ON THE STOCHASTIC LIE ALGEBRA

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Abstract. We study the structure of the Lie algebra $\mathfrak{s}(n,\mathbb{R})$ corresponding to the so-called stochastic Lie group $\mathcal{S}(n,\mathbb{R})$. We obtain the Levi decomposition of the Lie algebra, classify Levi factor and classify the representation of the factor in $\mathbb{R}^n$. We discuss isomorphism of $\mathcal{S}(n,\mathbb{R})$ with the group of invertible affine maps $\mathit{Aff}(n-1,\mathbb{R})$. We prove that $\mathfrak{s}(n,\mathbb{R})$ is generated by two generic elements.

1. Stochastic Lie group and stochastic Lie algebra

Let $\mathcal{S}_0^+(n,\mathbb{R})$ denote the space of *transition matrices* of size n, i.e., the space of real $n\times n$ matrices with all entries non-negative and row sums equal to 1.

One important motivation for the study of such matrices is their relation to Markov processes: It is easy to see that for any Markov process X with n possible states, the family

$$P(s,t)=[p_{i,j}(s,t)]_{1\leq i,j\leq n}, \qquad 0\leq s\leq t<+\infty,$$

where $p_{i,j}(s,t)$ is the probability of $X_t=j$, conditional on $X_s=i$, is a family of transition matrices such that

$$P_{s,t}=P_{u,t}P_{s,u}, \qquad \forall 0\leq s\leq u\leq t<+\infty. \tag{1.1}$$

Conversely, the Kolmogorov extension theorem (see e.g. [2], Theorem IV.4.18), states that for every family $\left\{P(s,t)\in\mathcal{S}_0^+(n,\mathbb{R})\right\}_{0\leq s\leq t<+\infty}$ satisfying (1.1), there exists a Markov process X such that $p_{i,j}(s,t)=\Pr\left\{X_t=j|\,X_s=i\right\}$ for every $i,j\leq n$ and every $0\leq s\leq t<+\infty$.

Let $\mathcal{S}^+(n,\mathbb{R})$ denote the space of *nonsingular* transition matrices. It is clear that $\mathcal{S}_0^+(n,\mathbb{R})$ is a semigroup with respect to matrix multiplication, and $\mathcal{S}^+(n,\mathbb{R})$ is a subsemigroup. However, $\mathcal{S}^+(n,\mathbb{R})$ is *not* a group, since the inverse of a transition matrix is not, in general, a transition matrix.

The smallest group containing $\mathcal{S}^+(n,\mathbb{R})$ is denoted by $\mathcal{S}(n,\mathbb{R})$. Due to the considerations above, this is called the *stochastic group* [5]. It can be shown that

$$\mathcal{S}(n,\mathbb{R})=\left\{P\in\mathbb{R}^{n\times n}:\mathrm{Det}(P)\neq 0, P\mathbf{1}=\mathbf{1}\right\},$$

Manuel Guerra was partly supported by FCT/MEC through the project CEMAPRE UID/MULTI/00491/2013.

where $\mathbf{1}$ is the n-dimensional vector with all entries equal to 1. It follows that $\mathcal{S}(n,\mathbb{R})$, provided with the topology inherited from the usual topology of $\mathbb{R}^{n\times n}$, is a $n\times(n-1)$ dimensional analytic Lie group.

The Lie algebra of $\mathcal{S}(n,\mathbb{R})$ is called *stochastic Lie algebra*, and is denoted by $\mathfrak{s}(n,\mathbb{R})$. Notice that $\mathfrak{s}(n,\mathbb{R})$ is isomorphic to the tangent space of $\mathcal{S}(n,\mathbb{R})$ at the identity

$$\mathfrak{s}(n,\mathbb{R})\sim T_{Id}\mathcal{S}(n,\mathbb{R})=\left\{A\in\mathbb{R}^{n\times n}:A\mathbf{1}=0\right\},$$

$\mathfrak{s}(n,\mathbb{R})$ is provided with the matrix commutator $[A,B]=AB-BA$.

We introduce the subset

$$\mathfrak{s}^{+}(n,\mathbb{R})=\{A\in\mathfrak{s}(n,\mathbb{R}):a_{i,j}\geq 0,\ \forall i\neq j\}.$$

It is clear that $\mathfrak{s}^{+}(n,\mathbb{R})$ is not a subalgebra of $\mathfrak{s}(n,\mathbb{R})$, but it is a convex cone with nonempty interior in $\mathfrak{s}(n,\mathbb{R})$. Since $\mathcal{S}^{+}(n,\mathbb{R})$ is invariant under the flow by ODE's of type

$$\dot{P}_t=P_tA,$$

with $A\in\mathfrak{s}^{+}(n,\mathbb{R})$, it follows that $\mathcal{S}^{+}(n,\mathbb{R})$ has nonempty interior in $\mathcal{S}(n,\mathbb{R})$.

