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# NEARLY PARALLEL $G_2$ -STRUCTURES WITH LARGE SYMMETRY GROUP

FABIO PODESTÀ

ABSTRACT. We prove the existence of a one-parameter family of nearly parallel  $G_2$ -structures on the manifold  $S^3 \times \mathbb{R}^4$ , which are mutually non isomorphic and invariant under the cohomogeneity one action of the group  $SU(2)^3$ . This family connects the two locally homogeneous nearly parallel  $G_2$ -structures which are induced by the homogeneous ones on the sphere  $S^7$ .

## 1. INTRODUCTION

A nearly parallel  $G_2$ -structure (NP-structure for brevity) on a 7-dimensional manifold  $M$  is given by a positive 3-form  $\varphi \in \Omega^3(M)$  such that  $d\varphi = \lambda *_{\varphi} \varphi$  for some (non-zero)  $\lambda \in \mathbb{R}$ , where  $*_{\varphi}$  denotes the Hodge star operator relative to the associated Riemannian metric  $g$ . The name “nearly parallel” comes from the fact that only a 1-dimensional component of  $\nabla\varphi$  is different from zero (see [14]), where  $\nabla$  is the Levi Civita connection of  $g$ , and these structures are also said to have weak holonomy  $G_2$ , where this terminology goes back to Gray ([19]). The Riemannian manifold  $(M, g)$  is irreducible Einstein with scalar curvature given by  $\frac{21}{8}\lambda^2$  and the existence of an NP-structure is equivalent to the existence of a spin structure with a non-zero Killing spinor as well as to the existence of a torsion-free  $\text{Spin}(7)$ -structure on the cone  $C(M) := \mathbb{R}^+ \times M$  inducing the cone metric  $dr^2 + r^2g$  (see [4]). More precisely, an NP-structure on a compact simply connected manifold  $M$  will be called *proper* if the cone metric on  $C(M)$  has full holonomy  $\mathcal{H} = \text{Spin}(7)$ , or equivalently if the space of Killing spinors is one-dimensional. When the NP-structure is not proper and the metric  $g$  has not constant curvature, the holonomy  $\mathcal{H}$  reduces either to  $SU(4)$  or further to  $Sp(2)$ , corresponding to the existence of a Sasakian (but not 3-Sasakian) and a 3-Sasakian structure on  $M$  respectively. It is known (see [18]) that any 3-Sasakian manifold admits a second NP-structure which is proper and the squashed sphere  $S^7$  is an example of this situation.

In some sense NP-structures are a seven-dimensional analogue of nearly Kähler structures in six dimensions, which are automatically Einstein and admit a Killing spinor. Actually the cone metric on the cone over a six-dimensional strict nearly Kähler manifold  $N$  has holonomy inside  $G_2$  and moreover for both nearly Kähler and NP-structures their canonical metric connections  $\bar{\nabla}$  have  $\bar{\nabla}$ -parallel, totally skew-symmetric torsion. It is also known that given a six-dimensional strict nearly Kähler manifold  $N$ , the cone  $C(N)$  endowed with the sine-cone metric has an NP-structure (see e.g. [6]).

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In order to find possibly new examples, it is very natural to investigate manifolds endowed with special structures, as nearly Kähler or NP-structures, whose full automorphism group is ample. The classification of compact homogeneous NP-structures was achieved in [18], where also many useful results were proved on the full automorphism group, while later in [11] the classification of compact homogeneous nearly Kähler six-dimensional manifolds was obtained. In [24],[25] the study of compact six-dimensional nearly Kähler manifolds which admit a compact Lie group of automorphisms with generic orbits of codimension one was initiated and more recently Foscolo and Haskins ([16]) proved the existence of completely new, inhomogeneous nearly Kähler structures on the sphere  $S^6$  and on  $S^3 \times S^3$ , invariant under the cohomogeneity one action of the group  $SU(2) \times SU(2)$ . As for NP-structures, Cleyton and Swann ([12]) classified all manifolds which carry such a structure with a simple Lie group of automorphisms acting by cohomogeneity one; in strong contrast to the homogeneous case, they found that the standard sphere  $S^7$  and  $\mathbb{R}P^7$  acted on by the exceptional Lie group  $G_2$  are the only complete examples.

