}}\le\delta. \tag{2.82}$$

If, instead, $p\in[2,\infty)$, we choose θ so small that

$$c_1\theta^\kappa+c_3\theta^{\min\{1,\frac{2\alpha}{p'}\}}\le\delta\,, \tag{2.83}$$

and let $\varepsilon_\theta = \varepsilon(n, p, N, \delta)$ be fixed as in Proposition 2.20. In particular, inequalities (2.82) and (2.83) hold if $\theta = \big(c_1 + \max\{c_2, c_3\}\big)^{-\frac{1}{\kappa}}\delta^{\frac{1}{\kappa}}$ for small δ. Inequality (2.2) follows, modulo an application of Hölder's inequality. □

3. Proofs of the main results

This section is devoted to the proofs of Theorems 1.1, 1.3 and 1.6.

Proof of Theorem 1.1. Owing to Proposition 2.1, given any $\delta \in (0, 1)$, there exist constants $\theta \in (0, 1)$ and $c > 0$, both depending only on n, N, p, δ, such that

$$\Big(\fint_{\theta B} |\mathbf{A}(\nabla\mathbf{u}) - \langle\mathbf{A}(\nabla\mathbf{u})\rangle_{\theta B}|^{\min\{2,p'\}}\,dx\Big)^{\frac{1}{\min\{2,p'\}}} \leq \delta\Big(\fint_{B} |\mathbf{A}(\nabla\mathbf{u}) - \langle\mathbf{A}(\nabla\mathbf{u})\rangle_{B}|^{\min\{2,p'\}}\,dx\Big)^{\frac{1}{\min\{2,p'\}}} + c\Big(\fint_B |\mathbf{F} - \mathbf{F}_0|^{p'}\,dx\Big)^{\frac{1}{p'}}. \tag{3.1}$$

for every ball $B \subset \mathbb{R}^n$. Given any $x \in \mathbb{R}^n$, let B be any ball such that $x \in \theta B$. Owing to the definition of sharp maximal function (1.10), and to the arbitrariness of $\mathbf{F}_0$, we deduce from (3.1) that

$$M^{\sharp,\min\{p',2\}}(\mathbf{A}(\nabla\mathbf{u}))(x) \leq c\delta M^{\sharp,\min\{p',2\}}(\mathbf{A}(\nabla\mathbf{u}))(x) + cM^{\sharp,p'}(\mathbf{F})(x). \tag{3.2}$$

Note that $M^{\sharp,\min\{p',2\}}(\mathbf{A}(\nabla\mathbf{u}))(x) < \infty$ for a.e. $x \in \mathbb{R}^n$, since we are assuming that $\mathbf{u} \in V^{1,p}(\mathbb{R}^n)$. With the choice $\delta = (2c)^{-1}$, inequality (3.2) thus implies that

$$M^{\sharp,\min\{p',2\}}\big(\mathbf{A}(\nabla\mathbf{u})\big)(x) \leq cM^{\sharp,p'}(\mathbf{F})(x) \quad \text{for } x \in \mathbb{R}^n \quad \text{for a.e. } x \in \mathbb{R}^n.$$

Hence (1.11) follows, inasmuch as $M^{\sharp}(\mathbf{A}(\nabla\mathbf{u})) \leq M^{\sharp,\min\{p',2\}}(\mathbf{A}(\nabla\mathbf{u}))$. □

Proof of Theorem 1.3. Let $\beta \in \big(0, \min\{1, \frac{2\alpha}{p'}\}\big)$ be such that condition (1.14) is fulfilled by ω. Fix γ and κ such that

$$\beta < \gamma < \kappa < \min\{1, \tfrac{2\alpha}{p'}\}. \tag{3.3}$$

Given δ, let θ be the number obeying

$$\overline{c}\theta^\kappa = \delta\,, \tag{3.4}$$

for some constant $\overline{c} = \overline{c}(n, N, p, \kappa) > 1$, and such that inequality (2.2) holds. This is possible owing to (2.3). Thus, there exists $\theta_0 \in (0, 1)$ (depending on κ and γ, as well as on n, N, p) such that $\overline{c}\theta^{\kappa-\gamma} \leq 1$ for $\theta \leq \theta_0$, and hence

$$\delta \leq \theta^\gamma \tag{3.5}$$

for every $\delta \leq \delta_0$, where $\delta_0 = \overline{c}\theta_0^\kappa$. Now choose $\delta = \delta_0$ in Proposition 2.1. Thus, given any ball $B_r \subset \Omega$, an iteration of (2.2) tells us that

$$\Big(\fint_{B_{\theta^k r}} |\mathbf{A}(\nabla\mathbf{u}) - \langle\mathbf{A}(\nabla\mathbf{u})\rangle_{B_{\theta^k r}}|^{\min\{2,p'\}}\,dx\Big)^{\frac{1}{\min\{2,p'\}}} \leq \delta_0^k\Big(\fint_{B_r} |\mathbf{A}(\nabla\mathbf{u}) - \langle\mathbf{A}(\nabla\mathbf{u})\rangle_{B_r}|^{\min\{2,p'\}}\,dx\Big)^{\frac{1}{\min\{2,p'\}}} + c\theta^{-\frac{nk}{p'}}\Big(\fint_{B_r} |\mathbf{F} - \mathbf{F}_0|^{p'}\,dx\Big)^{\frac{1}{p'}} \tag{3.6}$$

for $k \in \mathbb{N}$. On setting $\rho = \theta^k$, and making use of (3.5), inequality (3.6) tells us that

$$\Big(\fint_{B_{\rho r}} |\mathbf{A}(\nabla\mathbf{u}) - \langle\mathbf{A}(\nabla\mathbf{u})\rangle_{B_{\rho r}}|^{\min\{2,p'\}}\,dx\Big)^{\frac{1}{\min\{2,p'\}}} \tag{3.7}$$

$$\leq \rho^\gamma \bigg(\fint_{B_r} |\mathbf{A}(\nabla \mathbf{u}) - \langle \mathbf{A}(\nabla \mathbf{u})\rangle_{B_r}|^{\min\{2,p'\}}\,\mathrm{d}x \bigg)^{\frac{1}{\min\{2,p'\}}} + c\rho^{-\frac{n}{p'}} \bigg(\fint_{B_r} |\mathbf{F}-\mathbf{F}_0|^{p'}\,\mathrm{d}x \bigg)^{\frac{1}{p'}}.$$

Dividing through by $\omega(\rho r)$ in (3.7) yields

$$\begin{aligned}
&\frac{1}{\omega(\rho r)} \bigg(\fint_{B_{\rho r}} |\mathbf{A}(\nabla \mathbf{u}) - \langle \mathbf{A}(\nabla \mathbf{u})\rangle_{B_{\rho r}}|^{\min\{2,p'\}}\,\mathrm{d}x \bigg)^{\frac{1}{\min\{2,p'\}}} \\
&\quad \leq \frac{\rho^\gamma \omega(r)}{\omega(\rho r)} \frac{1}{\omega(r)} \bigg(\fint_{B_r} |\mathbf{A}(\nabla \mathbf{u}) - \langle \mathbf{A}(\nabla \mathbf{u})\rangle_{B_r}|^{\min\{2,p'\}}\,\mathrm{d}x \bigg)^{\frac{1}{\min\{2,p'\}}} \\
&\qquad + c\frac{\omega(r)\rho^{-\frac{n}{p'}}}{\omega(\rho r)} \frac{1}{\omega(r)} \bigg(\fint_{B_r} |\mathbf{F}-\mathbf{F}_0|^{p'}\,\mathrm{d}x \bigg)^{\frac{1}{p'}}.
\end{aligned} \tag{3.8}$$

Let $k = k(n, N, p, \omega_\beta, \gamma, \beta)$ be the smallest integer such that $c_\omega \theta^{(\gamma-\beta)k} \leq \frac{1}{2}$. Hence,

$$\frac{\omega(r)}{\omega(\rho r)} \leq c_\omega \rho^{-\beta} \tag{3.9}$$

and

$$\frac{\rho^\gamma \omega(r)}{\omega(\rho r)} \leq \frac{1}{2}. \tag{3.10}$$

As a consequence of (3.8) and (3.9) we obtain that

$$\begin{aligned}
&\frac{1}{\omega(\rho r)} \bigg(\fint_{B_{\rho r}} |\mathbf{A}(\nabla \mathbf{u}) - \langle \mathbf{A}(\nabla \mathbf{u})\rangle_{B_{\rho r}}|^{\min\{2,p'\}}\,\mathrm{d}x \bigg)^{\frac{1}{\min\{2,p'\}}} \\
&\leq \frac{1}{2\omega(r)} \bigg(\fint_{B_r} |\mathbf{A}(\nabla \mathbf{u}) - \langle \mathbf{A}(\nabla \mathbf{u})\rangle_{B_r}|^{\min\{2,p'\}}\,\mathrm{d}x \bigg)^{\frac{1}{\min\{2,p'\}}} + \frac{c}{\omega(r)} \bigg(\fint_{B_r} |\mathbf{F}-\mathbf{F}_0|^{p'}\,\mathrm{d}x \bigg)^{\frac{1}{p'}}.
\end{aligned} \tag{3.11}$$

Now, let B be as in the statement, and let $x \in B$. Inequality (3.11), applied to any ball B_r such that $r < R$ and $B_{\rho r} \ni x$, tells us that

$$\begin{aligned}
&\sup_{r<\rho R} \frac{1}{\omega(r)} \bigg(\fint_{B_r} |\mathbf{A}(\nabla \mathbf{u}) - \langle \mathbf{A}(\nabla \mathbf{u})\rangle_{B_r}|^{\min\{2,p'\}}\,\mathrm{d}x \bigg)^{\frac{1}{\min\{2,p'\}}} \\
&\quad \leq \frac{1}{2} \sup_{r<R} \frac{1}{\omega(r)} \bigg(\fint_{B_r} |\mathbf{A}(\nabla \mathbf{u}) - \langle \mathbf{A}(\nabla \mathbf{u})\rangle_{B_r}|^{\min\{2,p'\}}\,\mathrm{d}x \bigg)^{\frac{1}{\min\{2,p'\}}} \\
&\qquad + \sup_{r<R} \frac{c}{\omega(r)} \bigg(\fint_{B_r} |\mathbf{F}-\mathbf{F}_0|^{p'}\,\mathrm{d}x \bigg)^{\frac{1}{p'}}.
\end{aligned} \tag{3.12}$$

