A TRANSMISSION PROBLEM ON A POLYGONAL PARTITION: REGULARITY AND SHAPE DIFFERENTIABILITY

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Abstract. We consider a transmission problem on a polygonal partition for the two-dimensional conductivity equation. For suitable classes of partitions we establish the exact behaviour of the gradient of solutions in a neighbourhood of the vertexes of the partition. This allows to prove shape differentiability of solutions and to establish an explicit formula for the shape derivative.

1. Introduction

In this paper we consider the conductivity equation in a bounded planar domain
\[
\text{div}(\sigma \nabla u) = 0 \quad \text{in } \Omega \subset \mathbb{R}^2.
\]
We assume the conductivity \(\sigma\) of the form
\[
\sigma = \sum_{i=1}^{M+1} \sigma_j \chi_{P_i},
\]
where \(P = \{P_i\}_{i=1}^M\) is a polygonal regular partition in the background medium \(\Omega\) and \(P_{M+1} = \Omega \setminus P\).

This assumption on the conductivity is rather natural and arises, for example, in applications to geophysics, medical imaging and nondestructive testing of materials where the medium under investigation contains regions with different conducting properties. Moreover, piecewise constant coefficients represent a class of unknown functions in which Lipschitz stable reconstruction from boundary data can be expected (see [2], [5], [11], for example) and it appears in many finite-element scheme used for effective reconstruction.

Our main goal is to study the differentiability properties of solutions to the conductivity equation (1.1) with respect to movements of the partition \(P\) i.e. to establish the existence of the shape derivative of \(u\).

This analysis is motivated by the study of the inverse conductivity problem of recovering \(\sigma\) of the form (1.2) from boundary measurements. More precisely, in order to derive quantitative Lipschitz stability estimates for a conductivity parameter, satisfying (1.2), in terms of the Neumann to Dirichlet map \(\mathcal{N}_\sigma\), a crucial role is played by the differentiability properties of the map
\[
F : \sigma \to \mathcal{N}_\sigma
\]
with respect to movements of the partition and by the knowledge of an explicit

In [6] we performed a first step proving differentiability of $F$ in the case of a single
polygonal inclusion $\mathcal{P}$ contained in $\Omega$ and we derived rigorously for the first time
an explicit formula for the shape derivative of $F$ expressed in terms of an integral
on the boundaries of the polygons in $\mathcal{P}$. One of the main issues in the study of
shape differentiability is the regularity of the solution $u$ of the elliptic pde. The
coefficients we consider have jumps on polygonal boundaries. The related solutions
are Hölder continuous in the interior of the domain $\Omega$ (see [10] and [13]) and smooth
(in fact analytic) in the interior of each polygon. Across the sides of the polygons
the solutions are continuous and have continuous conormal derivative (transmission
conditions). Moreover, $\nabla u$ has a Lipschitz continuous extension from the interior
of the polygon to the internal part of each side of the polygon ([12]). When
approaching the vertexes of the polygons the gradient becomes more singular and
an analysis of the exact behaviour of gradients of solutions in a neighbourhood of
vertexes of $\mathcal{P}$ is needed. In the case of a single polygonal inclusion we used the
analysis derived in [3].

In the more general case considered in this paper the situation is far more compli-
cated. In this case, again, a crucial step is played by the analysis of the differenti-
ability properties of the solutions in a neighbourhood of the points of intersection
of the sides of the polygons but the behaviour of $u$ depends on how the sides of
elements of the partition intersect at those points.

In fact, from [14], it is known that for solutions of (1.1) with conductivities $\sigma \in
L^\infty(\Omega)$ satisfying

$$\lambda \leq \sigma \leq \Lambda \text{ a.e. in } \Omega \subset \mathbb{R}^2$$

the Hölder exponent $\alpha$ can be computed explicitly and has the form

$$\alpha = \frac{4}{\pi} \arctan \left( \sqrt{\frac{\lambda}{\Lambda}} \right).$$

This represents the worse Hölder exponent for solutions to (1.1) and it is attained
for solutions corresponding to partitions meeting in a vertex with four sides at a
right angle. So, in general, the regularity of solutions to (1.1) and (1.2) does not
allow us to prove shape differentiability of $u$.

In this paper we succeed in determining classes of partitions for which the regularity
of the solutions and its gradients at the points of intersection of the polygons is
enough to guarantee differentiability of solutions $u$. Furthermore, we establish an
explicit formula for the shape derivative of $u$, $u'$, on the boundary of $\Omega$. The paper
is organized as follows: in Section 2 we prove the estimate on the behaviour of
$\nabla u$ in a neighbourhood of the points of the partition with no more than 3 sides
intersecting. In Section 3 we use this estimate to prove the existence of the shape
derivative $u'$ with respect to movements of the partition and to find an explicit
representation formula on the boundary of $\Omega$. We also obtain (Theorem 3.4) a
formula for partial derivatives of the Neumann-to-Dirichlet map.
2. Behaviour of $\nabla u$ in a neighbourhood of a vertex of certain classes of partitions