In this work we investigate the existence of  $G$ -invariant NP-structures on the manifold  $M \cong S^3 \times \mathbb{R}^4$ , which admits a cohomogeneity one (almost effective) action of the group  $G = SU(2)^3$ . The manifold  $M$  can be realized as the complement  $M = S^7 \setminus \Sigma$ , where  $\Sigma \cong S^3$  is one of the two singular orbits for a cohomogeneity one action of  $G$  on  $S^7$ , and it is special in the sense that it already admits a complete  $G$ -invariant metric with full holonomy  $G_2$ , namely the well-known example constructed by Bryant and Salamon ([10]) on the spin bundle over  $S^3$ . The group  $G$  appears in the list of possible groups which can act by cohomogeneity one preserving a  $G_2$ -structure ([12]) and actually it is (locally) isomorphic to the full isometry group of the Bryant-Salamon metric. Moreover, in view of the results in [18], the automorphism group of an NP-structure on a compact manifold acts transitively on it whenever its dimension is at least 10, so that the group  $G$  has the highest possible dimension to allow non-homogeneous examples. Principal  $G$ -orbits are diffeomorphic to  $Y := S^3 \times S^3$  and the non-trivial isotropy representation of a principal isotropy subgroup allows to easily determine the space of invariant 2- and 3-forms on  $M$ . A  $G$ -invariant NP-structure on  $M_o \cong \mathbb{R}^+ \times Y$  given by a 3-form  $\varphi$  induces a family of so called nearly half-flat  $G$ -invariant  $SU(3)$ -structures  $(\omega, \psi_+, \psi_-)$  on  $Y$  (see [15]); the 2-form  $\omega$  is forced to lie in a one-dimensional subspace of invariant 2-forms on  $Y$  and when these  $SU(3)$ -structures are all nearly Kähler structures on  $Y$  we obtain the well-known example of the sine-cone over the homogeneous nearly Kähler manifold  $Y$  (see [5],[15]).

In our main result Theorem 4.1 we prove the existence of a one-parameter family  $\mathcal{F}_a$  ( $a \in \mathbb{R}^+$ ) of  $G$ -invariant NP-structures on  $M$ , mutually non isomorphic, connecting the two locally homogeneous NP-structures on  $M$  induced by the known homogeneous NP-structures on  $S^7$ ; the parameter  $a \in \mathbb{R}^+$  gives a measure of the size of the singular orbit  $S^3$ . The problem of understanding which of these structures extends over a  $G$ -equivariant compactification  $\overline{M}$  is unsolved, albeit there is some numerical evidence that no such structure might exist besides the homogeneous ones. In case a global  $G$ -invariant NP-structure on  $S^7$  should exist, we prove that it would be proper and distinct from any of the Einstein metrics of cohomogeneity one on  $S^7$  found by Böhm ([7]). One might expect to find more invariant NP-structures by reducing the group to  $SU(2)^2 \times U(1)$ , by analogy with what happens for

$G_2$ -holonomy metrics on  $M$  (see the recent results in [17]), or further to  $SU(2)^2$  and this can be the object of further investigations.

The work is structured as follows. In the second section we describe the manifold  $M$  together with the  $G$ -action as well as all the  $G$ -invariant  $G_2$ -structures. In section 3 we write down the equations defining the  $G$ -invariant NP-structures. We continue describing the special solutions to the system (3.3) given by the sine-cone construction over the nearly Kähler homogeneous manifold  $S^3 \times S^3$  and by the two well-known homogeneous NP-structures on  $S^7$ . We then analyze the symmetries of the system (3.3), providing (Prop.3.5) the existence of a two-dimensional family of mutually non isomorphic and non locally homogeneous NP-structures on an open tubular neighborhood of a  $G$ -principal orbit. In the last subsection of section 3, we give sufficient and necessary conditions on the solutions of the system (3.3) on the regular part so that the corresponding NP-structures extend smoothly to an NP-structure on the whole  $M$ . In the last section we prove our main Theorem 4.1 and the main properties of a global solution in Prop.4.6.