In [1], it is stated that the Levi decomposition

$$\mathfrak{s}(n,\mathbb{R})=\mathfrak{l}\oplus\mathfrak{r}, \tag{1.2}$$

has the following components:

a) The radical $\mathfrak{r}$ is the linear subspace generated by the matrices

$$\hat{R}_i=E_i(n)-E_n(n),\quad i=1,\dots,n-1,\qquad \hat{Z}=Id-\frac{1}{n}J_n, \tag{1.3}$$

where $E_i(n)$ are the matrices with the elements in the i-th column equal to 1 and all other elements equal to zero, J_n is the matrix with all elements equal to 1;

b) The Levi subalgebra $\mathfrak{l}$ is the linear subspace of real traceless matrices with all row and column sums equal to zero.

The result is correct but the respective proof of [1, Proposition 3.3] seems to contain a logical gap in what regards the semisimplicity of $\mathfrak{l}$ and the maximality of $\mathfrak{r}$.

In what follows, we present an orthonormal basis for $\mathfrak{s}(n,\mathbb{R})$ which has interesting properties with respect to the Lie algebraic structure of $\mathfrak{s}(n,\mathbb{R})$. In particular, it allows for the explicit computation of the Killing form and therefore we prove semisimplicity of $\mathfrak{l}$ by application of Cartan criterion. We also obtain the Dynkin diagram of $\mathfrak{l}$, showing that it is isomorphic to $\mathfrak{sl}(n-1,\mathbb{R})$.

2. Basis for the Lie algebra $\mathfrak{s}(n,\mathbb{R})$

Choose an orthonormal basis $v_1,\dots,v_{n-1}$ of the hyperplane

$$\Pi_n=\{x\in\mathbb{R}^n:\ x_1+\ldots+x_n=0\},$$

and set $v_0 = \frac{1}{\sqrt{n}}(1,\dots,1) \in \mathbb{R}^n$. Recall that for $a, b \in \mathbb{R}^n$, the dyadic product $a \otimes b$ is the matrix:

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \otimes \begin{pmatrix} b_1 & \cdots & b_n \end{pmatrix} = \begin{pmatrix} a_1 b_1 & \cdots & a_1 b_n \\ \vdots & & \vdots \\ a_n b_1 & \cdots & a_n b_n \end{pmatrix}.$$

The matrices

$$Z = \frac{1}{\sqrt{n-1}} \left(I_n - v_0 \otimes v_0 \right), \tag{2.1}$$

$$R_i = v_0 \otimes v_i, \qquad i = 1, \dots, n-1 \tag{2.2}$$

span the same linear subspace as the matrices (1.3).

We take the $(n-1)(n-2)$-dimensional linear subspace

$$\mathcal{A} = \operatorname{span}\left\{ A_{ij},\ i = 1, \dots n-1,\ j = 1, \dots, n-1,\ i \neq j \right\},$$

spanned by the rank-1 matrices

$$A_{ij} = v_i \otimes v_j. \tag{2.3}$$

Since $v_i \in \Pi$, there holds $v_0^*(v_i \otimes v_j) = (v_0 \cdot v_i) v_j^* = 0$. Similarly, $(v_i \otimes v_j) v_0 = 0$. Hence the matrices A_{ij} have zero row and column sums. Since $\operatorname{Tr}(v_i \otimes v_j) = v_i \cdot v_j = 0$, the matrices A_{ij} are traceless.

Now, consider the linear subspace

$$\mathcal{H} = \left\{ H = \sum_{\ell=1}^{n-1} \gamma_\ell (v_\ell \otimes v_\ell) \,\middle|\, \sum_{\ell=1}^{n-1} \gamma_\ell = 0 \right\}. \tag{2.4}$$

The row and column sums of each $(v_\ell \otimes v_\ell)$ are zero, and the trace of $H \in \mathcal{H}$ equals $\sum_{\ell=1}^{n-1} \gamma_\ell = 0$.

We set

$$\mathfrak{l} = \mathcal{A} \oplus \mathcal{H}. \tag{2.5}$$

We introduce a basis of $\mathcal{H}$:

$$H_k = \sum_{\ell=1}^{n-1} \gamma_\ell^k (v_\ell \otimes v_\ell), \qquad k = 1, \dots, (n-2), \tag{2.6}$$

where $\gamma^k = (\gamma_1^k, \dots, \gamma_{n-1}^k)$, $k = 1, \dots, (n-2)$, form an orthonormal basis for the subspace

$$\Pi_{n-1} = \left\{ x \in \mathbb{R}^{n-1} : x_1 + \dots + x_{n-1} = 0 \right\}.$$

Using the definition of dyadic product and elementary properties of the trace, it is straightforward to check that the matrices

$$Z, \quad R_i \ (i = 1, \dots, n-1),$$
$$A_{ij} \ (i, j = 1, \dots, n-1, i \neq j), \quad H_i \ (i = 1, \dots, n-2)$$

form an orthonormal system with respect to the matrix scalar product $\langle A, B\rangle = \mathrm{Tr}(AB^*)$.