Let $B$ be the open disk of radius $r_0$ centered at the origin $O = (0,0)$ and let $\sigma$ be a piecewise constant coefficient defined in $\overline{B}$ expressed in polar coordinates by

$$\sigma(\rho, \theta) = \begin{cases} 
\sigma_1 & \text{for} \quad \beta_0 := 0 \leq \theta < \beta_1, \\
\sigma_2 & \text{for} \quad \beta_1 \leq \theta < \beta_2, \\
\sigma_3 & \text{for} \quad \beta_2 \leq \theta < \beta_3 := 2\pi,
\end{cases}$$

where

$$0 < \sigma_0 \leq \sigma_k \leq \sigma_0^{-1}, \text{ for } k = 1, 2, 3.$$ 

Let $u \in H^1(B)$ be a solution to

$$\text{div}(\sigma \nabla u) = 0 \text{ in } B.$$ 

For $k = 1, 2, 3$, let us denote by

$$D_k = \{(\rho, \theta) : 0 < \rho < r_0, \beta_{k-1} \leq \theta \leq \beta_k\}$$

and by

$$u_k = u|_{D_k}.$$ 

Each function $u_k$ is harmonic in $D_k$ and transmission conditions at the boundaries of $D_k$ hold, that is, $u$ and $\sigma \frac{\partial u}{\partial n_k}$ are continuous across these boundaries. Moreover, by Theorem 1.1 in [12] each function $u_k$ can be extended as a $C^{1,\alpha}$ function up to the boundary of the sector $D_k$ and $C^{1,\alpha}$ norm of $u_k$ can be bounded in terms of the $L^2$ norm of $u$ uniformly on subsets of $\overline{D_k}$ that have positive distance from the origin.

**Theorem 2.1.** If, for some $\overline{\beta} \in (0, \frac{\pi}{3}]$,

$$2\overline{\beta} \leq \beta_k - \beta_{k-1} \leq \pi - \overline{\beta}, \text{ for } k = 1, 2, 3,$$

there exist $C > 0$ and $\gamma > 1/2$ depending only on $\overline{\beta}$, $r_0$ and $\sigma_0$, such that

$$|\nabla u_k(x, y)| \leq C\|u\|_{H^1(B)} \text{dist}((x, y), O)^{\gamma - 1}, \text{ for } (x, y) \in D_k.$$ 

In order to prove Theorem 2.1, let us show the following expansion for solution $u$.

**Proposition 2.2.** Under the same assumptions of Theorem 2.1 the following expansion holds for $0 < r \leq \frac{r_0}{2}$ and $k = 1, 2, 3$

$$u_k(r, \theta) = u_k(0) + \sum_{j=1}^{\infty} r^{\gamma_j} \left( A_j^k \cos(\gamma_j \theta) + B_j^k \sin(\gamma_j \theta) \right) \text{ for } \theta \in (\beta_{k-1}, \beta_k).$$

The series are convergent uniformly in $0 < r \leq \frac{r_0}{2}$ and their first derivatives are absolutely convergent in the same set. The sequence $\gamma_j$ is monotone increasing, there are $c_1$ and $c_2$ such that

$$0 < c_1 \leq \frac{\gamma_j}{j} \leq c_2 \text{ for all } j \in \mathbb{N},$$

and

$$\gamma_1 > \frac{1}{2}.$$
Proof. We follow the outline of [3]. Let us define the function $a(\theta) = \sigma(r_0, \theta)$ for $\theta \in [0, 2\pi]$ and introduce the weighted spaces $L^2_a(S^1), H^1_a(S^1)$ with norms

$$
\|v\|_{L^2_a(S^1)} = \left( \int_0^{2\pi} a(\theta)|v(\theta)|^2 \, d\theta \right)^{1/2},
$$

$$
\|v\|_{H^1_a(S^1)} = \left( \int_0^{2\pi} a(\theta) \left( \left| \frac{\partial v}{\partial \theta}(\theta) \right|^2 + |v(\theta)|^2 \right) \, d\theta \right)^{1/2}.
$$

Define

$$
L v = \frac{1}{a} \frac{\partial}{\partial \theta} \left( a \frac{\partial}{\partial \theta} v \right).
$$

$L$ is an unbounded, selfadjoint, positive elliptic operator with dense domain in $L^2_a(S^1)$, and $(L + 1)^{-1}$ is compact. Let us denote by $\gamma_j^2$, $(\gamma_j \geq 0)$ the positive eigenvalues of $L$ that constitute its spectrum. We denote the corresponding complete orthonormal sequence by \{v^{(j)}\}, which is a basis for $L^2_a(S^1)$. The solution $u$ can be written, for $0 < r < r_0$ as

$$
(2.7) \quad u(r, \theta) = u(0) + \sum_{j=1}^{\infty} C_j r^{\gamma_j} v^{(j)}(\theta).
$$

Since $u_r \in L^2_a(S^1)$ for $r = r_0$, we have

$$
(2.8) \quad K := \sum_{j=1}^{\infty} C_j^{2\gamma_j} \sigma_0^{2\gamma_j} < \infty.
$$

The asymptotic behaviour of eigenvalues (2.4) is obtained from the variational formulation for the eigenvalues: see, for example, [9, Example 4.6.1].