**Notation.** Lie groups and their Lie algebras will be indicated by capital and gothic letters respectively. Given a Lie group  $L$  acting on a manifold  $N$ , for every  $X \in \mathfrak{l}$  we will denote by  $\hat{X}$  the corresponding vector field on  $N$  induced by the one-parameter subgroup  $\exp(tX)$ .

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## 2. PRELIMINARIES

In this section we first consider the non-compact 7-dimensional manifold  $M$  together with the action of the group  $G \cong SU(2)^3$  with generic orbits of codimension one. We will then describe the space of all  $G$ -invariant  $G_2$ -structures on  $M$ .

**2.1. The manifold  $M$  and the group action of  $G$ .** We start with the standard (almost effective) action of the compact group  $U = Sp(2) \times Sp(1)$  on  $\mathbb{H}^2$  given by  $(A, q) \cdot v = Av\bar{q}$ , where  $(A, q) \in U$  and  $v \in \mathbb{H}^2$ . The sphere  $S^7 \subset \mathbb{H}^2$  can be written as the quotient space  $U/K^+$  with  $K^+ = \{(\text{diag}(q, q'), q) \in U\} \cong Sp(1) \times Sp(1)$  being the isotropy subgroup at the point  $e_1 = (1, 0) \in \mathbb{H}^2$ .

We consider the action of  $G := \{(\text{diag}(q_1, q_2), q_3) \in U \mid q_1, q_2, q_3 \in Sp(1)\} \cong Sp(1)^3$  on  $S^7$ . The curve  $\gamma : t \mapsto (\cos t, \sin t) \in S^7$  is transverse to the  $G$ -orbits and we easily see that

$$G_{\gamma(t)} = Sp(1)_{\text{diag}} =: \mathfrak{h} \quad t \in (0, \pi/2),$$

$$G_{\gamma(0)} = K^+, \quad G_{\gamma(\pi/2)} = \{(q, q', q') \in G\} =: K^-.$$

It then follows that  $G$  acts on  $S^7$  by cohomogeneity one with principal orbits diffeomorphic to  $S^3 \times S^3$ . We also fix an  $\text{Ad}(\mathfrak{h})$ -invariant decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus V^+ \oplus V^-, \quad \mathfrak{m} := V^+ \oplus V^-$$

where

$$V^+ := \{(X, -2X, X) \mid X \in \mathfrak{sp}(1)\}, \quad V^- := \{(-2X, X, X) \mid X \in \mathfrak{sp}(1)\}.$$

Note that  $\mathfrak{k}^\pm = \mathfrak{h} \oplus V^\pm$ . We fix the standard basis of  $\mathfrak{sp}(1) \cong \mathfrak{su}(2)$  given by

$$h := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad e := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad v := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

with

$$[h, e] = 2v, \quad [h, v] = -2e, \quad [e, v] = 2h$$

and we consider the maps  $j_\pm : \mathfrak{sp}(1) \rightarrow V^\pm$  given by  $j_+(X) = (X, -2X, X)$  and  $j_-(X) = (-2X, X, X)$ . We then define a basis of  $\mathfrak{m}$  as follows

$$\begin{aligned} e_2 &:= j_+(h), \quad e_3 := j_+(e), \quad e_4 := j_+(v), \\ e_5 &:= j_-(h), \quad e_6 := j_-(e), \quad e_7 := j_-(v). \end{aligned}$$

We consider the manifold  $M := G \times_{K^+} \mathbb{H}$ , where  $K^+$  acts on  $\mathbb{H}$  via its standard representation. Then  $M$  can be identified with  $S^7 \setminus (G \cdot \gamma(\frac{\pi}{2}))$  and it is an  $\mathbb{R}^4$ -bundle over the singular orbit  $G \cdot \gamma(0) = G/K^+ \cong S^3$ , namely it is diffeomorphic to  $S^3 \times \mathbb{R}^4$ . The regular open subset  $M_o$  of  $M$  is  $G$ -equivariantly diffeomorphic to  $(0, \frac{\pi}{2}) \times G/H$ . In the open manifold  $M_o$  we can identify the tangent spaces  $T_{\gamma(t)}M_o = \mathbb{R}\gamma'(t) \oplus \widehat{\mathfrak{m}}$ . Along the curve  $\gamma$  we have a frame, again denoted by  $\{e_1, \dots, e_7\}$ , that is given by  $\mathcal{B}_t := \{\gamma'(t), \hat{e}_2|_{\gamma(t)}, \dots, \hat{e}_7|_{\gamma(t)}\}$  and its dual coframe will be denoted by  $\{e^1, \dots, e^7\}$ . For basic information on cohomogeneity one manifolds we refer e.g. to [1], [2].