The following Lemma presents the multiplication table for our basis. Its proof is accomplished by a direct computation.

Lemma 2.1. *For meaningful values of the indexes i, j, k, ℓ there holds:*

$$[Z, R_i] = \tfrac{-1}{n-1} R_i;$$
$$[Z, A_{ij}] = 0;$$
$$[Z, H_i] = 0;$$
$$[R_i, R_j] = 0;$$
$$[R_i, A_{j,k}] = \begin{cases} R_k, & \text{if } i = j, \\ 0, & \text{if } i \neq j; \end{cases}$$
$$[R_i, H_j] = \gamma_i^j R_i;$$
$$[A_{ij}, A_{k\ell}] = \begin{cases} (v_i \otimes v_i) - (v_j \otimes v_j) = \sum_{r=1}^{n-2} \left(\gamma_i^r - \gamma_j^r\right) H_r, & \text{if } i = \ell, j = k, \\ A_{i\ell}, & \text{if } i \neq \ell, j = k, \\ -A_{kj}, & \text{if } i = \ell, j \neq k, \\ 0, & \text{if } i \neq \ell, j \neq k; \end{cases}$$
$$[A_{ij}, H_k] = \left(\gamma_j^k - \gamma_i^k\right) A_{ij};$$
$$[H_i, H_j] = 0. \;\square$$

Remark 2.2. Lemma 2.1 shows that the orthogonal subspaces $\mathcal{A}$, $\mathcal{H}$ possess remarkable properties:

1. $\mathcal{H}$ is a Cartan subalgebra of $\mathfrak{l}$.
2. $[\mathcal{H}, \mathcal{A}] \subset \mathcal{A}$. The adjoint action of $\mathcal{H}$ on $\mathcal{A}$ is diagonal, for $H \in \mathcal{H}$:
$$\mathrm{ad}H A_{ij} = (\gamma_i - \gamma_j) A_{ij}.$$
3. $[A_{ij}, A_{ji}] = v_i \otimes v_i - v_j \otimes v_j = H_{ij} \in \mathcal{H}$.
4. For $i \neq j$, $\{A_{ij}, A_{ji}, [A_{ij}, A_{ji}]\}$ spans a 3-dimensional Lie subalgebra:
$$[H_{ij}, A_{ij}] = 2A_{ij}, \quad [H_{ij}, A_{ji}] = -2A_{ji}.$$
5. For any $(ij), (k\ell)$ the commutator $[A_{ij}, A_{k\ell}] = \mathrm{ad}A_{ij} A_{k\ell}$ is orthogonal to $A_{k\ell}$ with respect to the matrix scalar product. $\square$

3. Semisimplicity of $\mathfrak{l}$

In this section, we prove semisimplicity of $\mathfrak{l}$ by direct computation of the Killing form $\mathfrak{B}$.

Proposition 3.1. *The Killing form $\mathfrak{B}$ satisfies:*

i) $\mathfrak{B}(\mathcal{A}, \mathcal{H}) = 0$,
ii) $\mathfrak{B}(H_i, H_j) = 2(n-1)\langle H_i, H_j\rangle$, *for* $i, j = 1, \ldots, n-2$,
iii) $\mathfrak{B}(A_{ij}, A_{k\ell}) = \begin{cases} 0, & \text{if } (i,j) \neq (\ell, k), \\ 2(n-1), & \text{if } (i,j) = (\ell, k). \;\square \end{cases}$

According to Cartan criterion for semisimplicity, we get

Corollary 3.2. *The Killing form $\mathfrak{B}$ is non-degenerate and the algebra $\mathfrak{l}$ is semisimple.* □

Proof of Proposition 3.1. (i) Take A_{ij}, H_k from the basis of $\mathcal{A}$ and $\mathcal{H}$, respectively.

Since $\mathcal{H}$ is Abelian, $(\mathrm{ad}A_{ij}\mathrm{ad}H_k)_{\mathcal{H}} = 0$.

Due to Lemma 2.1, for any $A_{\ell m}$, $\mathrm{ad}A_{ij}\mathrm{ad}H_k A_{\ell m} = C\mathrm{ad}A_{ij}A_{\ell m}$. By property 5 in Remark 2.2, the last matrix is orthogonal to $A_{\ell m}$ and therefore the trace of the restriction $(\mathrm{ad}A_{i,j}\mathrm{ad}H_k)|_{\mathcal{A}}$ is null, and we can conclude that $\mathfrak{B}(\mathcal{A},\mathcal{H}) = 0$.

(ii) Choose $H_k, H_\ell \in \mathcal{H}$. As far as $(\mathrm{ad}H_k\mathrm{ad}H_\ell)|_{\mathcal{H}} = 0$, we only need to compute the trace of $(\mathrm{ad}H_k\mathrm{ad}H_\ell)|_{\mathcal{A}}$.