We now want to estimate from below the first positive eigenvalue of $L$. Let $v \in H^1_a(S^1)$ be solution to

$$
(2.9) \quad L v + \gamma^2 v = 0
$$

such that

$$
\int_0^{2\pi} a v^2(\theta) \, d\theta = 1
$$

and $\gamma > 0$. The function $v(\theta)$ satisfies the equation

$$
\frac{\partial}{\partial \theta} \left( a(\theta) \frac{\partial}{\partial \theta} v(\theta) \right) + \gamma^2 a(\theta) v(\theta) = 0, \quad \text{for } 0 \leq \theta \leq 2\pi,
$$

with

$$
v(0) = v(2\pi).
$$

Let $v_k = v_{[\beta_k-1,\beta_k]}$ for $k = 1, 2, 3$. By considering the equation in $[\beta_0, \beta_1]$ we have

$$
v_1(\theta) = v_1(0) \cos(\gamma \beta_1) + \gamma^{-1} v_1'(0) \sin(\gamma \beta_1).
$$

By the transmission conditions at $\theta = \beta_1$ we get

$$
\frac{\sigma_2}{\sigma_1} v_2'(\beta_1) = \frac{\sigma_1}{\sigma_2} v_1'(\beta_1) = \frac{\sigma_1}{\sigma_2} \left\{ -v_1(0) \gamma \sin(\gamma \beta_1) + v_1'(0) \cos(\gamma \beta_1) \right\},
$$

$$
v_2(\beta_1) = v_1(\beta_1) = v_1(0) \cos(\gamma \beta_1) + \gamma^{-1} v_1'(0) \sin(\gamma \beta_1),
$$

$$
v_2'(\beta_1) = \frac{\sigma_1}{\sigma_2} v_1'(\beta_1) = \frac{\sigma_1}{\sigma_2} \left\{ -v_1(0) \gamma \sin(\gamma \beta_1) + v_1'(0) \cos(\gamma \beta_1) \right\}.
that can be written as
\[
\begin{pmatrix}
v_2(\beta_1)
v'_2(\beta_1)
\end{pmatrix} = M_1 \begin{pmatrix}
v_1(0)
v'_1(0)
\end{pmatrix}
\]
where
\[
M_1 = \begin{pmatrix}
\cos \gamma(\beta_1 - \beta_0) & \gamma^{-1} \sin \gamma(\beta_1 - \beta_0) \\
-\frac{\sigma_1}{\sigma_2} \gamma \sin \gamma(\beta_1 - \beta_0) & \frac{\sigma_1}{\sigma_2} \cos \gamma(\beta_1 - \beta_0)
\end{pmatrix}.
\]
In the same way, by writing explicitly the solution of the ordinary differential equation in \([\beta_1, \beta_2]\), exploiting the transmission conditions at \(\theta = \beta_2\), considering the solution in \([\beta_2, \beta_3]\) and, finally, using the transmission conditions at \(\theta = \beta_3 = 2\pi\) we get
\[
\begin{pmatrix}
v_1(0)
v'_1(0)
\end{pmatrix} = M_3 M_2 M_1 \begin{pmatrix}
v_1(0)
v'_1(0)
\end{pmatrix}
\]
where
\[
M_j = \begin{pmatrix}
\cos \gamma(\beta_j - \beta_{j-1}) & \gamma^{-1} \sin \gamma(\beta_j - \beta_{j-1}) \\
-\frac{\sigma_1}{\sigma_2} \gamma \sin \gamma(\beta_j - \beta_{j-1}) & \frac{\sigma_1}{\sigma_2} \cos \gamma(\beta_j - \beta_{j-1})
\end{pmatrix}.
\]
Hence the eigenvalue problem is equivalent to
\[
det (M_3 M_2 M_1 - I) = 0.
\]
The determinant above can be explicitly evaluated and has the form
\[
det (M_3 M_2 M_1 - I) =
2(1 - \cos 2\pi \gamma) + \mu_2 \sin \gamma \beta_1 \sin \gamma(2\pi - \beta_2) \cos \gamma(\beta_2 - \beta_1) +
+ \mu_1 \sin \gamma(\beta_2 - \beta_1) \sin \gamma(2\pi - \beta_2) \cos \gamma \beta_1 + \mu_3 \sin \gamma \beta_1 \sin \gamma(\beta_2 - \beta_1) \cos \gamma(2\pi - \beta_2),
\]
where
\[
\mu_2 = \frac{\sigma_3}{\sigma_1} + \frac{\sigma_1}{\sigma_3} - 2, \quad \mu_1 = \frac{\sigma_3}{\sigma_2} + \frac{\sigma_2}{\sigma_3} - 2, \quad \mu_3 = \frac{\sigma_2}{\sigma_1} + \frac{\sigma_1}{\sigma_2} - 2.
\]
Note that the coefficients \(\mu_j\) are non negative and \(1 - \cos 2\pi \gamma > 0\) for \(\gamma \in (0, 1)\), hence,
\[
det (M_3 M_2 M_1 - I) > 0
\]
for
\[
0 < \gamma \leq \frac{1}{2} \min \left\{ \frac{\pi}{\beta_2 - \beta_1}, \frac{\pi}{\beta_1}, \frac{\pi}{2\pi - \beta_2} \right\}.
\]
Since, by assumption (2.1)
\[
\frac{1}{2} \min \left\{ \frac{\pi}{\beta_2 - \beta_1}, \frac{\pi}{\beta_1}, \frac{\pi}{2\pi - \beta_2} \right\} \geq \frac{1}{2} \frac{\pi}{2\pi - \beta} > \frac{1}{2},
\]
we have that the first non zero eigenvalue \(\gamma_1\) is strictly larger than \(\frac{1}{2}\). \(\square\)