**2.2. Invariant  $G_2$ -structures.** We start recalling some basic facts about  $G_2$ -structures. Given a 7-dimensional manifold  $M$  and its frame bundle  $L(M) \rightarrow M$ , a  $G_2$ -structure is a reduction of  $L(M)$  to a subbundle  $P$  with structure group  $G_2 \subset \mathrm{SO}(7)$ . It is known that  $G_2$ -structures are in one to one correspondence with smooth sections of the associated bundle  $\Lambda_+^3(M) := L(M) \times_{\mathrm{GL}(7, \mathbb{R})} \Lambda_+^3(\mathbb{R}^7) \subset \Lambda^3(M)$ , where  $\Lambda_+^3(\mathbb{R}^7) \subset \Lambda^3(\mathbb{R}^7)$  is the open orbit  $\mathrm{GL}(7, \mathbb{R}) \cdot \varphi_o$  through a 3-form  $\varphi_o$  with stabilizer  $\mathrm{GL}(7, \mathbb{R})_{\varphi_o} = G_2$  (see e.g. [18], [8]). A smooth section  $\varphi$  of  $\Lambda_+^3(M)$  (hence a  $G_2$ -structure on  $M$ ) determines a Riemannian metric  $g_\varphi$  as follows: at each point  $p \in M$  we consider the non-degenerate symmetric bilinear map

$$b_\varphi : T_p M \times T_p M \rightarrow \Lambda^7(T_p M^*), \quad (v, w) \mapsto \frac{1}{6} \iota_v \varphi \wedge \iota_w \varphi \wedge \varphi$$

and if  $\{v_1, \dots, v_7\}$  is any basis of  $T_p M$  with dual basis  $\{v^1, \dots, v^7\}$  then for  $i, j = 1, \dots, 7$

$$b_\varphi(v_i, v_j) = \beta_\varphi(v_i, v_j) v^1 \wedge \dots \wedge v^7$$

for some non-degenerate matrix  $B_\varphi := (\beta_\varphi(v_i, v_j))_{i,j=1, \dots, 7}$ ; the Riemannian metric  $g_\varphi$  is then given by (see e.g. [20])

$$g_\varphi(v_i, v_j) = (\det(B_\varphi))^{-1/9} \beta_\varphi(v_i, v_j).$$

In order to investigate  $G$ -invariant  $G_2$ -structures on  $M = G \times_{K^+} \mathbb{H}$ , we start considering invariant  $G_2$ -structures on the open dense submanifold  $M_o$ .

The description of  $G$ -invariant 3-forms on  $M_o$  is reduced to the study of the space of  $H$ -invariant 3-forms  $\Lambda^3(V^*)$ , where  $V := \mathbb{R}e_1 + \mathfrak{m} \cong T_{\gamma(t)}M$  ( $t \in (0, \frac{\pi}{2})$ ). We first note that

$$\Lambda^3(V^*)^H \cong \Lambda^2(\mathfrak{m}^*)^H + \Lambda^3((V^+)^*) + \Lambda^3((V^-)^*) +$$

$$+[\Lambda^2((V^+)^*) \otimes (V^-)^*]^H + [(V^+)^* \otimes \Lambda^2((V^-)^*)]^H.$$

Using the standard notation  $e^{i_1 i_2 \dots i_k} = e^{i_1} \wedge \dots \wedge e^{i_k}$ , we immediately see that the space  $\Lambda^2(\mathfrak{m}^*)^H$  is generated by the form  $\omega := e^{25} + e^{36} + e^{47}$  and that the space  $\Lambda^3(V^*)^H$  is generated by the invariant 3-forms

$$e^1 \wedge \omega, \varphi_1 := e^{234}, \varphi_2 := e^{567}, \varphi_3 := e^{237} - e^{246} + e^{345}, \varphi_4 := e^{267} - e^{357} + e^{456}.$$