On the other hand, owing to (2.10) and (1.14),

$$\begin{aligned}
&\sup_{\rho R<r<R} \frac{1}{\omega(r)} \bigg(\fint_{B_r} |\mathbf{A}(\nabla \mathbf{u}) - \langle \mathbf{A}(\nabla \mathbf{u})\rangle_{B_r}|^{\min\{2,p'\}}\,\mathrm{d}x \bigg)^{\frac{1}{\min\{2,p'\}}} \\
&\quad \leq c\frac{\omega(2R)}{\omega(r)} \rho^{-\frac{n}{\min\{2,p'\}}} \frac{1}{\omega(2R)} \bigg(\fint_{2B} |\mathbf{A}(\nabla \mathbf{u}) - \langle \mathbf{A}(\nabla \mathbf{u})\rangle_{2B}|^{\min\{2,p'\}}\,\mathrm{d}x \bigg)^{\frac{1}{\min\{2,p'\}}}.
\end{aligned} \tag{3.13}$$

$$\leq c'\rho^{-\beta-\frac{n}{\min\{2,p'\}}}\frac{1}{\omega(2R)}\bigg(\fint_{2B}|\mathbf{A}(\nabla\mathbf{u})-\langle\mathbf{A}(\nabla\mathbf{u})\rangle_{2B}|^{\min\{2,p'\}}\,\mathrm{d}x\bigg)^{\frac{1}{\min\{2,p'\}}}.$$

Coupling (3.12) with (3.13) tells us that

$$\begin{aligned}\sup_{r<R}\frac{1}{\omega(r)}\bigg(\fint_{B_r}|\mathbf{A}(\nabla\mathbf{u})-\langle\mathbf{A}(\nabla\mathbf{u})\rangle_{B_r}|^{\min\{2,p'\}}\,\mathrm{d}x\bigg)^{\frac{1}{\min\{2,p'\}}}\\ \leq\frac{1}{2}\sup_{r<R}\frac{1}{\omega(r)}\bigg(\fint_{B_r}|\mathbf{A}(\nabla\mathbf{u})-\langle\mathbf{A}(\nabla\mathbf{u})\rangle_{B_r}|^{\min\{2,p'\}}\,\mathrm{d}x\bigg)^{\frac{1}{\min\{2,p'\}}}\\ +\sup_{r<R}\frac{c}{\omega(r)}\bigg(\fint_{B_r}|\mathbf{F}-\mathbf{F}_0|^{p'}\,\mathrm{d}x\bigg)^{\frac{1}{p'}}\\ +\frac{c}{\omega(2R)}\bigg(\fint_{2B}|\mathbf{A}(\nabla\mathbf{u})-\langle\mathbf{A}(\nabla\mathbf{u})\rangle_{2B}|^{\min\{2,p'\}}\,\mathrm{d}x\bigg)^{\frac{1}{\min\{2,p'\}}}.\end{aligned}\tag{3.14}$$

Inequality (1.15) follows from (3.14), via the very definition of localized weighted sharp maximal operator (1.13). □

Proof of Theorem 1.6. Without loss of generality, we may assume that

$$\int_0^R\bigg(\fint_{B_\varrho(x)}\bigg(\frac{|\mathbf{F}-\langle\mathbf{F}\rangle_{B_\varrho(x)}|}{\rho}\bigg)^{p'}\mathrm{d}y\bigg)^{\frac{1}{p'}}d\varrho<\infty.\tag{3.15}$$

Fix $\delta=\frac{1}{2}$, and let $\theta=\theta(n,N,p)$ be the corresponding value provided by Proposition 2.1. Given any $k\in\mathbb{N}$, one can show, via a telescope sum argument, that

(3.16)

$$\Big|\fint_{B_{\theta^kR}(x)}\mathbf{A}(\nabla\mathbf{u})\,\mathrm{d}y-\fint_{B_R(x)}\mathbf{A}(\nabla\mathbf{u})\,\mathrm{d}y\Big|\leq\theta^{-n}\sum_{i=0}^{k-1}\fint_{B_{\theta^iR}(x)}|\mathbf{A}(\nabla\mathbf{u})-\langle\mathbf{A}(\nabla\mathbf{u})\rangle_{B_{\theta^iR}(x)}|\,\mathrm{d}y\,.$$

Inequality (2.2) implies that

$$\begin{aligned}\sum_{i=1}^{k}\bigg(\fint_{B_{\theta^iR}(x)}|\mathbf{A}(\nabla\mathbf{u})-\langle\mathbf{A}(\nabla\mathbf{u})\rangle_{B_{\theta^iR}(x)}|^{\min\{p',2\}}\,\mathrm{d}y\bigg)^{\frac{1}{\min\{p',2\}}}\\ \leq\frac{1}{2}\sum_{i=0}^{k-1}\bigg(\fint_{B_{\theta^iR}(x)}|\mathbf{A}(\nabla\mathbf{u})-\langle\mathbf{A}(\nabla\mathbf{u})\rangle_{B_{\theta^iR}(x)}|^{\min\{p',2\}}\,\mathrm{d}y\bigg)^{\frac{1}{\min\{p',2\}}}\\ +c\sum_{i=0}^{k-1}\bigg(\fint_{B_{\theta^iR}(x)}|\mathbf{F}-\langle\mathbf{F}\rangle_{B_{\theta^iR}(x)}|^{p'}\,\mathrm{d}y\bigg)^{\frac{1}{p'}}.\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{i=0}^{k}\bigg(\fint_{B_{\theta^iR}(x)}|\mathbf{A}(\nabla\mathbf{u})-\langle\mathbf{A}(\nabla\mathbf{u})\rangle_{B_{\theta^iR}(x)}|^{\min\{p',2\}}\,\mathrm{d}y\bigg)^{\frac{1}{\min\{p',2\}}}\\ \leq c\bigg(\fint_{B_R(x)}|\mathbf{A}(\nabla\mathbf{u})-\langle\mathbf{A}(\nabla\mathbf{u})\rangle_{B_R(x)}|^{\min\{p',2\}}\,\mathrm{d}y\bigg)^{\frac{1}{\min\{p',2\}}}\end{aligned}\tag{3.17}$$

$$+ c\sum_{i=0}^{k-1}\bigg(\fint_{B_{\theta^i R}(x)} |\mathbf{F}-\langle \mathbf{F}\rangle_{B_{\theta^i R}(x)}|^{p'}\,\mathrm{d}y\bigg)^{\frac{1}{p'}}. \tag{3.18}$$

Note that

$$\sum_{i=0}^{k}\bigg(\fint_{B_{\theta^i R}(x)} |\mathbf{F}-\langle \mathbf{F}\rangle_{B_{\theta^i R}(x)}|^{p'}\,\mathrm{d}y\bigg)^{\frac{1}{p'}} \le c\int_0^R \bigg(\fint_{B_\varrho(x)} \Big(\frac{|\mathbf{F}-\langle \mathbf{F}\rangle_{B_\varrho(x)}|}{\rho}\Big)^{p'}\,\mathrm{d}y\bigg)^{\frac{1}{p'}}\,d\varrho.$$

Therefore, letting k go to ∞ in (3.17) tells us that

$$\begin{aligned}\sum_{i=0}^{\infty}\bigg(\fint_{B_{\theta^i R}(x)} &|\mathbf{A}(\nabla\mathbf{u})-\langle \mathbf{A}(\nabla\mathbf{u})\rangle_{B_{\theta^i R}(x)}|^{\min\{p',2\}}\,\mathrm{d}y\bigg)^{\frac{1}{\min\{p',2\}}}\\ &\le c\bigg(\fint_{B_R(x)} |\mathbf{A}(\nabla\mathbf{u})-\langle \mathbf{A}(\nabla\mathbf{u})\rangle_{B_R(x)}|^{\min\{p',2\}}\,\mathrm{d}y\bigg)^{\frac{1}{\min\{p',2\}}}\\ &\quad + c\int_0^R \bigg(\fint_{B_\varrho(x)} \Big(\frac{|\mathbf{F}-\langle \mathbf{F}\rangle_{B_\varrho(x)}|}{\rho}\Big)^{p'}\,\mathrm{d}y\bigg)^{\frac{1}{p'}}\,d\varrho.\end{aligned} \tag{3.19}$$

Owing to (3.15), the series in (3.19) is convergent. Thus, in particular,

$$\lim_{i\to\infty}\bigg(\fint_{B_{\theta^i R(x)}} |\mathbf{A}(\nabla\mathbf{u})-\langle \mathbf{A}(\nabla\mathbf{u})\rangle_{B_{\theta^i R}(x)}|^{\min\{p',2\}}\,\mathrm{d}y\bigg)^{\frac{1}{\min\{p',2\}}} = 0. \tag{3.20}$$