By Lemma 2.1, $\mathrm{ad}H_k\mathrm{ad}H_\ell A_{ij} = \mathrm{ad}H_k(\gamma_i^\ell - \gamma_j^\ell)A_{ij} = (\gamma_i^\ell - \gamma_j^\ell)(\gamma_i^k - \gamma_j^k)A_{ij}$. Hence,

$$\begin{aligned}\mathfrak{B}(H_k, H_\ell) = &\sum_{i,j}(\gamma_i^\ell - \gamma_j^\ell)(\gamma_i^k - \gamma_j^k) = \\ =&(n-1)\sum_i \gamma_i^\ell\gamma_i^k - \sum_i \gamma_i^\ell \sum_j \gamma_j^k - \sum_j \gamma_j^\ell \sum_i \gamma_i^k + (n-1)\sum_j \gamma_j^\ell\gamma_j^k.\end{aligned}$$

Since $\sum\limits_i \gamma_i^k = 0$, it follows that

$$\mathfrak{B}(H_k, H_\ell) = 2(n-1)\sum_i \gamma_i^\ell\gamma_i^k = 2(n-1)\langle H_k, H_\ell\rangle.$$

(iii) Pick $A_{ij}, A_{k\ell}$. For every H_m

$$\mathrm{ad}A_{ij}\mathrm{ad}A_{k\ell}H_m = \mathrm{ad}A_{ij}(\gamma_\ell^m - \gamma_k^m)A_{k\ell}, \tag{3.1}$$

lies in $\mathcal{A}$ whenever $(k,\ell) \neq (j,i)$. This implies

$$\mathrm{Tr}\ (\mathrm{ad}A_{ij}\mathrm{ad}A_{k\ell})|_{\mathcal{H}} = 0,\ \text{ for } (k,\ell) \neq (j,i).$$

To compute $\mathrm{Tr}\ (\mathrm{ad}A_{ij}\mathrm{ad}A_{k\ell})|_{\mathcal{A}}$, notice that

$$\begin{aligned}\langle A_{\alpha\beta}, \mathrm{ad}A_{ij}\mathrm{ad}A_{k\ell}A_{\alpha\beta}\rangle &= v_\alpha^*\left(A_{ij}\mathrm{ad}A_{k\ell}A_{\alpha\beta} - (\mathrm{ad}A_{k\ell}A_{\alpha\beta})A_{ij}\right)v_\beta = \\ =&(v_\alpha\cdot v_i)v_j^*\left(A_{k\ell}A_{\alpha\beta} - A_{\alpha\beta}A_{ij}\right)v_\beta - (v_\beta\cdot v_j)v_\alpha^*\left(A_{k\ell}A_{\alpha\beta} - A_{\alpha\beta}A_{k\ell}\right)v_i.\end{aligned}$$

Since $i \neq j$ and $k \neq \ell$, $v_j^*A_{\alpha\beta}A_{ij}v_\beta = v_\alpha^*A_{k\ell}A_{\alpha\beta}v_i = 0$, and therefore

$$\begin{aligned}&\langle A_{\alpha\beta}, \mathrm{ad}A_{ij}\mathrm{ad}A_{k\ell}A_{\alpha\beta}\rangle = \\ &=(v_j\cdot v_k)(v_i\cdot v_\alpha)(v_\ell\cdot v_\alpha) + (v_i\cdot v_\ell)(v_j\cdot v_\beta)(v_k\cdot v_\beta),\end{aligned}\tag{3.2}$$

which is zero whenever $(k,\ell) \neq (j,i)$.

For $(k,\ell) = (j,i)$, the equality (3.1) and Lemma 2.1 yield

$$\begin{aligned}\langle H_m, \mathrm{ad}A_{ij}\mathrm{ad}A_{ji}H_m\rangle =&(\gamma_i^m - \gamma_j^m)\langle H_m, \mathrm{ad}A_{ij}A_{ji}\rangle = \\ =&(\gamma_i^m - \gamma_j^m)\langle H_m, v_i\otimes v_i - v_j\otimes v_j\rangle = (\gamma_i^m - \gamma_j^m)^2,\end{aligned}$$

and $\mathrm{Tr}\ (\mathrm{ad}A_{ij}\mathrm{ad}A_{ji})|_{\mathcal{H}} = \sum\limits_{m=1}^{n-2}(\gamma_i^m - \gamma_j^m)^2$.