**Proof of Theorem 2.1.** Let us consider the series expansion (2.7) where \(v^{(j)}\) are eingenvfunctions related to eigenvalue \(\gamma_j\) with
\[
(2.10) \quad \int_0^{2\pi} a(v^{(j)})^2 d\theta = 1.
\]
The weak form of equation (2.9) gives
\[
\int_0^{2\pi} \left( a \frac{\partial v^{(j)}}{\partial \theta} \frac{\partial w}{\partial \theta} - a \gamma^2 v^{(j)} w \right) d\theta = 0 \text{ for every } w \in H^1_d(S^1).
\]
By choosing $w = v^{(j)}$ we have, by (2.10),
\[
\int_0^{2\pi} a \left( \frac{\partial v^{(j)}}{\partial \theta} \right)^2 d\theta = \gamma_j^2 \int_{S^1} a(v^{(j)})^2 d\theta = \gamma_j^2.
\]
Now we recall (see [8]) that for some universal constant $c$
\[
|v^{(j)}(\theta)| \leq c\|v^{(j)}\|_{H^1(S^1)},
\]
and, hence, since $\gamma_j > 1/2$, there is a constant $C$ depending only on $\sigma_0$ such that
\[
(2.11) \quad |v^{(j)}(\theta)| \leq C\gamma_j \text{ for } 0 \leq \theta \leq 2\pi.
\]
From (2.7) and (2.11), by Hölder inequality and by (2.8), we have for $0 < r \leq \frac{r_0}{2}$
\[
|u_r(r, \theta)| \leq C r^{\gamma_j - 1} \sum_{j=1}^{\infty} |C_j r^{\gamma_j - \gamma_j^2}|
\leq \frac{C}{r_0} \left( \frac{r}{r_0} \right)^{\gamma_j - 1} \left( \sum_{j=1}^{\infty} \left( \frac{r}{r_0} \right)^{2\gamma_j} \gamma_j^2 \left( \sum_{j=1}^{\infty} C_j^2 r_0^{2\gamma_j} \gamma_j^2 \right)^{1/2} \right)\left( \sum_{j=1}^{\infty} C_j^2 r_0^{2\gamma_j} \gamma_j^2 \right)^{1/2}
\leq \frac{C \sqrt{CK}}{r_0} \left( \frac{r}{r_0} \right)^{\gamma_j - 1},
\]
(2.12)
where $\tilde{C} = \sum_{j=1}^{\infty} 2^{-2\gamma_j} \gamma_j^2$ (the convergence of this series is a consequence of (2.4)).
Moreover, by equation (2.9) we get that
\[
(2.13) \quad \left( \frac{\partial^2 v^{(j)}}{\partial \theta^2} \right)(\theta) = \gamma_j^2 v^{(j)}(\theta) \text{ in } (0, 2\pi) \setminus \{\beta_1, \beta_2\},
\]
and, by (2.11), we get
\[
(2.14) \quad \left| \left( \frac{\partial^2 v^{(j)}}{\partial \theta^2} \right)(\theta) \right| \leq C\gamma_j^3 \text{ in } (0, 2\pi) \setminus \{\beta_1, \beta_2\}.
\]
By an interpolation inequality in each subset of $[\beta_{k-1}, \beta_k]$ in which $a$ is constant, we have
\[
\left\| \frac{\partial v^{(j)}}{\partial \theta} \right\|_\infty \leq \frac{C}{\beta_k - \beta_{k-1}} \left\| v^{(j)} \right\|^{1/2}_\infty \left( \left\| v^{(j)} \right\|^{1/2}_\infty + (\beta_k - \beta_{k-1}) \left\| \frac{\partial^2 v^{(j)}}{\partial \theta^2} \right\|_\infty^{1/2} \right)
\]
hence, by (2.11) and (2.14),
\[
(2.15) \quad \left\| \left( \frac{\partial v^{(j)}}{\partial \theta} \right) \right\|_{L^\infty((0, 2\pi))} \leq \frac{C}{\beta} \gamma_j^2,
\]
where $C$ depends on $\sigma_0$. Then, proceeding as before,
\[
(2.16) \quad \frac{1}{r} |u_{\theta}(r, \theta)| \leq \frac{C}{\beta r_0} \left( \frac{r}{r_0} \right)^{\gamma_j - 1} \sqrt{CK}.
\]
From (2.12) and (2.16), for $0 < r < \frac{r_0}{2}$, we have
\[
(2.17) \quad |\nabla u| \leq \frac{C}{\beta r_0} \left( \frac{r}{r_0} \right)^{\gamma_j - 1} \sqrt{CK}
\]
on each $D_k$ for $k = 1, 2, 3$. By (2.8), $\sqrt{K}$ can be bounded in terms of $\|u\|_{H^1(B)}$. □
Remark 2.3. Estimate (2.2) holds true also if coefficient $\sigma$ attains only two different values on two non degenerate sectors, see [3]. Nevertheless, if we consider a vertex at which more than 3 sides intersects, then the estimate is not true anymore. A counterexample of this estimate can be easily constructed in the case of four equal sectors. See [14, Lemma 1]. Moreover, if assumption (2.1) is not satisfied, the first positive eigenvalue can be smaller than $1/2$: for example, if $\beta_1 = \pi/6$, $\beta_2 = \pi/3$, $\sigma_1 = 10^{-1}$, $\sigma_2 = 10^{3}$ and $\sigma_3 = 10$, direct calculation shows that, for $\gamma = 1/2$, $\det(M_3M_2M_1-I) < 0$, hence the first positive eigenvalue is smaller that $1/2$.