If we denote by  $\varphi$  a  $G$ -invariant 3-form on  $M_o$ , its restriction along  $\gamma$  can be written as

$$(2.1) \quad \varphi|_{\gamma(t)} = f_0 (e^{125} + e^{136} + e^{147}) + f_1 e^{234} + f_2 e^{567} + f_3 (e^{237} - e^{246} + e^{345}) + f_4 (e^{267} - e^{357} + e^{456}),$$

for suitable  $f_i \in \mathcal{C}^\infty((0, \frac{\pi}{2}))$ . Let us fix the volume form  $e^{1234567}$  along  $\gamma$ , so that we get an identification  $\Lambda^7(V^*) \cong \mathbb{R}$ . Then, the matrix  $B_\varphi$  associated with the symmetric bilinear form  $b_\varphi$  with respect to  $\mathcal{B}_t$  is given by (here  $\mathbb{I}$  denotes the  $3 \times 3$ -matrix)

$$B_\varphi = f_0 \begin{pmatrix} -f_0^2 & 0 & 0 \\ 0 & b_1 \mathbb{I} & b_3 \mathbb{I} \\ 0 & b_3 \mathbb{I} & b_2 \mathbb{I} \end{pmatrix},$$

where

$$b_1 := f_1 f_4 - f_3^2, \quad b_2 := f_2 f_3 - f_4^2, \quad b_3 := \frac{1}{2}(f_1 f_2 - f_3 f_4).$$

The 3-form  $\varphi$  defines a  $G_2$ -structure if and only if  $B_\varphi$  is definite. In such a case,  $g_\varphi = (\det(B_\varphi))^{-1/9} B_\varphi$  is positive definite and

$$\det(B_\varphi) = \frac{1}{64} f_0^9 (f_1^2 f_2^2 - 6 f_1 f_2 f_3 f_4 + 4 f_1 f_4^3 + 4 f_2 f_3^3 - 3 f_3^2 f_4^2)^3 \neq 0.$$

In this case, we will suppose that the parameter  $t$  is the arc length parameter along the curve  $\gamma$  (hence throughout the following the parametr  $t$  will vary in some interval  $I = (0, T)$ ), i.e.

$$g_\varphi(e_1, e_1) = 1,$$

so that  $\det(B_\varphi) = -f_0^{27}$  or equivalently

$$(2.2) \quad f_0^2 = - \left( \frac{f_1^2 f_2^2 - 6 f_1 f_2 f_3 f_4 + 4 f_1 f_4^3 + 4 f_2 f_3^3 - 3 f_3^2 f_4^2}{4} \right)^{\frac{1}{3}}.$$

This implies that  $g_\varphi$  can be expressed as a block matrix

$$g_\varphi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & g_1 \mathbb{I} & g_3 \mathbb{I} \\ 0 & g_3 \mathbb{I} & g_2 \mathbb{I} \end{pmatrix},$$

where

$$g_1 := \frac{f_3^2 - f_1 f_4}{f_0^2}, \quad g_2 = \frac{f_4^2 - f_3 f_2}{f_0^2}, \quad g_3 := \frac{f_3 f_4 - f_1 f_2}{2 f_0^2}$$

together with the positivity condition, which in view of (2.2) can be written as

$$(2.3) \quad f_3^2 - f_1 f_4 > 0, \quad f_4^2 - f_2 f_3 > 0.$$

We can now compute the expression of the 4-form  $*_{\varphi}\varphi$ , where  $*_{\varphi}$  denotes the Hodge operator w.r.t.  $g_{\varphi}$ . We easily obtain