To compute the last expression, let us form the matrix

$$\Gamma = \begin{pmatrix} \gamma_1^1 & \cdots & \gamma_1^{n-2} \\ \vdots & & \vdots \\ \gamma_{n-1}^1 & \cdots & \gamma_{n-1}^{n-2} \end{pmatrix}. \tag{3.3}$$

Then $\Gamma\Gamma^*$ is the matrix of the orthogonal projection of $\mathbb{R}^{n-1}$ onto the subspace Π_{n-1}. Take a standard basis $e_1, \dots, e_{n-1}$ in $\mathbb{R}^{n-1}$, and note that $e_i - e_j \in \Pi_{n-1}$. Then

$$\begin{aligned} \mathrm{Tr}\,(\mathrm{ad}A_{ij}\mathrm{ad}A_{ji})|_{\mathcal{H}} &= \sum_{m=1}^{n-2}(\gamma_i^m - \gamma_j^m)^2 = \\ = (e_i - e_j)^*\Gamma\Gamma^*(e_i - e_j) &= (e_i - e_j)^*(e_i - e_j) = 2. \end{aligned}$$

In what regards $\mathrm{Tr}\,(\mathrm{ad}A_{ij}\mathrm{ad}A_{ji})|_{\mathcal{A}}$, then by (3.2):

$$\langle A_{\alpha\beta}, \mathrm{ad}A_{ij}\mathrm{ad}A_{ji}A_{\alpha\beta}\rangle = (v_i \cdot v_\alpha) + (v_j \cdot v_\beta).$$

Hence,

$$\mathrm{Tr}\,(\mathrm{ad}A_{ij}\mathrm{ad}A_{ji})|_{\mathcal{A}} = \sum_{\substack{\alpha,\beta \le n-1 \\ \alpha \ne \beta}} ((v_i \cdot v_\alpha) + (v_j \cdot v_\beta)) = 2(n-2),$$

and therefore $\mathrm{Tr}\,(\mathrm{ad}A_{ij}\mathrm{ad}A_{ji}) = 2(n-1)$. □

4. Classification of the Levi subalgebra $\mathfrak{l}$

Now we wish to prove the following result concerning the type of the semisimple subalgebra $\mathfrak{l}$.

Theorem 4.1. *The Levi subalgebra $\mathfrak{l}$ is isomorphic to the special linear Lie algebra $\mathfrak{sl}(n-1, \mathbb{R})$.* □

Proof. As stated in Remark 2.2, $\mathcal{H}$ is a Cartan subalgebra of $\mathfrak{l}$. From Lemma 2.1, we see that the nonzero characteristic functions of $\mathfrak{l}$ with respect to $\mathcal{H}$ are the linear functionals $\alpha_{ij} : \mathcal{H} \mapsto \mathbb{R}$ such that

$$\alpha_{ij}(H_k) = \gamma_i^k - \gamma_j^k, \qquad \text{for } 1 \le k \le n-2, \quad 1 \le i, j \le n-1, \quad i \ne j,$$

and the corresponding characteristic spaces are

$$\mathcal{A}_{ij} = \{tA_{ij} : t \in \mathbb{R}\} \qquad 1 \le i, j \le n-1, \quad i \ne j.$$

Thus, $\mathfrak{l}$ is split as

$$\mathfrak{l} = \mathcal{H} \oplus \bigoplus_{i \ne j} \mathcal{A}_{ij}.$$

Hence the set, $\mathcal{R} = \{\alpha_{ij} : 1 \le i \le n-1,\ 1 \le j \le n-1,\ i \ne j\}$ is a root system of $\mathfrak{l}$.

Since the Killing form restricted to $\mathcal{H}$ is diagonal, the dual space $\mathcal{H}^*$ is provided with the inner product uniquely defined by

$$\langle \alpha_{ij}, \alpha_{\ell,m} \rangle = \sum_{k=1}^{n-2} \left(\gamma_i^k - \gamma_j^k \right) \left(\gamma_\ell^k - \gamma_m^k \right) = (e_i - e_j)^* (e_\ell - e_m)$$

for every $\alpha_{ij}, \alpha_{\ell m} \in \mathcal{R}$. Thus, $\mathcal{R}$ is isomorphic to the root system

$$\mathcal{E} = \{ e_i - e_j : 1 \leq i \leq n-1,\ 1 \leq j \leq n-1,\ i \neq j \}$$

on the hyperplane Π_{n-1}. Since

$$e_\ell - e_m = \begin{cases} \sum\limits_{i=\ell}^{m-1} (e_i - e_{i+1}), & \text{if } \ell < m, \\ \sum\limits_{i=m}^{\ell-1} -(e_i - e_{i+1}), & \text{if } \ell > m, \end{cases}$$

it follows that the set $\Delta = \left\{ \alpha_{12}, \alpha_{23}, \alpha_{34}, \ldots, \alpha_{(n-2)(n-1)} \right\}$ is a system of positive simple roots. Further,

$$\left\langle \alpha_{i(i+1)}, \alpha_{i(i+1)} \right\rangle = 2 \qquad 1 \leq i \leq n-2,$$

$$2 \frac{\left\langle \alpha_{i(i+1)}, \alpha_{j(j+1)} \right\rangle}{\left\langle \alpha_{i(i+1)}, \alpha_{i(i+1)} \right\rangle} = \begin{cases} -1 & \text{if } |i-j| = 1, \\ 0 & \text{if } |i-j| > 1. \end{cases}$$