Now fix any $0 < s < r \le \frac{1}{\theta}R$. Then there exists $k \in \mathbb{N}$ such that $\theta^{1-k}s < r \le \theta^{-k}s$. Set $r_s = \theta^{-k}s$ and observe that there exists $h \in \mathbb{N}$ such that $\theta^h R < r_s \le \theta^{h-1}R$. In particular, $\theta^{h+k}R < s \le \theta^{h+k-1}R$ and $\theta^{h+1}R < r \le \theta^{h-1}R$. Therefore

$$\begin{aligned}\fint_{B_r(x)} |\mathbf{A}(\nabla\mathbf{u})-\langle \mathbf{A}(\nabla\mathbf{u})\rangle_{B_s(x)}|\,\mathrm{d}y &\le \frac{c}{\theta^n}\fint_{B_{r_s}(x)} |\mathbf{A}(\nabla\mathbf{u})-\langle \mathbf{A}(\nabla\mathbf{u})\rangle_{B_s(x)}|\,\mathrm{d}y\\ &\le \frac{c}{\theta^n}\fint_{B_{r_s}(x)} |\mathbf{A}(\nabla\mathbf{u})-\langle \mathbf{A}(\nabla\mathbf{u})\rangle_{B_{r_s}(x)}|\,\mathrm{d}y + \frac{c}{\theta^n}|\langle \mathbf{A}(\nabla\mathbf{u})\rangle_{B_{r_s}(x)}-\langle \mathbf{A}(\nabla\mathbf{u})\rangle_{B_s(x)}|\\ &\le \frac{c}{\theta^{2n}}\fint_{B_{\theta^{h-1}R}(x)} |\mathbf{A}(\nabla\mathbf{u})-\langle \mathbf{A}(\nabla\mathbf{u})\rangle_{B_{\theta^{h-1}R}(x)}|\,\mathrm{d}y + \frac{c}{\theta^n}|\langle \mathbf{A}(\nabla\mathbf{u})\rangle_{B_{r_s}(x)}-\langle \mathbf{A}(\nabla\mathbf{u})\rangle_{B_{\theta^k r_s}(x)}|.\end{aligned} \tag{3.21}$$

The last addend on the right-hand side of (3.21) can be estimated via (3.16) and (3.19), with R replaced by r_s. As a consequence, we obtain that

$$\begin{aligned}\fint_{B_r(x)} |\mathbf{A}(\nabla\mathbf{u})-\langle \mathbf{A}(\nabla\mathbf{u})\rangle_{B_s(x)}|\,\mathrm{d}y &\le c\int_0^{r_s} \bigg(\fint_{B_\varrho(x)} \Big(\frac{|\mathbf{F}-\langle \mathbf{F}\rangle_{B_\varrho(x)}|}{\rho}\Big)^{p'}\,\mathrm{d}y\bigg)^{\frac{1}{p'}}\,d\varrho\\ &+ \bigg(\fint_{B_{\theta^{h-1}R}(x)} |\mathbf{A}(\nabla\mathbf{u})-\langle \mathbf{A}(\nabla\mathbf{u})\rangle_{B_{\theta^{h-1}R}(x)}|^{\min\{p',2\}}\,\mathrm{d}y\bigg)^{\frac{1}{\min\{p',2\}}}\\ &+ \bigg(\fint_{B_{r_s}(x)} |\mathbf{A}(\nabla\mathbf{u})-\langle \mathbf{A}(\nabla\mathbf{u})\rangle_{B_{r_s}(x)}|^{\min\{p',2\}}\,\mathrm{d}y\bigg)^{\frac{1}{\min\{p',2\}}}\end{aligned} \tag{3.22}$$

$$
\begin{aligned}
&\leq c\int_0^{r_s}\bigg(\fint_{B_\varrho(x)}\Big(\frac{|\mathbf{F}-\langle\mathbf{F}\rangle_{B_\varrho(x)}|}{\rho}\Big)^{p'}\,\mathrm{d}y\bigg)^{\frac{1}{p'}}\,d\varrho.\\
&+\bigg(\fint_{B_{\theta^{h-1}R}(x)}|\mathbf{A}(\nabla\mathbf{u})-\langle\mathbf{A}(\nabla\mathbf{u})\rangle_{B_{\theta^{h-1}R}(x)}|^{\min\{p',2\}}\,\mathrm{d}y\bigg)^{\frac{1}{\min\{p',2\}}}\\
&+\bigg(\frac{c}{\theta^n}\fint_{B_{\theta^{h-1}R}(x)}|\mathbf{A}(\nabla\mathbf{u})-\langle\mathbf{A}(\nabla\mathbf{u})\rangle_{B_{\theta^{h-1}R}(x)}|^{\min\{p',2\}}\,\mathrm{d}y\bigg)^{\frac{1}{\min\{p',2\}}}.
\end{aligned}
$$

Since $\theta^{h-1}R\leq\frac{r}{\theta}$ and $r_s\leq\frac{r}{\theta}$, equations (3.22) and (3.20) ensure that, given any $\varepsilon>0$ there exists $r_\varepsilon>0$ such that, if $0<s<r<r_\varepsilon$, then

$$
\Big|\fint_{B_s(x)}\mathbf{A}(\nabla\mathbf{u})\,\mathrm{d}y-\fint_{B_r(x)}\mathbf{A}(\nabla\mathbf{u})\,\mathrm{d}y\Big|\leq\fint_{B_r(x)}|\mathbf{A}(\nabla\mathbf{u})-\langle\mathbf{A}(\nabla\mathbf{u})\rangle_{B_s(x)}|\,\mathrm{d}y<\varepsilon. \tag{3.23}
$$

This shows that the function $r\mapsto\fint_{B_r(x)}\mathbf{A}(\nabla\mathbf{u})\,\mathrm{d}y$ satisfies the Cauchy property, whence there exists its limit as $r\to0^+$, and is finite. On denoting by $\mathbf{A}(\nabla\mathbf{u}(x))$ such limit, we infer from (3.23) that

$$
\lim_{r\to0}\fint_{B_r(x)}|\mathbf{A}(\nabla\mathbf{u}(y))-\mathbf{A}(\nabla\mathbf{u}(x))|dy=0, \tag{3.24}
$$

and hence x is a Lebesgue point of $\mathbf{A}(\nabla\mathbf{u})$.
On making use of inequalities (3.16) and (3.19), and of Hölder's inequality, and passing to the limit as $k\to\infty$, one can show that

$$
\begin{aligned}
\Big|\mathbf{A}(\nabla\mathbf{u}(x))-\fint_{B_R(x)}\mathbf{A}(\nabla\mathbf{u})\,\mathrm{d}y\Big|&\leq c\int_0^R\bigg(\fint_{B_\varrho(x)}\Big(\frac{|\mathbf{F}-\langle\mathbf{F}\rangle_{B_\varrho(x)}|}{\rho}\Big)^{p'}\,\mathrm{d}y\bigg)^{\frac{1}{p'}}\,d\varrho\\
&+c\bigg(\fint_{B_R(x)}|\mathbf{A}(\nabla\mathbf{u})-\langle\mathbf{A}(\nabla\mathbf{u})\rangle_{B_R(x)}|^{\min\{p',2\}}\,\mathrm{d}y\bigg)^{\frac{1}{\min\{p',2\}}}
\end{aligned} \tag{3.25}
$$

for every $x\in\Omega$ such that $B_R(x)\subset\Omega$ and (3.24) holds. An application of inequality (3.25) with R replaced with $\frac{R}{2}$, and of Corollary 2.5 with $\mathbf{P}=0$ and $B=B_{\frac{R}{2}}(x)$, tell us that

(3.26)

$$
\begin{aligned}
&|\mathbf{A}(\nabla\mathbf{u}(x))|\\
&\leq c\bigg(\fint_{B_{\frac{R}{2}}(x)}|\mathbf{A}(\nabla\mathbf{u})|^{\min\{p',2\}}\,\mathrm{d}y\bigg)^{\frac{1}{\min\{p',2\}}}+c\int_0^R\bigg(\fint_{B_\varrho(x)}\Big(\frac{|\mathbf{F}-\langle\mathbf{F}\rangle_{B_\varrho(x)}|}{\rho}\Big)^{p'}\,\mathrm{d}y\bigg)^{\frac{1}{p'}}\,d\varrho\\
&\leq c\fint_{B_R(x)}|\mathbf{A}(\nabla\mathbf{u})|\,\mathrm{d}y+c\int_0^R\bigg(\fint_{B_\varrho(x)}\Big(\frac{|\mathbf{F}-\langle\mathbf{F}\rangle_{B_\varrho(x)}|}{\rho}\Big)^{p'}\,\mathrm{d}y\bigg)^{\frac{1}{p'}}\,d\varrho\\
&\quad+c\bigg(\fint_{B_R(x)}|\mathbf{F}-\langle\mathbf{F}\rangle_{B_R(x)}|^{p'}\,\mathrm{d}y\bigg)^{\frac{1}{p'}}
\end{aligned}
$$

for a.e. $x \in \Omega$ such that $B_R(x) \subset \Omega$. Since $|\mathbf{A}(\nabla \mathbf{u}(x))| = |\nabla \mathbf{u}(x)|^{p-1}$, and the last term in (3.26) can be estimated (up to a multiplicative constant) by the last but one, inequality (1.17) follows. □

4. Estimates in norms depending on the size of functions

Here, we are concerned with gradient estimates for solutions to (1.1) involving rearrangement-invariant norms. Loosely speaking, a rearrangement-invariant norm is a norm on the space of measurable functions which only depends on their "size", or, more precisely, on the measure of their level sets. In particular, a "reduction principle" is established via Theorem 1.1, which turns the problem of bounds for this kind of norms of the gradient in terms of norms of the same kind of the right-hand side into a couple of one-dimensional Hardy type inequalities. This general principle is then specialized to various customary classes of norms.
For ease of presentation, here we limit our discussion to local solutions to (1.1) when $\Omega = \mathbb{R}^n$. Analogous results hold, however, in arbitrary open sets Ω.

Let $f : \Omega \to \mathbb{R}$ be a measurable function. We denote by $f^* : [0, \infty) \to [0, \infty]$ the decreasing rearrangement of f, defined as

$$f^*(s) = \inf\{t \geq 0 : |\{x \in \Omega : |f(x)| > t\}| \leq s\} \qquad \text{for } s \geq 0.$$

Moreover, we set

$$f^{**}(s) = \frac{1}{s}\int_0^s f^*(r)dr \quad \text{for } s > 0.$$

In other words, f^* is the (unique) non increasing, right-continuous function in $[0, \infty)$ equimeasurable with f, and f^{**} is a maximal function of f^*. Observe that

$$f^*(s) \leq f^{**}(s) \quad \text{for } s > 0. \tag{4.1}$$

We say that $\|\cdot\|_{X(0,\infty)}$ is a rearrangement-invariant functional defined on the set of real-valued measurable functions on $(0, \infty)$ if it takes values into $[0, \infty]$ and satisfies the following properties:
(i) $\|\varphi\|_{X(0,\infty)} \leq \|\psi\|_{X(0,\infty)}$ whenever $0 \leq \varphi \leq \psi$ a.e. in $(0, \infty)$,
(ii) $\|\varphi\|_{X(0,\infty)} = \|\psi\|_{X(0,\infty)}$ whenever $\varphi^* = \psi^*$.