$$\begin{aligned} *_{\varphi}\varphi &= A e^1 \wedge [(f_1^2 f_2 - 3 f_1 f_3 f_4 + 2 f_3^3) e^{234} - (f_1 f_2^2 - 3 f_2 f_3 f_4 + 2 f_4^3) e^{567} \\ &\quad + (f_1 f_2 f_3 - 2 f_1 f_4^2 + f_3^2 f_4) (e^{237} - e^{246} + e^{345}) \\ &\quad - (f_1 f_2 f_4 - 2 f_2 f_3^2 + f_3 f_4^2) (e^{267} - e^{357} + e^{456})] \\ &\quad + \left( \frac{f_1^2 f_2^2 - 6 f_1 f_2 f_3 f_4 + 4 f_1 f_4^3 + 4 f_2 f_3^3 - 3 f_3^2 f_4^2}{4} \right)^{\frac{1}{3}} (e^{2356} + e^{2457} + e^{3467}), \end{aligned}$$

where

$$A := f_0 2^{\frac{1}{3}} (f_1^2 f_2^2 - 6 f_1 f_2 f_3 f_4 + 4 f_1 f_4^3 + 4 f_2 f_3^3 - 3 f_3^2 f_4^2)^{-\frac{2}{3}}.$$

Using (2.2), we see that  $A = \frac{1}{2} f_0^{-3}$ . Consequently, the 4-form  $*_{\varphi}\varphi$  can be rewritten as follows

$$\begin{aligned} (2.4) \quad *_{\varphi}\varphi &= \frac{1}{2 f_0^3} e^1 \wedge [(f_1^2 f_2 - 3 f_1 f_3 f_4 + 2 f_3^3) e^{234} - (f_1 f_2^2 - 3 f_2 f_3 f_4 + 2 f_4^3) e^{567} \\ &\quad + (f_1 f_2 f_3 - 2 f_1 f_4^2 + f_3^2 f_4) (e^{237} - e^{246} + e^{345}) \\ &\quad - (f_1 f_2 f_4 - 2 f_2 f_3^2 + f_3 f_4^2) (e^{267} - e^{357} + e^{456})] \\ &\quad - f_0^2 (e^{2356} + e^{2457} + e^{3467}). \end{aligned}$$

In order to compute  $d\varphi$ , we need some preliminary remarks. First of all we note that

$$\Lambda^4(V^*)^{\text{H}} = \mathbb{R}e^1 \wedge \Lambda^3(\mathfrak{m}^*)^{\text{H}} + [\Lambda^2((V^+)^*) \otimes \Lambda^2((V^-)^*)]^{\text{H}},$$

where the last summand is generated by the invariant form  $\alpha := e^{2356} + e^{2457} + e^{3467}$ . The next lemma follows by straightforward computations.

**Lemma 2.1.** *We have the following commutators for  $x, y \in \mathfrak{sp}(1)$*

$$[j_{\pm}(x), j_{\pm}(y)]_{\mathfrak{m}} = -j_{\pm}([x, y]),$$

$$[j_+(x), j_-(y)]_{\mathfrak{m}} = j_+([x, y]) + j_-([x, y]).$$

Using the standard Koszul's formula for the differential of an invariant form  $\psi \in \Lambda^k(\mathfrak{m})^{\text{H}}$ , namely for  $X_0, X_1, \dots, X_k \in \mathfrak{m}$

$$d\psi(X_0, X_1, \dots, X_k) = \sum_{i < j} (-1)^{i+j} \psi([X_i, X_j]_{\mathfrak{m}}, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_k),$$

(here the hat denotes omitted terms) we see that

$$d\varphi_1 = d\varphi_2 = 0, \quad d\omega = 6(\varphi_3 - \varphi_4), \quad d\varphi_3 = d\varphi_4 = 6\alpha.$$

Therefore we obtain

$$\begin{aligned} (2.5) \quad d\varphi|_{\gamma(t)} &= f_1' e^{1234} + f_2' e^{1567} + (f_3' - 6f_0) (e^{1237} - e^{1246} + e^{1345}) \\ &\quad + (f_4' + 6f_0) (e^{1267} - e^{1357} + e^{1456}) + 6(f_3 + f_4) (e^{2356} + e^{2457} + e^{3467}). \end{aligned}$$