Thus, the Dynkin diagram of $\mathfrak{l}$ is of type A_{n-2}, and therefore, $\mathfrak{l}$ is isomorphic to $\mathfrak{sl}(n-1, \mathbb{R})$ (see, e.g., [6, Chapter 14]). □

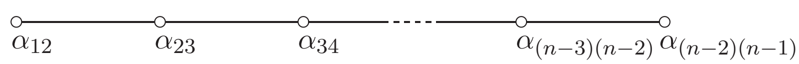

Figure 1. Dynkin diagram of $\mathfrak{l}$

5. Representation of the Levi factor $\mathfrak{l}$ in $V = \mathbb{R}^n$

Considering $\mathfrak{l}$ as a subalgebra of the stochastic (matrix) algebra $\mathfrak{s}(n, \mathbb{R})$ defines its representation $\phi : \mathfrak{l} \mapsto \mathfrak{gl}(n)$ in $V = \mathbb{R}^n$. To characterize it, let us pick the basis $v_0, v_1, \ldots, v_{n-1}$, introduced in Section 2, and consider the matrix $M \in \mathbb{R}^{n \times n}$: $M = \left(\begin{array}{c|c|c|c} v_0 & v_1 & \cdots & v_{n-1} \end{array} \right)$.

By construction, M is orthogonal and the mapping

$$\forall y \in \mathfrak{l} : \qquad y \mapsto M^* \phi(y) M,$$

defines an isomorphic representation of $\mathfrak{l}$ in $V = \mathbb{R}^n$.

Note that the subspace $V_1 = \operatorname{span}\{v_1, \ldots, v_{n-1}\}$ is invariant under $\phi(\mathfrak{l})$ and therefore we get:

$$\forall y \in \mathfrak{l} : \qquad M^* \phi(y) M = \begin{pmatrix} 0 & 0 \\ 0 & M_1^* \phi(y) M_1 \end{pmatrix}, \tag{5.1}$$

where $M_1 = \left(\begin{array}{c|c|c|c} v_1 & v_2 & \cdots & v_{n-1} \end{array} \right) \in \mathbb{R}^{n \times (n-1)}$.

The mapping

$$y \mapsto \phi_1(y) = M_1^* \phi(y) M_1$$

is a faithful representation of $\mathfrak{l}$ in $V_1 = \mathbb{R}^{n-1}$.

Formula (5.1) identifies the representation of the semisimple Levi factor $\mathfrak{l}$ in $\mathbb{R}^n$ by stochastic matrices with a direct sum of the faithful representation ϕ_1 in $\mathbb{R}^{n-1}$ and the null 1-dimensional representation.

Besides

$$\begin{aligned} M_1^* A_{ij} M_1 &= e_i \otimes e_j \qquad \text{for } i,j \in \{1,2,\dots,n-1\},\ i \neq j, \\ M_1^* H_i M_1 &= \mathrm{diag}(\gamma^i) \qquad \text{for } i = 1,2,\dots,n-2. \end{aligned}$$

Therefore ϕ_1 maps isomorphically the Cartan subalgebra $\mathcal{H}$ onto the space of traceless diagonal $(n-1)\times(n-1)$ matrices, while $\phi_1(\mathcal{A})$ coincides with the space of $(n-1)\times(n-1)$ matrices with vanishing diagonal.

6. Affine group and affine Lie algebra

It is noticed in [5] that the group of $\mathcal{S}(n,\mathbb{R})$ is isomorphic to the group $\mathit{Aff}(n-1,\mathbb{R})$ of the affine maps $S : x \to Ax + B,\ x \in \mathbb{R}^{n-1}$. We wish to discuss this relation, in the light of the results obtained above. We also discuss the relation between the elements of $\mathcal{S}(n,\mathbb{R})$ and finite state space Markov processes outlined in Section 1.

Let $(\mathbb{R}^n)^*$ be the dual of $\mathbb{R}^n$. As usual, elements of $\mathbb{R}^n$ are identified with column vectors, and elements of $(\mathbb{R}^n)^*$ are identified with row vectors. Further, we identify any vector $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ with the function $x : i \mapsto x_i$, with the domain $D_n = \{1,2,\dots,n\}$, and identify any dual vector $p = (p_1, p_2, \dots, p_n) \in (\mathbb{R}^n)^*$ with the (signed) measure p on the set D_n such that $p\{i\} = p_i$, for $i = 1,2,\dots,n$. Thus, the product px is identified with the integral $\int_{D_n} x\, dp$.

Each $S \in \mathcal{S}(n,\mathbb{R})$ can be identified either with the linear endomorphism of $\mathbb{R}^n$, $S : x \mapsto Sx$ or with the linear endomorphism of $(\mathbb{R}^n)^*$, $S : p \mapsto pS$.