Let $m \in \mathbb{N}$. We say that $\|\cdot\|_{X(\Omega)}$ is a rearrangement-invariant functional on the set of measurable functions from Ω into $\mathbb{R}^m$ if there exists a rearrangement-invariant functional $\|\cdot\|_{X(0,\infty)}$ on $(0, \infty)$ such that

$$\|\mathbf{f}\|_{X(\Omega)} = \||\mathbf{f}|^*\|_{X(0,\infty)} \tag{4.2}$$

for every such function $\mathbf{f}$.

A rearrangement-invariant functional $\|\cdot\|_{X(0,\infty)}$ is called a rearrangement-invariant function norm, if, for every φ, ψ and $\{\varphi_j\}_{j\in\mathbb{N}}$, and every $\lambda \geq 0$, the following additional properties hold:

(P1) $\|\varphi\|_{X(0,\infty)} = 0$ if and only if $\varphi = 0$; $\|\lambda\varphi\|_{X(0,\infty)} = \lambda\|\varphi\|_{X(0,\infty)}$;
$\|\varphi + \psi\|_{X(0,\infty)} \leq \|\varphi\|_{X(0,\infty)} + \|\psi\|_{X(0,\infty)}$;
(P2) $\varphi_j \nearrow \varphi$ a.e. implies $\|\varphi_j\|_{X(0,\infty)} \nearrow \|\varphi\|_{X(0,\infty)}$;
(P3) $\|\chi_E\|_{X(0,\infty)} < \infty$ if $E \subset (0, \infty)$ with $|E| < \infty$;
(P4) $\int_E \varphi(s)\,ds \leq C\|\varphi\|_{X(0,\infty)}$ if $E \subset (0, \infty)$ with $|E| < \infty$, for some constant C independent of φ.

If $\|\cdot\|_{X(\Omega)}$ is a rearrangement-invariant functional built upon a rearrangement-invariant function norm $\|\cdot\|_{X(0,\infty)}$, we denote by $X(\Omega)$ the collection of all measurable functions $\mathbf{f} : \Omega \to \mathbb{R}^m$ such that $\|\mathbf{f}\|_{X(\Omega)} < \infty$. The functional $\|\cdot\|_{X(\Omega)}$ defines a norm on $X(\Omega)$, and the latter is a Banach space endowed with this norm, which is called a rearrangement-invariant space. The function norm $\|\cdot\|_{X(0,\infty)}$ is called a representation norm of $\|\cdot\|_{X(\Omega)}$. In particular, $X(0, \infty)$ is a rearrangement-invariant space itself.

Given a measurable subset G of Ω, we denote by χ_G the characteristic function of G, and define

$$\|\mathbf{f}\|_{X(G)} = \|\mathbf{f}\chi_G\|_{X(\Omega)}$$

for any measurable function $\mathbf{f} : \Omega \to \mathbb{R}^m$.
Moreover, we denote by $X_{\rm loc}(\Omega)$ the space of measurable functions $\mathbf{f}$ such that $\|\mathbf{f}\|_{X(G)} < \infty$ for every compact set $G \subset \Omega$.
Given a rearrangement-invariant functional $\|\cdot\|_{X(\Omega)}$ and any number $q \in (0,\infty)$, the functional $\|\mathbf{f}\|_{X^q(\Omega)}$, defined as

$$\|\mathbf{f}\|_{X^q(\Omega)} = \||\mathbf{f}|^q\|_{X(\Omega)}^{\frac{1}{q}} \tag{4.3}$$

for any measurable function $\mathbf{f} : \Omega \to \mathbb{R}^m$, is also a rearrangement-invariant functional. Moreover, if $\|\cdot\|_{X(\Omega)}$ is a rearrangement-invariant norm, and $q \geq 1$, then $\|\cdot\|_{X^q(\Omega)}$ is a rearrangement-invariant norm as well [48].

Theorem 4.1. [Reduction to one-dimensional inequalities] *Let $\|\cdot\|_{X(\mathbb{R}^n)}$ and $\|\cdot\|_{Y(\mathbb{R}^n)}$ be rearrangement-invariant functionals. Assume that there exists a constant c such that*

$$\left\|\frac{1}{s}\int_0^s \varphi(r)dr\right\|_{X^{\frac{1}{p'}}(0,\infty)} \leq c\|c\varphi\|_{X^{\frac{1}{p'}}(0,\infty)} \tag{4.4}$$

and

$$\left\|\int_s^\infty \varphi(r)\frac{dr}{r}\right\|_{Y(0,\infty)} \leq c\|c\varphi\|_{X(0,\infty)} \tag{4.5}$$

for every nonnegative function $\varphi \in X(0,\infty)$. Then there exists a constant $c' = c'(n,N,p,c)$ such that

$$\||\nabla\mathbf{u}|^{p-1}\|_{Y(\mathbb{R}^n)} \leq c'\|c'\mathbf{F}\|_{X(\mathbb{R}^n)} \tag{4.6}$$

for every local weak solution $\mathbf{u} \in V^{1,p}(\mathbb{R}^n)$ to system (1.1) *with $\Omega = \mathbb{R}^n$.*

Remark 4.2. Let us briefly comment on assumptions (4.4) and (4.5) in Theorem 4.1. The first one amounts to requiring that the functional $\|\cdot\|_{X(\mathbb{R}^n)}$ is stronger, in a qualified sense, than $\|\cdot\|_{L^{p'}(\mathbb{R}^n)}$. As recalled in Section 1, this is a borderline norm for estimates for the gradient to weak solutions to system (1.1). Assumption (4.5) concerns, instead, the opposite endpoint in the scale of admissible gradient estimates, which fail if the norm $\|\cdot\|_{X(\mathbb{R}^n)}$ is too strong, namely if the latter is "too close" to $\|\cdot\|_{L^\infty(\mathbb{R}^n)}$. In particular, if this is the case, (4.5) tells us how much the $\|\cdot\|_{Y(\mathbb{R}^n)}$ norm of $|\nabla\mathbf{u}|^{p-1}$ has to be weaker than the $\|\cdot\|_{X(\mathbb{R}^n)}$ norm of $|\mathbf{F}|$, for inequality (4.6) to hold.

The proof of Theorem 4.1 requires the following proposition.

Proposition 4.3. *Let $\|\cdot\|_{X(\mathbb{R}^n)}$ and $\|\cdot\|_{Y(\mathbb{R}^n)}$ be rearrangement-invariant functionals. Assume that inequality* (4.5) *holds. Then*

$$\|\psi^*\|_{Y(0,\infty)} \leq c\|c(\psi^{**} - \psi^*)\|_{X(0,\infty)} \tag{4.7}$$

for every nonnegative measurable function ψ in $(0,\infty)$ such that $\psi < \infty$ a.e. and $\lim_{s\to\infty}\psi^*(s) = 0$. Here, c denotes the constant appearing in (4.5).

Proof. Given any function ψ as in the statement, there exists a nonnegative Radon measure ν on $(0,\infty)$ such that

$$\psi^*(s) = \int_s^\infty d\nu(r) \quad \text{for } s > 0. \tag{4.8}$$

Thus, by Fubini's Theorem,

$$\psi^{**}(s) - \psi^*(s) = \frac{1}{s}\int_0^s r\,d\nu(r) \quad \text{for } s > 0. \tag{4.9}$$

By (4.8) and (4.9), inequality (4.7) will follow if we show that

$$\left\| \int_s^\infty d\nu(r) \right\|_{Y(0,\infty)} \le c \left\| \frac{c}{s} \int_0^s r\,d\nu(r) \right\|_{X(0,\infty)} \tag{4.10}$$

for every nonnegative Radon measure ν in $(0,\infty)$ such that $\int_s^\infty d\nu(r) < \infty$ for $s > 0$, or, equivalently, that

$$\left\| \int_s^\infty \frac{d\mu(r)}{r} \right\|_{Y(0,\infty)} \le c \left\| \frac{c}{s} \int_0^s d\mu(r) \right\|_{X(0,\infty)} \tag{4.11}$$

for every nonnegative Radon measure μ on $(0,\infty)$ such that

$$\int_s^\infty \frac{d\mu(r)}{r} < \infty \quad \text{for } s > 0. \tag{4.12}$$

In order to prove (4.11), set

$$\mu(0,s) = \int_0^s d\mu(r) \quad \text{for } s > 0,$$

and observe that integration by parts yields

$$\int_s^t \frac{d\mu(r)}{r} = \frac{\mu(0,t)}{t} - \frac{\mu(0,s)}{s} + \int_s^t \frac{\mu(0,r)}{r^2} dr \quad \text{if } 0 < s < t. \tag{4.13}$$

By (4.12) and (4.13), $\lim_{t\to\infty} \frac{\mu(0,t)}{t}$ exists, is finite, and

$$\int_s^\infty \frac{\mu(0,r)}{r^2} dr < \infty. \tag{4.14}$$

From equations (4.12)–(4.14) one can deduce that $\lim_{t\to\infty} \frac{\mu(0,t)}{t} = 0$. Thus, passing to the limit as $t \to \infty$ in (4.13) tells us that

$$\int_s^\infty \frac{d\mu(r)}{r} \le \int_s^\infty \frac{\mu(0,r)}{r^2} dr \quad \text{for } s > 0. \tag{4.15}$$

Hence, inequality (4.11) is a consequence of the fact that

$$\left\| \int_s^\infty \frac{g(r)}{r^2} dr \right\|_{Y(0,\infty)} \le c \left\| c \frac{g(s)}{s} \right\|_{X(0,\infty)} \tag{4.16}$$

for every nonnegative measurable function g in $(0,\infty)$, which, in turn, follows from (4.5). □