3. INVARIANT NEARLY PARALLEL G<sub>2</sub>-STRUCTURES AND THEIR EQUATIONS

Recall that a G<sub>2</sub>-structure is *nearly parallel* if the defining 3-form  $\varphi$  satisfies the equation

$$(3.1) \quad d\varphi = \lambda *_{\varphi} \varphi,$$

for some non-zero real constant  $\lambda$ . In this case, the Riemannian metric  $g_{\varphi}$  induced by  $\varphi$  is Einstein with scalar curvature  $\text{Scal}(g_{\varphi}) = \frac{21}{8}\lambda^2$ . We now consider a G<sub>2</sub>-structure induced by a G-invariant 3-form  $\varphi$ , which can be described as in (2.1). Then  $\varphi$  defines an NP-structure if and only if  $f_0, f_1, f_2, f_3, f_4$  satisfy the following equations

$$(3.2) \quad \begin{cases} f'_1 = \lambda \frac{1}{f_0^3} \left( f_1 \frac{f_1 f_2 - f_3 f_4}{2} - f_3 (f_1 f_4 - f_3^2) \right), & (1) \\ f'_2 = \lambda \frac{1}{f_0^3} \left( f_4 (f_2 f_3 - f_4^2) - f_2 \frac{f_1 f_2 - f_3 f_4}{2} \right), & (2) \\ f'_3 = 6f_0 + \lambda \frac{1}{2f_0^3} (f_1 (f_2 f_3 - f_4^2) - f_4 (f_1 f_4 - f_3^2)), & (3) \\ f'_4 = -6f_0 + \lambda \frac{1}{2f_0^3} (f_3 (f_2 f_3 - f_4^2) - f_2 (f_1 f_4 - f_3^2)), & (4) \\ f_4 + f_3 = -\frac{1}{6} \lambda f_0^2, & (5) \\ f_0^6 = (f_1 f_4 - f_3^2) (f_2 f_3 - f_4^2) - \frac{1}{4} (f_1 f_2 - f_3 f_4)^2 > 0, & (6) \\ 0 > f_1 f_4 - f_3^2, \quad 0 > f_2 f_3 - f_4^2. \end{cases}$$

We use equation (5) in equation (4) and compare it with equation (3). We then get the expression of  $f'_0$  in terms of  $f_0, \dots, f_4$  and the system of equations can be written as follows

$$(3.3) \quad \begin{cases} f'_1 = \lambda \frac{1}{f_0^3} \left( f_1 \frac{f_1 f_2 - f_3 f_4}{2} - f_3 (f_1 f_4 - f_3^2) \right), & (1) \\ f'_2 = \lambda \frac{1}{f_0^3} \left( f_4 (f_2 f_3 - f_4^2) - f_2 \frac{f_1 f_2 - f_3 f_4}{2} \right), & (2) \\ f'_3 = 6f_0 + \lambda \frac{1}{2f_0^3} (f_1 (f_2 f_3 - f_4^2) - f_4 (f_1 f_4 - f_3^2)), & (3) \\ f'_4 = -6f_0 + \lambda \frac{1}{2f_0^3} (f_3 (f_2 f_3 - f_4^2) - f_2 (f_1 f_4 - f_3^2)), & (4) \\ f'_0 = -\frac{3}{2f_0^4} ((f_1 + f_3)(f_2 f_3 - f_4^2) - (f_2 + f_4)(f_1 f_4 - f_3^2)), & (5) \\ f_4 + f_3 + \frac{1}{6} \lambda f_0^2 = 0, & (6) \\ f_0^6 - (f_1 f_4 - f_3^2) (f_2 f_3 - f_4^2) + \frac{1}{4} (f_1 f_2 - f_3 f_4)^2 = 0, & (7) \\ 0 > f_1 f_4 - f_3^2, \quad 0 > f_2 f_3 - f_4^2, \quad f_0 \neq 0. & (8) \end{cases}$$

The following Lemma can be easily verified using a direct computation.

**Lemma 3.1.** *Equations (6) and (7) in (3.3) hold for all  $t \in I$  if and only if they hold at one point in  $I$  and equations (1)-(5) are satisfied for all  $t \in I$ .*

As an immediate corollary, we note that the algebro-differential system (3.3) can be reduced to the system of ODE's formed by equations (1)-(5) in (3.3) coupled with initial conditions at a fixed point  $t_o \in I$  satisfying equations (6) and (7) at  $t_o$ , together with the inequalities (8). We will use this point of view when we will construct families of mutually non-isometric and non locally homogeneous NP-structures in a suitable neighbourhood of homogeneous solutions, which we describe in the following subsection.