Let Y be a D_n-valued Markov process and $S \in \mathcal{S}^+(n,\mathbb{R})$ be defined by $s_{ij} = \Pr\{Y_t = j |\, Y_s = i\}$ for every $i,j \in D_n$ ($0 \le s \le t < +\infty$, fixed). Then the vector Sx is identified with the function $i \mapsto \mathbb{E}[x(Y_t)|\, Y_s = i]$, while the covector pS is identified with the probability law of Y_t assuming the probability law of Y_s is p.

For every $S \in \mathcal{S}(n,\mathbb{R})$, the map $p \mapsto pS$ preserves each affine space of the form $\{p \in (\mathbb{R}^n)^* : p\mathbf{1} = C\}$ ($C \in \mathbb{R}$, fixed), which is the space of signed measures on D_n such that $p(D_n) = C$. Note that, $\{t\mathbf{1} : t \in \mathbb{R}\}$ is the unique affine (linear) proper subspace of $\mathbb{R}^n$ which is preserved by all the maps $x \mapsto Sx$ with $S \in \mathcal{S}(n,\mathbb{R})$.

Now, consider the group of invertible affine maps $S : q \mapsto qA + B$, $q \in \left(\mathbb{R}^{n-1}\right)^*$.[1] The group can be identified with the subgroup $\mathbf{A}\left(\left(\mathbb{R}^{n-1}\right)^*\right)$ of $GL\left((\mathbb{R}^n)^*\right)$:

$$\mathbf{A}\left(\left(\mathbb{R}^{n-1}\right)^*\right) = \left\{ \begin{pmatrix} 1 & B \\ 0 & A \end{pmatrix} \middle| \; A \in \mathbb{R}^{(n-1)\times(n-1)} \text{ is nonsingular} \right\}.$$

The Lie algebra $\mathfrak{a}\left(\left(\mathbb{R}^{n-1}\right)^*\right)$ of $\mathbf{A}\left(\left(\mathbb{R}^{n-1}\right)^*\right)$ consists of matrices

$$\begin{pmatrix} 0 & B \\ 0 & A \end{pmatrix}.$$

Now, fix $S \in \mathcal{S}(n, \mathbb{R})$. By the results of Section 2, S can be written as

$$S = \beta_0 Z + \sum_{i=1}^{n-1} \beta_i R_i + A,$$

with $\beta_0, \beta_1, \dots, \beta_{n-1} \in \mathbb{R}$, $A \in \mathfrak{l}$. Taking into account that

$$Zv_0 = 0,$$

$$Zv_i = \frac{1}{\sqrt{n-1}} v_i, \quad R_i v_j = \begin{cases} v_0, & \text{if } j = i, \\ 0, & \text{if } j \neq i, \end{cases} \qquad \text{for } i = 1, 2, \dots, n-1,$$

we get

$$M^* S M = \begin{pmatrix} 0 & \beta^* \\ 0 & M_1^* A M_1 + \frac{\beta_0}{\sqrt{n-1}} Id \end{pmatrix},$$

where $\beta^* = (\beta_1, \beta_2, \dots, \beta_{n-1})$. Thus, the similarity $S \mapsto M^* S M$ is an isomorphism from $\mathcal{S}(n, \mathbb{R})$ into $\mathfrak{a}\left(\left(\mathbb{R}^{n-1}\right)^*\right)$. In particular, the radical of $\mathfrak{a}\left(\left(\mathbb{R}^{n-1}\right)^*\right)$ is the linear space of matrices

$$\begin{pmatrix} 0 & \beta^* \\ 0 & \beta_0 Id \end{pmatrix}, \qquad \beta_0, \beta_1, \dots, \beta_{n-1} \in \mathbb{R},$$

while the Levi subalgebra of $\mathfrak{a}\left(\left(\mathbb{R}^{n-1}\right)^*\right)$ consists of matrices

$$\begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}, \qquad A \in \mathfrak{sl}(n-1, \mathbb{R}).$$

Thus, the Levi splitting of $\mathfrak{a}\left(\left(\mathbb{R}^{n-1}\right)^*\right)$ corresponds to two connected Lie subgroups of $\mathbf{A}\left(\left(\mathbb{R}^{n-1}\right)^*\right)$: The subgroup generated by the translations and rescalings of $\left(\mathbb{R}^{n-1}\right)^*$, and the subgroup of orientation and volume preserving linear transformations in $\left(\mathbb{R}^{n-1}\right)^*$.

[1] The mapping

$$q = (q_1, q_2, \dots, q_{n-1}) \mapsto \left(C - \sum_{i=1}^{n-1} q_i, q_1, q_2, \dots, q_{n-1} \right)$$

coordinatizes the affine subspace $\{p \in (\mathbb{R}^n)^* : p\mathbf{1} = C\}$.