3. Shape derivative of the solution of a Neumann problem with respect to movements of a polygonal partition

Let $\Omega \subset \mathbb{R}^2$ be a bounded open set such that $\partial \Omega$ is Lipschitz continuous with constants $r_0$ and $K_0$ and $\text{diam}(\Omega) \leq L$.

Let us consider a polygonal partition $\mathcal{P} \subset \Omega$ such that $\text{dist}(\mathcal{P}, \partial \Omega) \geq d_0$ and such that

$$\mathcal{P} = \bigcup_{i=1}^{M} \mathcal{P}_i,$$

where $\mathcal{P}_i$ is an open polygon.

Let us denote by $Q_1, \ldots, Q_N$ the vertexes of the polygons that compose $\mathcal{P}$. Let us also assume that:

- each $Q_j$ does not belong to more than three sides of polygons;
- $\text{dist}(Q_j, Q_k) \geq d_0$ if $j \neq k$;
- each polygon $\mathcal{P}_i$ contains a disk of radius greater than $r_1$

denoting by $\beta_j^k$, $k = 1, \ldots, k_j \leq 3$, the angles in the vertex $Q_j$, we assume there exists $\beta \in (0, \pi)$ such that

$$\begin{align*}
&\text{if } k_j = 2, \quad 0 < \beta^k < 2\pi - \beta^k \text{ for } k = 1, 2 \\
&\text{if } k_j = 3, \quad 0 < \beta^k < \pi - \beta^k \text{ for } k = 1, 2, 3.
\end{align*}$$

(3.1)

Let

$$\sigma_0 = \sum_{i=1}^{M} \sigma_i \chi_{\mathcal{P}_i},$$
where \( \mathcal{P}_{M+1} = \Omega \setminus \mathcal{P} \) and

\[
0 < c_0^{-1} < c_i < c_0, \quad \text{for every } i = 1, \ldots, M + 1.
\]

Let \( f \in H^{-1/2}(\partial \Omega) \) such that \( \int_{\partial \Omega} f = 0 \) and let \( u_0 \in H^1(\Omega) \) be the unique solution to the boundary value problem

\[
\begin{cases}
\text{div}(\sigma_0 \nabla u_0) = 0 \text{ in } \Omega, \\
\sigma_0 \frac{\partial u_0}{\partial \nu} = f \text{ on } \partial \Omega, \\
\int_{\partial \Omega} u_0 = 0,
\end{cases}
\]

where \( \nu \) denotes the unit outer normal to \( \partial \Omega \).

Let \( V = (v_1, \ldots, v_N) \in \mathbb{R}^2 \) be an arbitrary vector that represents the movements of vertexes of the polygons.

For \( t \geq 0 \) let \( \Psi^V \) be a function defined on \( \cup_{i=1}^M \partial \mathcal{P}_i \), such that, if \( Q_jQ_k \) is a side of one of the polygons, we have

\[
\Psi^V(x) := v_j + \frac{(x - Q_j) \cdot (Q_k - Q_j)}{|Q_k - Q_j|}(v_k - v_j) \text{ for } x \in Q_jQ_k.
\]

We extend the function \( \Psi^V \) to a \( W^{1,\infty} \) function with compact support in \( \Omega \) that, with a slight abuse of notation, we still denote by \( \Psi^V \).

Let \( \Phi_t(x) = x + t\Psi^V(x) \), denote by \( \mathcal{P}_t \) the polygon whose boundary is given by \( \Phi_t(\partial \mathcal{P}_i) \) and let \( \mathcal{P}^t = \cup_{i=1}^M \mathcal{P}_i^t \). The points \( Q_j^t = Q_j + tv_j \) for \( j = 1, \ldots, N \) are the vertexes of polygons in \( \mathcal{P}^t \).

For \( t \) sufficiently small (depending on \( V, r_1, \beta \) and \( d_0 \)) the new partition has the same properties of the original one, with slightly different constants.

Let

\[
\sigma_t(x) = \sum_{i=1}^{M+1} \sigma_t(x) \chi_{P_i^t}
\]

and let \( u_t \in H^1(\Omega) \) be the unique solution to the boundary value problem

\[
\begin{cases}
\text{div}(\sigma_t \nabla u_t) = 0 \text{ in } \Omega, \\
\sigma_t \frac{\partial u_t}{\partial \nu} = f \text{ on } \partial \Omega, \\
\int_{\partial \Omega} u_t = 0.
\end{cases}
\]

The aim of this section is to evaluate, for \( y \in \partial \Omega \), the derivative of \( u \) in the direction \( V \), that is

\[
u'(y) = \lim_{t \to 0} \frac{u_t(y) - u_0(y)}{t}.
\]

As in [6], thanks to Theorem 2.1, we can obtain this derivative by direct calculation, but, since the geometry of the problem makes these calculations quite involved, we follow here a different strategy.