Upper and lower estimates, in rearrangement form, for sharp maximal function operators, also play a key role in the proof of Theorem 4.1. Given $q \ge 1$ and any function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ such that $\mathbf{f} \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$, the classical maximal function $M^q\mathbf{f} : \mathbb{R}^n \to [0,\infty]$ is defined as

$$M^q\mathbf{f}(x) = \sup_{B \ni x} \left(\fint_B |\mathbf{f}|^q \, dy \right)^{\frac{1}{q}} \quad \text{for } x \in \mathbb{R}^n.$$

Clearly,

$$M^{\sharp,q}\mathbf{f}(x) \le 2M^q\mathbf{f}(x) \quad \text{for } x \in \mathbb{R}^n \tag{4.17}$$

for any such function $\mathbf{f}$. Furthermore, Riesz' inequality [6, Theorem 3.8, Chapter 3] tells us that there exists a constant $C = C(n)$ such that

$$(M^q\mathbf{f})^*(s) \le C(|\mathbf{f}|^q)^{**}(s)^{\frac{1}{q}} \quad \text{for } s > 0, \tag{4.18}$$

for $\mathbf{f} \in L^q_{\text{loc}}(\mathbb{R}^n)$. Coupling inequalities (4.17) and (4.18) tells us that

$$(M^{\sharp,q}\mathbf{f})^*(s) \le 2C(|\mathbf{f}|^q)^{**}(s)^{\frac{1}{q}} \quad \text{for } s > 0, \tag{4.19}$$

for $\mathbf{f} \in L^q_{\text{loc}}(\mathbb{R}^n)$.
On the other hand, as a consequence of [6, Theorem 7.3, Chapter 5], one can show that

$$f^{**}(s) - f^*(s) \le C(M^\sharp f)^*(s) \quad \text{for } s > 0, \tag{4.20}$$

for every locally integrable function $f : \mathbb{R}^n \to \mathbb{R}$.

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. By inequality (4.19),

$$(M^{\sharp,p'}(\mathbf{F}))^*(s) \le C\Big[\big(|\mathbf{F}|^{p'}\big)^{**}(s)\Big]^{\frac{1}{p'}} \quad \text{for } s > 0 \tag{4.21}$$

for some constant $C = C(n)$. Next, set $\mathbf{u} = (u^1, \cdots, u^N)$. Owing to inequality (4.20), applied with $f = |\nabla \mathbf{u}|^{p-2}u^j_{x_i}$, for $i = 1, \dots, n$, $j = 1, \dots, N$, one has that

$$\begin{aligned}\big(|\nabla \mathbf{u}|^{p-2}u^j_{x_i}\big)^{**}(s) - \big(|\nabla \mathbf{u}|^{p-2}u^j_{x_i}\big)^*(s) &\le C\big(M^\sharp(|\nabla \mathbf{u}|^{p-2}u^j_{x_i})\big)^*(s)\\ &\le C\big(M^\sharp(|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u})\big)^*(s) \quad \text{for } s > 0.\end{aligned} \tag{4.22}$$

Inequality (1.11) implies that

$$M^\sharp(|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u})^*(s) \le CM^{\sharp,p'}(\mathbf{F})^*(s) \quad \text{for } s > 0. \tag{4.23}$$

Combining inequalities (4.21)–(4.23) yields

$$\big(|\nabla \mathbf{u}|^{p-2}u^j_{x_i}\big)^{**}(s) - \big(|\nabla \mathbf{u}|^{p-2}u^j_{x_i}\big)^*(s) \le C\Big[\big(|\mathbf{F}|^{p'}\big)^{**}(s)\Big]^{\frac{1}{p'}} \quad \text{for } s > 0 \tag{4.24}$$

for some constant $C = C(n, N, p)$. Hence, if $\|\cdot\|_{X(0,\infty)}$ is any rearrangement-invariant functional,

$$\|\big(|\nabla \mathbf{u}|^{p-2}u^j_{x_i}\big)^{**} - \big(|\nabla \mathbf{u}|^{p-2}u^j_{x_i}\big)^*\|_{X(0,\infty)} \le C\left\|\Big[\big(|\mathbf{F}|^{p'}\big)^{**}\Big]^{\frac{1}{p'}}\right\|_{X(0,\infty)}. \tag{4.25}$$

Inequality (4.5) implies, via Proposition 4.3, that
(4.26)
$$\||\nabla \mathbf{u}|^{p-2}u^j_{x_i}\|_{Y(\mathbb{R}^n)} = \|\big(|\nabla \mathbf{u}|^{p-2}u^j_{x_i}\big)^*\|_{Y(0,\infty)} \le C\|C(\big(|\nabla \mathbf{u}|^{p-2}u^j_{x_i}\big)^{**} - \big(|\nabla \mathbf{u}|^{p-2}u^j_{x_i}\big)^*)\|_{X(0,\infty)}$$

for some constant C. Moreover, owing to inequality (4.4),

$$\left\|C\Big[\big(|\mathbf{F}|^{p'}\big)^{**}\Big]^{\frac{1}{p'}}\right\|_{X(0,\infty)} \le c\big\|c^{\frac{1}{p'}}C|\mathbf{F}|^*\big\|_{X(0,\infty)} = c\|cC\mathbf{F}\|_{X(\mathbb{R}^n)}. \tag{4.27}$$

Inequality (4.6) follows from (4.25)–(4.27). □

As a first application of Theorem 4.1, one can easily recover the by now standard gradient estimates in Lebesgue spaces. The fact that the relevant Lebesgue spaces satisfy assumptions (4.4) and (4.5) is a consequence of classical Hardy inequalities.

Proposition 4.4. [Lebesgue spaces] *Let $q \in (p', \infty)$. Then*

$$\|\nabla \mathbf{u}\|_{L^{q(p-1)}(\mathbb{R}^n)} \le C'\|\mathbf{F}\|^{\frac{1}{p-1}}_{L^q(\mathbb{R}^n)} \tag{4.28}$$

for every local weak solution $\mathbf{u} \in V^{1,p}(\mathbb{R}^n)$ to system (1.1) with $\Omega = \mathbb{R}^n$.

The Lorentz spaces provide a refinement of the Lebesgue spaces. If either $q \in (1,\infty)$ and $r \in [1,\infty]$, or $q = r = 1$, or $q = r = \infty$, the rearrangement-invariant functional $\|\cdot\|_{L^{q,r}(0,\infty)}$, defined as

$$\|\varphi\|_{L^{q,r}(0,\infty)} = \left\| s^{\frac{1}{q}-\frac{1}{r}} \varphi^*(s) \right\|_{L^r(0,\infty)} \tag{4.29}$$

is a rearrangement-invariant norm for any measurable function φ in $(0,\infty)$. The corresponding rearrangement-invariant space $L^{q,r}(\Omega)$ is called a Lorentz space. The Lebesgue spaces are special instances of Lorentz spaces, inasmuch as

$$L^{q,q}(\Omega) = L^q(\Omega)$$

for every $q \in [1,\infty]$.

Proposition 4.5. [Lorentz spaces] *Let $q \in (p',\infty)$ and $r \in [1,\infty]$. Then*

$$\|\nabla \mathbf{u}\|_{L^{q(p-1),r(p-1)}(\mathbb{R}^n)} \le C' \|\mathbf{F}\|^{\frac{1}{p-1}}_{L^{q,r}(\mathbb{R}^n)} \tag{4.30}$$

for every local weak solution $\mathbf{u} \in V^{1,p}(\mathbb{R}^n)$ to system (1.1) *with $\Omega = \mathbb{R}^n$.*

Proof. Let $X(\mathbb{R}^n) = Y(\mathbb{R}^n) = L^{q,r}(\mathbb{R}^n)$ in Theorem 4.1. Then, $X^{\frac{1}{p'}}(\mathbb{R}^n) = L^{\frac{q}{p'},\frac{r}{p'}}(\mathbb{R}^n)$, and condition (4.4) reads

$$\left(\int_0^\infty \left[\left(\frac{1}{(\cdot)} \int_0^{(\cdot)} \varphi(\tau) d\tau \right)^*(s) \right]^{\frac{r}{p'}} s^{\frac{r}{q}-1} \, ds \right)^{\frac{p'}{r}} \le C \left(\int_0^\infty \varphi^*(s) s^{\frac{r}{q}-1} \, ds \right)^{\frac{p'}{r}} \tag{4.31}$$

for every measurable function $\varphi : (0,\infty) \to [0,\infty)$. The Hardy-Littlewood inequality for rearrangements [6, Theorem 2.2, Chapter 2] ensures that $\frac{1}{s}\int_0^s \varphi(\tau)d\tau \le \frac{1}{s}\int_0^s \varphi^*(\tau)d\tau = \varphi^{**}(s)$ for $s > 0$. Hence,

$$\left(\frac{1}{(\cdot)} \int_0^{(\cdot)} \varphi(\tau) d\tau \right)^*(s) \le \varphi^{**}(s) \text{ for } s > 0.$$

Thus, inequality (4.31) follows from the inequality

$$\left(\int_0^\infty \varphi^{**}(s)^{\frac{r}{p'}} s^{\frac{r}{q}-1} \, ds \right)^{\frac{p'}{r}} \le C \left(\int_0^\infty \varphi^*(s) s^{\frac{r}{q}-1} \, ds \right)^{\frac{p'}{r}} \tag{4.32}$$

for every measurable function $\varphi : (0,\infty) \to [0,\infty)$, which holds, for instance, as a special case of [12, Theorem 4.1].
As for condition (4.5), we have that

$$\begin{aligned} \sup_{\varphi \ge 0} \frac{\| \int_s^\infty \varphi(\tau) \frac{d\tau}{\tau} \|_{L^{q,r}(0,\infty)}}{\|\varphi\|_{L^{q,r}(0,\infty)}} &\approx \sup_{\varphi \ge 0} \sup_{\psi \ge 0} \frac{\int_0^\infty \psi(s) \int_s^\infty \varphi(\tau) \frac{d\tau}{\tau} \, ds}{\|\psi\|_{L^{q',r'}(0,\infty)} \|\varphi\|_{L^{q,r}(0,\infty)}} \\ = \sup_{\psi \ge 0} \sup_{\varphi \ge 0} \frac{\int_0^\infty \frac{\varphi(\tau)}{\tau} \int_0^\tau \psi(s) \, ds \, d\tau}{\|\psi\|_{L^{q',r'}(0,\infty)} \|\varphi\|_{L^{q,r}(0,\infty)}} &\approx \sup_{\psi \ge 0} \frac{\| \frac{1}{\tau} \int_0^\tau \psi(s) \, ds \|_{L^{q',r'}(0,\infty)}}{\|\psi\|_{L^{q',r'}(0,\infty)}} < \infty \,, \end{aligned}$$

where the equivalences hold due to Hölder type inequalities in Lorentz spaces, the equality to Fubini's theorem, and the last inequality to weighted Hardy type inequalities [50, Theorem 1.3.2/1].
The conclusion now follows from Theorem 4.1. □