7. Minimal number of generators of $\mathfrak{s}(n,\mathbb{R})$

Finally we prove

Theorem 7.1. *The Lie algebra $\mathfrak{s}(n,\mathbb{R})$ is generated by two matrices.* □

The argument in our proof is an adaptation of the argument used in [4] to prove that every semisimple Lie algebra is generated by two elements. We will use the following lemma:

Lemma 7.2. *For every integer $n \geq 2$ there is a vector $\gamma \in \mathbb{R}^n$ such that*

a) $\sum_{i=1}^{n} \gamma_i = 0$;
b) $\gamma_i \neq 0, \qquad i = 1, \ldots, n$;
c) $\gamma_i \neq \gamma_j, \qquad \forall i, j \in \{1, \ldots, n\}, i \neq j$;
d) $\gamma_i - \gamma_j \neq \gamma_k - \gamma_\ell, \qquad \forall i, j, k, \ell \in \{1, \ldots, n\}, i \neq j, k \neq \ell, (i,j) \neq (k,\ell)$.

For every γ satisfying (a)–(d) and every $\lambda \in \mathbb{R} \setminus \{0\}$, $\lambda\gamma$ satisfies (a)–(d). □

Proof. For $n = 2$, the Lemma holds with $\gamma = (1, -1)$.

Suppose that the Lemma holds for some $n \geq 2$, and fix $\gamma \in \mathbb{R}^n$ satisfying *(a)–(d)*. Let

$$\tilde{\gamma} = (\gamma_1, \ldots, \gamma_{n-1}, \gamma_n - \varepsilon, \varepsilon) .$$

Since there are only finitely many values of ε such that $\tilde{\gamma}$ fails at least one condition *(a)–(d)*, we see that the Lemma holds for $n + 1$.

The last statement in the Lemma is obvious, since the equations in conditions *(a)–(d)* are homogeneous. □

Proof of Theorem 7.1. Pick a vector $\gamma \in \mathbb{R}^{n-1}$ satisfying conditions *(a)–(d)* of Lemma 7.2, let Γ be the matrix (3.3), and $\beta = (\beta_1, \ldots, \beta_{n-2}) = \gamma^T\Gamma$. Let Z, R_i, A_{ij}, H_i be elements of our basis of $\mathfrak{s}(n,\mathbb{R})$, and consider the matrices

$$X = Z + \sum_{k=1}^{n-2} \beta_k H_k, \qquad Y = R_1 + \sum_{i \neq j} A_{ij}.$$

Using the Lemma 2.1, we obtain

$$\begin{aligned}\mathrm{ad}XY =&[Z, R_1] + \sum_{i \neq j}[Z, A_{ij}] + \sum_{k=1}^{n-2} \beta_k[H_k, R_1] + \sum_{k=1}^{n-2} \beta_k[H_k, A_{ij}] = \\ =&\frac{-1}{n-1}R_1 + 0 - \gamma_1 R_1 + \sum_{i \neq j}(\gamma_i - \gamma_j)A_{ij} = \\ =&-\left(\frac{1}{n-1} + \gamma_1\right) R_1 + \sum_{i \neq j}(\gamma_i - \gamma_j)A_{ij}.\end{aligned}$$

Multiplying γ by an appropriate non zero constant we can make $\gamma_1 = \frac{-1}{n-1}$, and thus

$$\mathrm{ad}XY = \sum_{i \neq j} (\gamma_i - \gamma_j) A_{ij}.$$

Iterating, we see that

$$\mathrm{ad}^k XY = \sum_{i \neq j} (\gamma_i - \gamma_j)^k A_{ij} \qquad \forall k \in \mathbb{N}.$$

Let $m = (n-1)(n-2)$. Since

$$\det \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & \gamma_1 - \gamma_2 & \cdots & \gamma_{n-1} - \gamma_{n-2} \\ 0 & 0 & (\gamma_1 - \gamma_2)^2 & \cdots & (\gamma_{n-1} - \gamma_{n-2})^2 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & (\gamma_1 - \gamma_2)^m & \cdots & (\gamma_{n-1} - \gamma_{n-2})^m \end{pmatrix} \neq 0,$$

we see that the matrices $X, Y, \mathrm{ad}XY, \ldots, \mathrm{ad}^m XY$ span the same subspace as the matrices $X, R_1, A_{ij},\ i, j \leq n-1, i \neq j$, and this subspace lies in $\mathfrak{Lie}\{X, Y\}$, the Lie algebra generated by X, Y.

By the Lemma 2.1, $[R_1, A_{1i}] = R_i$, for $i = 1, 2, \ldots, n-1$. Hence

$$\{R_2, \ldots, R_{n-1}\} \subset \mathfrak{Lie}\{X, Y\}.$$

Finally, also by the Lemma 2.1, $[A_{ij}, A_{ji}] = \sum_{r=1}^{n-2} (\gamma_i^r - \gamma_j^r) H_r$. This implies that $[A_{1j}, A_{j1}],\ j = 2, \ldots, n-1$ are $n-2$ linearly independent elements of $\mathcal{H}$. Hence, $\mathcal{H} \subset \mathfrak{Lie}\{X, Y\}$ and $Z \in \mathfrak{Lie}\{X, Y\}$. □

8. Acknowledgement

The authors are grateful to A.A.Agrachev for stimulating remarks.

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