Let \( \hat{u}_t(x) = u_t \circ \Phi_t(x) \) and let us evaluate the material derivative \( \hat{u} \), that is the limit for \( t \to 0 \) of \( \frac{\hat{u}_t(y) - \hat{u}_0(y)}{t} \) in \( H^1(\Omega) \). Then, from the material derivative \( \hat{u} \) we obtain the boundary values of the shape derivative \( u' \).

Note that for sufficiently small \( t \) ( \( t \leq \frac{1}{\|\Psi^V\|_{W^{1,\infty}}} \)) the function \( \Phi_t^{-1} \) exists in \( \Omega \). Let us define

\[
A(t) = (D\Phi_t^{-1})^T (D\Phi_t^{-1}) \quad \text{and}
\]

\[
\mathcal{A} = \frac{dA}{dt} \bigg|_{t=0} = \text{div}(\Psi^V)Id - (D\Psi^V + (D\Psi^V)^T)
\]

(3.2)

(3.3)
where $D\Phi_\ell^{-1}$ and $D\Psi$ represent the Jacobian matrices of $\Phi_\ell^{-1}$ and $\Psi$. 

Let $\tilde{u}_t(x) = u_t \circ \Phi_\ell(x)$.

**Lemma 3.1.** There exist a constant $C$ and a function $\omega(t)$, independent of $f$ and of $V/\|V\|$, such that $\omega(t) \to 0$ as $t \to 0$ and

$$\left\| \frac{\tilde{u}_t - u_t}{t} - \dot{u} \right\|_{H^1(\Omega)} \leq C\omega(t)\|f\|_{H^{3/2}(\partial\Omega)}$$

where $\dot{u} \in H^1(\Omega)$ is the material derivative of $u$ that solves

$$\int_\Omega \sigma_0 \nabla \dot{u} \cdot \nabla w = - \int_\Omega \sigma_0 A \nabla u \cdot \nabla w \quad \forall w \in H^1(\Omega)$$

with $\int_{\partial\Omega} \dot{u} = 0$.


We now want to write equation (3.4) in a different way by integration by parts. Since the functions involved are not regular enough to perform this integration, we need to analyze carefully what happens close to vertexes. This is the point where Theorem 2.1 comes into play.

**Proposition 3.2.** Let us denote by $S_k$ for $k = 1, \ldots, M_1$ the sides of the polygons in $P$. For each $v \in H^1(\Omega)$ solution of

$$\text{div} (\sigma_0 \nabla v) = 0 \text{ in } \Omega,$$

we have,

$$\int_\Omega \sigma_0 \nabla \dot{u} \cdot \nabla v = \sum_{k=1}^{M_1} \int_{S_k} [\sigma_0 b] \cdot n_k ds,$$

where

$$b = (\Psi^V \cdot \nabla u_0) \nabla v + (\Psi^V \cdot \nabla v) \nabla u_0 - (\nabla u_0 \cdot \nabla v) \Psi^V,$$

$n_k$ is a normal unit vector to $S_k$ and $[\sigma_0 b] = \sigma^- b^- - \sigma^+ b^+$ where $\sigma^-$, $b^-$ are the functions $\sigma_0$, $b$ restricted to the polygon with side $S_k$ and with outer normal $n_k$ while $\sigma^+$, $b^+$ are the functions $\sigma_0$, $b$ restricted to the polygon with side $S_k$ and with inner normal $n_k$.

**Proof.** For $0 < \varepsilon < \frac{d\Phi}{d\ell}$, let

$$B_\varepsilon = \bigcup_{j=1}^N B(P_j, \varepsilon),$$

and let us denote by

$$u_i = u_{0i} |_{P_i} \text{ and } v_i = v_{0i} |_{P_i} \text{ for } i = 1, \ldots, M.$$ 

Each of these functions is harmonic in $P_i$; moreover $u_i, v_i \in H^2(P_i \setminus B_\varepsilon)$ and, by the regularity estimates in [12], $u_i, v_i \in W^{1,\infty}(P_i \setminus B_\varepsilon)\setminus B_\varepsilon\textit{ and belong to } H^2(P_1 \setminus B_\varepsilon) \text{ and to } W^{1,\infty}(\overline{P_j \setminus (B_\varepsilon \cap \{x \in \Omega : d(x, \partial\Omega) < \varepsilon\})})$. Let us now consider equation (3.4) with $w = v$ and write