Let now focus on gradient estimates in Orlicz spaces, an extension of the Lebesgue spaces in a different direction. The notion of Orlicz space involves that of Young functions. A Young function is a (non-trivial) convex function $\Phi : [0,\infty) \to [0,\infty]$, vanishing at 0. Its Young conjugate $\Phi^\sim$ is again a Young function, and is defined as

$$\Phi^\sim(t) = \sup\{ts - \Phi(s) : s \ge 0\} \quad \text{for } t \ge 0.$$

The Orlicz space $L^{\Phi}(\Omega)$ is the rearrangement-invariant space associated with the Luxemburg rearrangement-invariant norm $\|\cdot\|_{L^{\Phi}(\Omega)}$ given by

$$\|\varphi\|_{L^{\Phi}(0,\infty)} = \inf\left\{\lambda > 0 : \int_0^\infty \Phi\left(\frac{|\varphi(s)|}{\lambda}\right) ds \leq 1\right\} \tag{4.33}$$

for every measurable function φ in $(0,\infty)$.

Proposition 4.6. [Orlicz spaces] *Let Φ be a Young function such that*

$$\operatorname*{ess\,inf}_{t\in(0,\infty)} \frac{t\Phi'(t)}{\Phi(t)} > p', \tag{4.34}$$

and

$$\int_0 \frac{\Phi^{\sim}(r)}{r^2} dr < \infty. \tag{4.35}$$

Let Ψ be the Young function defined by

$$\Psi(t^{\frac{1}{p-1}}) = \left(t\int_0^t \frac{\Phi^{\sim}(r)}{r^2} dr\right)^{\sim} \quad \text{for } t \geq 0. \tag{4.36}$$

Then there exists a constant $C = C(n, N, p, \Phi)$ such that

$$\|\nabla \mathbf{u}\|_{L^{\Psi}(\mathbb{R}^n)} \leq C\|\mathbf{F}\|_{L^{\Phi}(\mathbb{R}^n)}^{\frac{1}{p-1}}, \tag{4.37}$$

and

$$\int_{\mathbb{R}^n} \Psi\big(|\nabla \mathbf{u}|\big)\, dx \leq \int_{\mathbb{R}^n} \Phi(C^{p-1}|\mathbf{F}|)\, dx. \tag{4.38}$$

for every local weak solution $\mathbf{u} \in V^{1,p}(\mathbb{R}^n)$ to system (1.1), with $\Omega = \mathbb{R}^n$.

Proof. Assumption (4.34) ensures that there exists $\varepsilon > 0$ such that the function $t \mapsto \frac{\Phi(t^{\frac{1}{p'}})}{t^{1+\varepsilon}}$ is increasing. Thus,

$$\int_0^t \frac{\Phi(s^{\frac{1}{p'}})}{s^2} ds = \int_0^t \frac{\Phi(s^{\frac{1}{p'}})}{s^{1+\varepsilon}} t^{\varepsilon-1} ds \leq \frac{\Phi(t^{\frac{1}{p'}})}{t^{1+\varepsilon}} \frac{t^\varepsilon}{\varepsilon} = \frac{\Phi(t^{\frac{1}{p'}})}{\varepsilon\, t}. \tag{4.39}$$

Hence, by [14, 38, 30], there exists a constant C, depending only on the left-hand side of (4.34), such that

$$\left\|\left(\frac{1}{s}\int_0^s \varphi(r)\, dr\right)^{\frac{1}{p'}}\right\|_{L^{\Phi}(0,\infty)} \leq C\big\|\varphi(s)^{\frac{1}{p'}}\|_{L^{\Phi}(0,\infty)} \tag{4.40}$$

for every measurable function $\varphi : (0,\infty) \to [0,\infty)$.
On the other hand, under assumption (4.35), [14, Lemma 1, Part (ii)] ensures that there exists an absolute constant C such that

$$\left\|\int_s^\infty \varphi(r)\, \frac{dr}{r}\right\|_{L^{\Theta}(0,\infty)} \leq C\big\|\varphi(s)\|_{L^{\Phi}(0,\infty)} \tag{4.41}$$

for every measurable function $\varphi : (0,\infty) \to [0,\infty)$, where Θ is the Young function given by

$$\Theta(t) = \left(t\int_0^t \frac{\Phi^{\sim}(r)}{r^2} dr\right)^{\sim} \quad \text{for } t \geq 0.$$

Owing to (4.40) and (4.41), Theorem 4.1 implies that

$$\||\nabla \mathbf{u}|^{p-1}\|_{L^{\Theta}(\mathbb{R}^n)} \leq C\|\mathbf{F}\|_{L^{\Phi}(\mathbb{R}^n)}. \tag{4.42}$$

Hence, inequality (4.37) follows.
Inequality (4.38) can be derived on applying (4.42) with Φ replaced with $\frac{\Phi}{\int_{\mathbb{R}^n} \Phi(C|\mathbf{F}|)\, dx}$. Such

a derivation makes use of the definition of the Luxemburg norm, and of the fact that the constant in (4.42) depends on Φ only through the infimum on the left-hand side of (4.34), and such infimum is invariant under replacements of Φ with $k\Phi$ for every positive constant k. $\square$

The content of the following result is a special case of Proposition 4.6. In the statement, $\exp_q L^\gamma(\mathbb{R}^n)$ denotes the Orlicz space associated with a Young function which is equivalent to e^{t^γ} for large t, and to t^q for small t.

Proposition 4.7. [Exponential spaces] *Let* $\gamma > 0$*, and* $q > p'$*. Then*

$$\|\nabla \mathbf{u}\|_{\exp_{q(p-1)} L^{\frac{\gamma(p-1)}{\gamma+1}}(\mathbb{R}^n)} \leq C \|\mathbf{F}\|_{\exp_q L^\gamma(\mathbb{R}^n)}^{\frac{1}{p-1}} \tag{4.43}$$

for every local weak solution $\mathbf{u} \in V^{1,p}(\mathbb{R}^n)$ *to system* (1.1) *with* $\Omega = \mathbb{R}^n$*.*

The last result of this section deals with a borderline case of Theorem 4.1. In the statement, $L_q^\infty(\mathbb{R}^n)$ denotes the Orlicz space built upon a Young function which equals ∞ for large t, and is equivalent to t^q for small t.

Proposition 4.8. [Borderline exponential spaces] *Let* $q > p'$*. Then*

$$\|\nabla \mathbf{u}\|_{\exp_{q(p-1)} L^{p-1}(\mathbb{R}^n)} \leq C \|\mathbf{F}\|_{L_q^\infty(\mathbb{R}^n)}^{\frac{1}{p-1}} \tag{4.44}$$

for every local weak solution $\mathbf{u} \in V^{1,p}(\mathbb{R}^n)$ *to system* (1.1) *with* $\Omega = \mathbb{R}^n$*.*

5. Estimates in norms depending on oscillations of functions

Let $\omega : [0, \infty) \to [0, \infty)$ be a function, let $q \geq 1$, and let $m \in \mathbb{N}$. The Campanato type space $\mathcal{L}^{\omega,q}(\Omega)$ on an open set $\Omega \subset \mathbb{R}^n$, is the space of those $\mathbb{R}^m$-valued functions $\mathbf{f} \in L^q_{\mathrm{loc}}(\Omega)$ for which the semi-norm

$$\|\mathbf{f}\|_{\mathcal{L}^{\omega,q}(\Omega)} = \sup_{B_r \subset \Omega} \frac{1}{\omega(r)} \left(\fint_{B_r} |\mathbf{f} - \langle \mathbf{f} \rangle_{B_r}|^q \, dx \right)^{\frac{1}{q}} \tag{5.1}$$

is finite. The space $\mathcal{L}^{\omega,q}_{\mathrm{loc}}(\Omega)$ is defined accordingly, as the set of all functions $\mathbf{f}$ such that $\|f\|_{\mathcal{L}^{\omega,q}(\Omega')} < \infty$ for every open set $\Omega' \subset\subset \Omega$. When $q = 1$, we shall simply denote $\mathcal{L}^{\omega,q}(\Omega)$ by $\mathcal{L}^{\omega}(\Omega)$, and similarly for local spaces.
In the special case when $\omega(r) = 1$, one has that

$$\mathcal{L}^{1,q}(\Omega) = \mathrm{BMO}(\Omega) \tag{5.2}$$

for every $q \geq 1$ [34]. Another customary instance corresponds to the choice $\omega(r) = r^\beta$, for some $\beta \in (0, 1]$. Indeed, Campanato's representation theorem tells us that, if Ω is regular enough, say a bounded Lipschitz domain, then

$$\mathcal{L}^{r^\beta,q}(\Omega) = C^\beta(\Omega) \tag{5.3}$$

for every $q \geq 1$ [11]. It is easily verified that, if ω is non-decreasing, then

$$C^\omega(\Omega) \subset \mathcal{L}^{\omega,q}(\Omega) \tag{5.4}$$

for $q \geq 1$. Here, $C^\omega(\Omega)$ denotes the space of those functions $\mathbf{f} : \Omega \to \mathbb{R}^m$ such that

$$\sup_{\substack{x, y \in \Omega \\ x \neq y}} \frac{|\mathbf{f}(x) - \mathbf{f}(y)|}{\omega(|x - y|)} < \infty \,,$$

a space of uniformly continuous functions, with modulus of continuity not exceeding ω, if $\lim_{r \to 0^+} \omega(r) = 0$. The reverse inclusion in (5.4) need not hold for an arbitrary ω, as shown, for example, by equation (5.2). Results from [55] ensure that, if ω is non-decreasing and decays at 0 so fast that

$$\int_0 \frac{\omega(r)}{r}\, dr < \infty, \tag{5.5}$$

then

$$\mathcal{L}^{\omega}(\Omega) \subset C^{\varpi}(\Omega) \tag{5.6}$$

for every regular domain Ω, where the function $\varpi : [0,\infty) \to [0,\infty)$ is defined as

$$\varpi(r) = \int_0^r \frac{\omega(\rho)}{\rho}\, d\rho \qquad \text{for } r \geq 0. \tag{5.7}$$

On the other hand, if (5.5) fails, and the function $r \mapsto \frac{\omega(r)}{r}$ is non-increasing near 0, then the space $\mathcal{L}^{\omega}(\Omega)$ is not even contained in $L^{\infty}_{\rm loc}(\Omega)$.

As an immediate consequence of Theorem 1.3, we have the following regularity result for the gradient of solutions to (1.1) in Campanato spaces.

Theorem 5.1. [Campanato spaces] *Let $n \geq 2$, $N \geq 1$ and $p \in (1,\infty)$. Let Ω be an open set in $\mathbb{R}^n$, and let $R > 0$. Then there exist constants $c_1 = c_1(n,N,p) > 0$, $c_2 = c_2(n,N,p,R) > 0$ such that, if the function $\omega(r)r^{-\beta}$ is (almost) decreasing for some $\beta \in \big(0, \min\{1, \frac{2\alpha}{p'}\}\big)$, $\mathbf{F} \in \mathcal{L}^{\omega,p'}_{\rm loc}(\Omega)$, and $B_{2R} \subset\subset \Omega$, then*

$$\||\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}\|_{\mathcal{L}^{\omega}(B_R)} \leq c_1 \|\mathbf{F}\|_{\mathcal{L}^{\omega,p'}(B_{2R})} + c_2 \|\nabla \mathbf{u}\|^{p-1}_{L^p(B_{2R})} \tag{5.8}$$

for every local weak solution $\mathbf{u} \in W^{1,p}_{\rm loc}(\Omega)$ to system (1.1). Here, $\alpha = \alpha(n,N,p)$ denotes the exponent appearing in Theorem 2.6.

Bounds for Hölder norms follow from Theorem 5.1 and equation (5.3).

Corollary 5.2. [Hölder spaces] *Let $n \geq 2$, $N \geq 1$ and $p \in (1,\infty)$. Let Ω be an open set in $\mathbb{R}^n$, and let $B_{2R} \subset\subset \Omega$. Then there exist constants $c_1 = c_1(n,N,p) > 0$, $c_2 = c_2(n,N,p,R) > 0$ such that, if $\beta < \min\{1, \frac{2\alpha}{p'}\}$, and $\mathbf{F} \in C^{\beta}_{\rm loc}(\Omega)$, then*

$$\||\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}\|_{C^{\beta}(B_R)} \leq c_1 \|\mathbf{F}\|_{C^{\beta}(B_{2R})} + c_2 \|\nabla \mathbf{u}\|^{p-1}_{L^p(B_{2R})} \tag{5.9}$$

for every local weak solution $\mathbf{u} \in W^{1,p}_{\rm loc}(\Omega)$ to system (1.1). Here, $\alpha = \alpha(n,N,p)$ denotes the exponent appearing in Theorem 2.6.

Corollary 5.2 is a special case of the next result, which can be deduced from Theorem 5.1 and embedding (5.6), and deals with estimates for more general moduli of continuity.

Corollary 5.3. [Spaces of uniformly continuous functions] *Let $n \geq 2$, $N \geq 1$ and $p \in (1,\infty)$. Let Ω be an open set in $\mathbb{R}^n$, and let $R > 0$. Then there exist constants $c_1 = c_1(n,N,p)$ and $c_2 = c_2(n,N,p,R)$ such that, if the function $\omega(r)r^{-\beta}$ is (almost) decreasing for some $\beta \in \big(0, \min\{1, \frac{2\alpha}{p'}\}\big)$, fulfills condition (5.5), $\mathbf{F} \in C^{\omega}_{\rm loc}(\Omega)$, and $B_{2R} \subset\subset \Omega$, then*

$$\||\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}\|_{C^{\varpi}(B_R)} \leq c_1 \|\mathbf{F}\|_{C^{\omega}(B_{2R})} + c_2 \|\nabla \mathbf{u}\|^{p-1}_{L^p(B_{2R})} \tag{5.10}$$

for every local weak solution $\mathbf{u} \in W^{1,p}_{\rm loc}(\Omega)$ to system (1.1). Here, $\alpha = \alpha(n,N,p)$ denotes the exponent appearing in Theorem 2.6.

Remark 5.4. Note that the norm $\|\mathbf{F}\|_{C^{\omega}(B_{2R})}$ on the right-hand side of (5.10) can be replaced with the (possibly slightly weaker) norm $\|\mathbf{F}\|_{\mathcal{L}^{\omega,p'}(B_{2R})}$.

Assumption (5.5) is minimal for $\nabla \mathbf{u}$ to be continuous, or even merely locally bounded, for every local solution $\mathbf{u}$ and every $\mathbf{F} \in C^{\omega}_{\rm loc}(\Omega)$. This is demonstrated by the following example, involving just the scalar Laplace operator, which is somehow inspired by a counterexample to Korn's inequality in Orlicz spaces exhibited in [8, remark 1.4].

Example 5.5. Given any function ω as above, which violates (5.5), namely such that

$$\int_0 \frac{\omega(r)}{r}\, dr = \infty, \tag{5.11}$$

we produce a solution u to the equation

$$-\Delta u = -\mathrm{div}\mathbf{F} \quad \text{in } B_1(0), \tag{5.12}$$

with $B_1(0) \subset \mathbb{R}^2$, and $\mathbf{F} \in C^\omega(B_1(0))$, but $|\nabla u| \notin L^\infty_{\mathrm{loc}}(B_1(0))$. To this purpose, define the function $\xi : (0,1) \to [0,\infty)$ as

$$\xi(r) = -\int_r^1 \frac{\omega(\rho)}{\rho}\, d\rho \quad \text{for } r \in (0,1),$$

and consider the function $u : B_1(0) \to \mathbb{R}$ given by

$$u(x) = x_2 \xi(|x|) \quad \text{for } x \in B_1(0).$$

One can verify that u fulfils (5.12), with

$$\mathbf{F}(x) = \Big(\frac{2x_1x_2}{|x|^2}\omega(|x|), \frac{x_2^2 - x_1^2}{|x|^2}\omega(|x|)\Big) \quad \text{for } x \neq 0,$$

so that $\mathbf{F} \in C^\omega(B_1(0))$, but

$$\nabla u(x) = \Big(\frac{x_1x_2}{|x|^2}\omega(|x|), \xi(|x|) + \frac{x_2^2}{|x|^2}\omega(|x|)\Big) \quad \text{for } x \neq 0,$$

and hence $\nabla u \notin L^\infty_{\mathrm{loc}}(B_1(0))$.

As shown by Example 5.5, continuity, and mere local boundedness, of the gradient of solution to system (1.1) is not guaranteed when $\mathbf{F} \in \mathcal{L}^{\omega,p'}_{\mathrm{loc}}(\Omega)$, or even $\mathbf{F} \in C^\omega_{\mathrm{loc}}(\Omega)$, if ω does not decay at 0 sufficiently fast to 0 for (5.5) to be satisfied. Still, it turns out that the degree of integrability of $|\nabla \mathbf{u}|$ is higher than that ensured from membership of $\mathbf{F}$ just to $L^\infty_{\mathrm{loc}}(\Omega)$ (see Proposition 4.8). This assertion can be precisely formulated in terms of Marcinkiewicz quasi-norms, also called weak-type quasi-norms. Recall that, given a non-decreasing function $\eta : [0,\infty) \to [0,\infty)$, the Marcinkiewicz functional $\|\cdot\|_{M^\eta(0,\infty)}$ is defined as

$$\|f\|_{M^\eta(0,\infty)} = \sup_{s>0} \eta(s) f^*(s)$$

for a measurable function $f : (0,\infty) \to \mathbb{R}$. Clearly, $\|\cdot\|_{M^\eta(0,\infty)}$ is a rearrangement-invariant functional.

Owing to [55, Theorem 1], if Ω is a bounded Lipschitz domain, $R_0 > \mathrm{diam}(\Omega)$, and ζ is given by

$$\zeta(r) = \bigg(\int_{r^{\frac{1}{n}}}^{R_0^{\frac{1}{n}}} \frac{\omega(\rho)}{\rho}\, d\rho\bigg)^{-1} \quad \text{for } r \in (0, R_0), \tag{5.13}$$

then

$$\mathcal{L}^\omega(\Omega) \to M^\zeta(\Omega). \tag{5.14}$$

The following result is a straightforward consequence of Theorem 5.1 and embedding (5.14). In what follows, ζ_p denotes the function given by

$$\zeta_p(r) = \zeta^{\frac{1}{p-1}}(r) \quad \text{for } r \in (0, R_0).$$

Corollary 5.6. [Borderline Marcinkiewicz spaces] *Let $n \geq 2$, $N \geq 1$ and $p \in (1,\infty)$. Let Ω be an open set in $\mathbb{R}^n$, let $R > 0$, and let $R_0 > R$. Then there exist constants $c_1 = c_1(n,N,p,R,R_0) > 0$ and $c_2 = c_2(n,N,p,R,R_0) > 0$ such that, if the function $\omega(r)r^{-\beta}$ is (almost) decreasing for some $\beta \in \big(0, \min\{1, \frac{2\alpha}{p'}\}\big)$, $\mathbf{F} \in \mathcal{L}^{\omega,p'}_{\rm loc}(\Omega)$, and $B_{2R} \subset\subset \Omega$, then*

$$\|\nabla \mathbf{u}\|_{\mathcal{M}^{\zeta_p}(B_R)} \leq c_1 \|\mathbf{F}\|^{\frac{1}{p-1}}_{\mathcal{L}^{\omega,p'}(B_{2R})} + c_2 \|\nabla \mathbf{u}\|^{\frac{1}{p-1}}_{L^{p'}(B_{2R})} \tag{5.15}$$

for every local weak solution $\mathbf{u} \in W^{1,p}_{\rm loc}(\Omega)$ to system (1.1). Here, $\alpha = \alpha(n,N,p)$ denotes the exponent appearing in Theorem 2.6.