$$\int_\Omega \sigma_0 \nabla \dot{u} \cdot \nabla v = - \int_{\Omega \setminus B_\varepsilon} \sigma_0 A \nabla u_0 \cdot \nabla v - \int_{B_\varepsilon} \sigma_0 A \nabla u_0 \cdot \nabla v.$$
In each set $P_i \setminus B_{\epsilon}$ we have that
\begin{equation}
-\mathbf{A} \nabla u_0 \cdot \nabla v = \text{div}(b)
\end{equation}
for $b$ given by (3.6). Here we also used the fact that $\Delta u_i = \Delta v_i = 0$ in $P_i$.
Now, we integrate by parts in each $P_j \setminus B_{\epsilon}$ and, recalling that $\Psi^V$ and, hence, $b$ have compact support in $\Omega$, we have
\begin{equation}
-\int_{\Omega \setminus B_{\epsilon}} \sigma_0 \mathbf{A} \nabla u_0 \cdot \nabla v = \sum_{i=1}^{M+1} \int_{P_i} \sigma_i \text{div}(b) = \sum_{k=1}^{M_1} \int_{S_k \setminus B_{\epsilon}} [\sigma_0 b] \cdot n_k + \int_{\partial B_{\epsilon}} \sigma_0 b \cdot n,
\end{equation}
where $n$ is the exterior normal to $\partial B_{\epsilon}$. By putting together (3.7) and (3.9) we have
\begin{equation}
\int_{\Omega} \sigma_0 \nabla \dot{u} \cdot \nabla v = \sum_{k=1}^{M_1} \int_{S_k \setminus B_{\epsilon}} [\sigma_0 b] \cdot n_k + \int_{\partial B_{\epsilon}} \sigma_0 \mathbf{A} \nabla u_0 \cdot \nabla v.
\end{equation}
Functions $u_0$ and $v$ both solve the same equation and, hence, for the assumption (3.1) on the polygons, they satisfy estimate (2.2). Then, we have
\begin{equation}
\left| \int_{B_{\epsilon}} \sigma_0 \mathbf{A} \nabla u_0 \cdot \nabla v \right| \leq C\varepsilon^{2\gamma}
\end{equation}
and
\begin{equation}
\left| \int_{\partial B_{\epsilon}} \sigma_0 b \cdot n \right| \leq C\varepsilon^{2\gamma-1}.
\end{equation}
Since $\gamma > 1/2$ (see Theorem 2.1), both the integrals in the rightmost hand side of (3.10) tend to zero for $\varepsilon \to 0$. Moreover, again by (2.2), for $\varepsilon \to 0$
\begin{equation}
\int_{S_k \setminus B_{\epsilon}} [\sigma_0 b] \cdot n_k \to \int_{S_k} [\sigma_0 b] \cdot n_k.
\end{equation}
By (3.10), (3.11), (3.12) and (3.13) we have (3.5).

**Remark 3.3.** Let us evaluate more precisely the jump $[\sigma_0 b]$. Denoting by $\tau_k$ a direction orthogonal to $n_k$ we have,
\begin{equation}
[\sigma_0 b] \cdot n_k = \left[ \sigma_0 \left( \Psi^V \cdot \nabla u_0 \right) \frac{\partial v}{\partial n_k} + \sigma_0 \left( \Psi^V \cdot \nabla u_0 \right) \frac{\partial u_0}{\partial n_k} \right]
- \sigma_0 \left( \nabla u_0 \cdot \nabla v \right) \Psi^V \cdot n_k
= (\Psi^V \cdot n_k) \left[ \sigma_0 \frac{\partial u_0}{\partial n_k} \frac{\partial v}{\partial n_k} - \sigma_0 \frac{\partial u_0}{\partial \tau_k} \frac{\partial v}{\partial \tau_k} \right]
+ (\Psi^V \cdot \tau_k) \left[ \sigma_0 \frac{\partial u_0}{\partial \tau_k} \frac{\partial v}{\partial n_k} + \sigma_0 \frac{\partial u_0}{\partial n_k} \frac{\partial v}{\partial \tau_k} \right].
\end{equation}
By transmission conditions across $S_k$ for solution of the equation $\text{div}(\sigma_0 \nabla u)$, we have
\begin{equation}
\left[ \sigma_0 \frac{\partial u_0}{\partial \tau_k} \frac{\partial v}{\partial n_k} + \sigma_0 \frac{\partial u_0}{\partial n_k} \frac{\partial v}{\partial \tau_k} \right] = 0,
\end{equation}
and
\begin{equation}
\left[ \sigma_0 \frac{\partial u_0}{\partial n_k} \frac{\partial v}{\partial n_k} - \sigma_0 \frac{\partial u_0}{\partial \tau_k} \frac{\partial v}{\partial \tau_k} \right] = (\sigma^- - \sigma^+) \left( \frac{\sigma^+ \partial u^+}{\sigma^- \partial n_k} \frac{\partial v^+}{\partial n_k} + \frac{\partial u^+}{\partial \tau_k} \frac{\partial v^+}{\partial \tau_k} \right).
\end{equation}
3.1. **Boundary values of the shape derivative.** We now want to obtain the boundary values of the shape derivatives $u'$. Since, by chain rule,

$$u' = \dot{u} - \Psi^V \cdot \nabla u$$

and $\Psi^V$ has compact support in $\Omega$, it is enough to get the boundary values of $\dot{u}$. Let us now consider the Neumann function $N$ with pole at the boundary of $\Omega$, that is, for $y \in \partial\Omega$ the unique solution to the boundary value problem

$$\begin{cases}
\text{div}(\sigma_0 \nabla N(\cdot, y)) = 0 \text{ in } \Omega, \\
\sigma_0 \frac{\partial N}{\partial n}(\cdot, y) = -\delta_y(\cdot) + \frac{1}{|\partial\Omega|} \text{ on } \partial\Omega,
\end{cases}$$

Let $y$ be a fixed point on $\partial\Omega$. It is well known that $N(\cdot, y)$ is in $W^{1,1}(\Omega)$. Then, since $\Psi^V$ has compact support in $\Omega$ and $P \subseteq \Omega_{d_k}$, it is possible to construct a sequence $v_m \in C^1(\Omega)$ that converges to $N(\cdot, y)$ in $W^{1,1}(\Omega)$ and in $C^1(\Omega_{d_k})$. Moreover since $\dot{u}$ is smooth near $\partial\Omega$ we can insert $v_m$ into (3.5) and pass to the limit, concluding that

$$u'(y) = \dot{u}(y) = \sum_{k=1}^{M_1} \int_{S_k} (\sigma^- - \sigma^+) \left( \frac{\sigma^+}{\sigma^-} w^+_n y, \cdot \right) + u^+_n N^+_n(y, \cdot) (\Psi^V \cdot n_k) ds,$$

which is the same formula we have in [6, Theorem 4.6] for $g = -\delta_y + \frac{1}{|\partial\Omega|}$.

3.2. **Derivative of the Neumann-to-Dirichlet map.** The Neumann-to-Dirichlet map is the operator $N_{\sigma_0} : H^{-1/2}_0(\partial\Omega) \rightarrow H^{1/2}_0(\partial\Omega)$, defined by

$$(3.18) \quad N_{\sigma_0}(f) = u|_{\partial\Omega},$$

where $H^s_0(\partial\Omega) = \{ f \in H^s(\partial\Omega) : \int_{\partial\Omega} f = 0 \}$, $g \in H^{-1/2}_0(\partial\Omega)$ and $u$ is the unique $H^1(\Omega)$ weak solution of the Dirichlet problem for the conductivity equation

$$\nabla \cdot (\sigma_0 \nabla u) = 0 \text{ on } \Omega, \quad \sigma_0 \frac{\partial u}{\partial n} \bigg|_{\partial\Omega} = f,$$

satisfying the normalization condition

$$\int_{\partial\Omega} u d\sigma = 0,$$

where $\nu$ is the outer normal of $\partial\Omega$.

Let $P$ denote a partition of vertices $Q = (Q_1, Q_2, \ldots, Q_N)$ and denote by $Q$ the subset of points $Q \in \Omega_{d_k}$ satisfying the assumptions stated at the beginning of Section 3. Let us denote by $s_{\sigma_0}$ the coefficient corresponding to the partition $P$ with vertices $Q = (Q_1, Q_2, \ldots, Q_N)$.

**Theorem 3.4.** The map $Q \rightarrow N_{\sigma_0}$ is differentiable. Moreover if $f, g \in H^{-1/2}_0(\partial\Omega)$ and $Q^t = Q + tV$, then

$$(3.20) \quad < g, \frac{dN_{\sigma_0}}{dt} \bigg|_{t=0} f > = \sum_{k=1}^{M_1} \int_{S_k} (\sigma^- - \sigma^+) \left( \frac{\sigma^+}{\sigma^-} w^+_n w^+_n + u^+_n w^+_n \right) (\Psi^V \cdot n_k) ds,$$

where $u, w \in H^1(\Omega)$ solve

$$\nabla \cdot (\sigma_0 \nabla u) = \nabla \cdot (\sigma_0 \nabla w) = 0 \text{ on } \Omega,$$

with $\int_{\partial\Omega} u = \int_{\partial\Omega} w = 0$ and with Neumann data $f$ and $g$ respectively.
Proof. Define the linear operator \( \tilde{L} : H^{-1/2}_0(\partial\Omega) \to H^{1/2}_0(\partial\Omega) \) by
\[
<g, \tilde{L}(f) > = -\int_{\Omega} \sigma_0 A \nabla w \nabla u
\]
where \( A \) is defined as in (3.3). Notice that \( \tilde{L} \) is linear in \( V \).

Let \( N_{\sigma_t} \) be the Neumann-to-Dirichlet map corresponding to the partition of vertices \( Q^t = Q + tV \). Since \( u_t = \tilde{u}_t \) on \( \partial\Omega \) and by Lemma 3.1, we have
\[
<g, N_{\sigma_t}(f) - N_{\sigma_0}(f) - t\tilde{L}(f) > = t < g, \frac{\tilde{u}_t - u}{t} - \dot{u} >
\]
hence, again by Lemma 3.1 and the trace theorem, we have
\[
\left| g, N_{\sigma_t}(f) - N_{\sigma_0}(f) - t\tilde{L}(f) \right| \leq C t \| g \|_{H^{-1/2}(\partial\Omega)} \left\| \frac{\tilde{u}_t - u}{t} - \dot{u} \right\|_{H^1(\Omega)}
\]
where \( C \) and \( \omega(t) \) do not depend on the \( V/\|V\| \). This implies that the Neumann-to-Dirichlet map is differentiable.

Moreover
\[
<g, \frac{dN_{\sigma_t}}{dt} \bigg|_{t=0} f > = -\int_{\Omega} \sigma_0 A \nabla w \nabla u.
\]
By Lemma 3.1, Proposition 3.2 and by Remark 3.3 we get (3.20).

The exact form of the derivative is particularly useful for studying stability for the inverse problem of recovering the conductivity \( \sigma_0 \) from knowledge of the Neumann-to-Dirichlet map. In particular, estimates from below of the derivative would lead to Lipschitz stability estimates. See for example [4] for the application of the corresponding formula for the derivative in the case of Helmholtz equation.

Remark 3.5. Proposition 3.2 holds true also in different assumptions on the geometry of the domain. For example if there is more than one polygonal partition \( P \) inside the domain (see Figure 2 on the left) or if the polygons are nested (see Figure 2 on the right). The only condition on the partition is that each vertex has positive distance from the boundary of \( \Omega \) and from the other vertexes and that there are no more that 3 sides intersecting at each vertex.

References


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