A local version of Proposition 4.8 tells us that, if $\mathbf{F} \in L^\infty_{\rm loc}(\Omega)$, then $|\nabla \mathbf{u}| \in \exp L^{p-1}_{\rm loc}(\Omega)$. Since $L^\infty_{\rm loc}(\Omega)$ is the smallest (local) rearrangement-invariant space, this is the strongest integrability information on $|\nabla \mathbf{u}|$ which follows from membership of $\mathbf{F}$ to a rearrangement-invariant space. As observed above, Corollary 5.6 complements this results, and ensures that, if $\mathbf{F}$ belongs to a smaller space than $L^\infty_{\rm loc}(\Omega)$, namely a space of (locally) uniformly continuous functions $C^\omega_{\rm loc}(\Omega)$, or just to the Campanato type space $\mathcal{L}^{\omega,p'}_{\rm loc}(\Omega)$, then, even if ω does not fulfil (5.5), yet $|\nabla \mathbf{u}|$ belongs to the better (smaller) rearrangement-invariant space $M^{\zeta_p}_{\rm loc}(\Omega)$ than $\exp L^{p-1}_{\rm loc}(\Omega)$, as soon as

$$\lim_{r\to 0} \omega(r) = 0. \tag{5.16}$$

To verify this fact, note that the latter limit ensures that

$$\lim_{r\to 0} \frac{\zeta_p(r)}{\log^{\frac{1}{p-1}}(\frac{1}{r})} = 0. \tag{5.17}$$

On the other hand, it is well known (and easily verified) that

$$\exp L^{p-1}_{\rm loc}(\Omega) = M^{\log^{\frac{1}{p-1}}(\frac{1}{r})}_{\rm loc}(\Omega).$$

Equation (5.17) thus implies that

$$M^{\zeta_p}_{\rm loc}(\Omega) \subsetneqq \exp L^{p-1}_{\rm loc}(\Omega). \tag{5.18}$$

Example 5.7. Assume that $\mathbf{F} \in C^{\log^{-\sigma}(\frac{1}{r})}_{\rm loc}(\Omega)$ for some $\sigma > 0$. If $\sigma > 1$, then Corollary 5.3 entails that

$$|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u} \in C^{\log^{1-\sigma}(\frac{1}{r})}_{\rm loc}(\Omega).$$

If $0 < \sigma < 1$ we instead have, via Corollary 5.6,

$$|\nabla \mathbf{u}| \in \exp L^{\frac{p-1}{1-\sigma}}_{\rm loc}(\Omega).$$

In the borderline case when $\sigma = 1$, Corollary 5.6 again tells us that

$$|\nabla \mathbf{u}| \in \exp\exp L^{p-1}_{\rm loc}(\Omega).$$

By Remark 5.4, the same conclusions hold even under the slightly weaker assumption that $\mathbf{F} \in \mathcal{L}^{\log^{-\sigma}(\frac{1}{r})}_{\rm loc}(\Omega)$.

Assume now that (5.16) fails, namely that $\lim_{r\to 0}\omega(r) > 0$. Then $\mathcal{L}^{\omega,p'}_{\rm loc}(\Omega) = \mathrm{BMO}_{\rm loc}(\Omega)$, and Corollary 5.6 just yields $|\nabla \mathbf{u}| \in \exp L^{p-1}_{\rm loc}(\Omega)$. In other words, no difference seems to be reflected in the integrability of $|\nabla \mathbf{u}|$, depending on whether $\mathbf{F} \in \mathrm{BMO}_{\rm loc}(\Omega)$ or $\mathbf{F} \in L^\infty_{\rm loc}(\Omega)$. However, a difference appears in the scale of oscillation spaces, since, under the former assumption, the BMO estimate (1.8), even in local form, can be recovered as a special case of Theorem 5.1, owing to (5.2).

Corollary 5.8. [BMO] *Let $n \geq 2$, $N \geq 1$ and $p \in (1,\infty)$. Let Ω be an open set in $\mathbb{R}^n$, and let $R > 0$. Then there exist constants $c_1 = c_1(n,N,p) > 0$, $c_2 = c_2(n,N,p,R) > 0$ such that, if $\mathbf{F} \in \mathrm{BMO}_{\mathrm{loc}}(\Omega)$, and $B_{2R} \subset\subset \Omega$, then*

$$\| |\nabla \mathbf{u}|^{p-2} \nabla \mathbf{u} \|_{\mathrm{BMO(B_R)}} \leq c_1 \|\mathbf{F}\|_{\mathrm{BMO(B_{2R})}} + c_2 \|\nabla \mathbf{u}\|_{L^{p'}(B_{2R})} \tag{5.19}$$

for every local weak solution $\mathbf{u} \in W^{1,p}_{\mathrm{loc}}(\Omega)$ to system (1.1).

Theorem 5.1 can be used to show that *vanishing mean oscillation* (VMO) regularity of the datum $\mathbf{F}$ is also reflected into the same regularity of $|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}$. Recall that a locally integrable function $\mathbf{f} : \Omega \to \mathbb{R}^m$ is said to belong to $\mathrm{VMO}(\Omega)$ if

$$\lim_{\varrho \to 0^+} \Bigg(\sup_{\substack{B_r \subset \Omega \\ r \leq \varrho}} \fint_{B_r} |\mathbf{f} - \langle \mathbf{f} \rangle_{B_r}| \, dx \Bigg) = 0. \tag{5.20}$$

Clearly, $\mathrm{VMO}(\Omega) \subset \mathrm{BMO}(\Omega)$. We shall write $\mathbf{f} \in \mathrm{VMO}_{\mathrm{loc}}(\Omega)$ to denote that equation holds with Ω replaced by any open set $\Omega' \subset\subset \Omega$.

Corollary 5.9. [VMO] *Let $n \geq 2$, $N \geq 1$ and $p \in (1,\infty)$. Let Ω be an open set in $\mathbb{R}^n$, and let $R > 0$. If $\mathbf{F} \in \mathrm{VMO}_{\mathrm{loc}}(\Omega)$, then $|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u} \in \mathrm{VMO}_{\mathrm{loc}}(\Omega)$ for every local weak solution $\mathbf{u} \in W^{1,p}_{\mathrm{loc}}(\Omega)$ to system* (1.1).

Proof. Let Ω' be an open set such $\Omega' \subset\subset \Omega$. Fix any $\gamma \in (0,1)$. By Hölder's inequality

$$\begin{aligned} &\sup_{\substack{B_r \subset \Omega' \\ r \leq \varrho}} \Bigg(\fint_{B_r} |\mathbf{F} - \langle \mathbf{F} \rangle_{B_r}|^{p'} \, dx \Bigg)^{\frac{1}{p'}} \\ &\quad \leq \sup_{B_r \subset \Omega'} \Bigg(\fint_{B_r} |\mathbf{F} - \langle \mathbf{F} \rangle_{B_r}|^{\frac{p'-\gamma}{1-\gamma}} \, dx \Bigg)^{\frac{1-\gamma}{p'}} \sup_{\substack{B_r \subset \Omega' \\ r \leq \varrho}} \Bigg(\fint_{B_r} |\mathbf{F} - \langle \mathbf{F} \rangle_{B_r}| \, dx \Bigg)^{\frac{\gamma}{p'}} \\ &\quad = \|\mathbf{F}\|^{\frac{p'-\gamma}{p'}}_{\mathcal{L}^{\frac{p'-\gamma}{1-\gamma}}(\Omega')} \sup_{\substack{B_r \subset \Omega' \\ r \leq \varrho}} \Bigg(\fint_{B_r} |\mathbf{F} - \langle \mathbf{F} \rangle_{B_r}| \, dx \Bigg)^{\frac{\gamma}{p'}} \quad \text{for } \varrho > 0. \end{aligned} \tag{5.21}$$

Denote by $\sigma : (0,\infty) \to (0,\infty]$ the function defined for $\varrho \in (0,\infty)$ by the leftmost side of equation (5.21). Owing to the assumption that $\mathbf{F} \in \mathrm{VMO}_{\mathrm{loc}}(\Omega)$, to equations (5.2) and (5.21), one has that $\lim_{\varrho \to 0^+} \sigma(\varrho) = 0$. Next, given any exponent β as in Theorem 5.1, define the function $\omega : (0,\infty) \to (0,\infty)$ as

$$\omega(r) = r^\beta \sup_{\varrho \geq r} \frac{\{\sigma(\varrho)\}}{\varrho^\beta} \quad \text{for } r > 0. \tag{5.22}$$

Obviously, the function $r \mapsto \frac{\omega(r)}{r^\beta}$ is non-increasing, and one can verify that $\lim_{r \to 0^+} \omega(\varrho) = 0$. Since $\omega \geq \sigma$, equation (5.21) ensures that $\mathbf{F} \in \mathcal{L}^{\omega,p'}_{\mathrm{loc}}(\Omega)$. An application of Theorem 5.1 tells us that $|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u} \in \mathcal{L}^{\omega,p'}_{\mathrm{loc}}(\Omega)$ as well. In particular, this entails that $|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u} \in \mathrm{VMO}_{\mathrm{loc}}(\Omega)$. ☐

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