**Remark 3.2.** Note that under the rescaling  $\varphi \mapsto c \cdot \varphi$  ( $c \neq 0$ ), we have  $g_{c\varphi} = c^{2/3} \cdot g_\varphi$  and the constant in (3.1)  $\lambda \mapsto c^{-1/3} \lambda$ . This means that we can always fix  $\lambda$  to be any non zero real number. Alternatively, one can consider new functions

$$\tilde{f}_0(t) := \lambda^2 f_0(t/\lambda), \quad \tilde{f}_i(t) := \lambda^3 f_i(t/\lambda), \quad i = 1, \dots, 4$$

which satisfy the system (3.3) with  $\lambda = 1$ .

**Remark 3.3.** It is well known that, given a manifold  $X$  endowed with an NP-structure with 3-form  $\varphi$  and associated Riemannian metric  $g$ , a hypersurface  $f : S \rightarrow X$  inherits a so-called *nearly half-flat* SU(3)-structure given by a 2-form  $\omega$  and a 3-form  $\psi_+$  so that

$$\omega := \nu \varphi, \quad \psi_+ := -\nu * \varphi, \quad \psi_- := J\psi_+ = -f^* \varphi,$$

where  $\nu$  denotes the unit normal to  $S$  and  $J$  is the almost complex structure induced on  $S$  by the SU(3)-structure  $(\omega, \psi_+)$  (see [15]). This nearly half-flat structure  $(\omega, \psi_+)$  satisfies the condition

$$d\psi_- = -2\omega \wedge \omega,$$

when the 3-form  $\varphi$  satisfies  $d\varphi = 4 * \varphi$  (i.e.  $\lambda = 4$ ). In our situation the G-invariant nearly half-flat structures induced on the principal orbit  $G/H = S^3 \times S^3$  have proportional 2-forms  $\omega$ , as the isotropy representation of  $H$  forces the space of invariant 2-forms to be one-dimensional.

Viceversa given a smooth family of nearly half-flat structures  $(\omega(t), \psi_+(t))_{t \in \mathbb{R}}$  on a 6-dimensional manifold  $S$ , the 3-form  $\varphi := \omega \wedge dt - \psi_-$  on  $\mathbb{R} \times S$  defines an NP-structure (with  $\lambda = 4$ ) if and only if the following equations are fulfilled (Prop.5.2 in [15])

$$(3.4) \quad \begin{cases} \partial_t \psi_- = 4\psi_+ - d\omega, \\ d\psi_+ = -\frac{1}{2} \partial_t (\omega \wedge \omega). \end{cases}$$

In [27] it is proved that starting from a nearly half-flat structure on  $S$  it is possible to extend it to a smooth one-parameter family of nearly half-flat structures satisfying (3.4), hence obtaining an NP-structure on  $I \times S$  for some interval  $I \subseteq \mathbb{R}$  (see also [13]). We will prove a local existence result in Proposition 3.5.

**3.1. Particular solutions.** In this subsection we will describe three special solutions to the system (3.3), corresponding to known NP-structures. More precisely they are the sine-cone over the homogeneous nearly Kähler manifold  $S^3 \times S^3$  and the two homogeneous NP-structures on the sphere  $S^7$ .

(a) It is known (see [5],[6]) that the sine-cone  $C_s(Y) = (0, \pi) \times Y$  over a nearly Kähler 6-dimensional manifold  $Y$  carries an NP-structure inducing the sine-cone metric  $dt^2 + \sin^2 t \cdot g_Y$ . The homogeneous nearly Kähler structure on  $S^3 \times S^3$  is known to be invariant under the group  $SU(2)^3$  (see [11]) and therefore it gives rise to a solution  $(f_0, \dots, f_4)$  of the system (3.3) for  $t \in (0, \pi)$ , namely

$$\begin{aligned} \lambda &= 4, \quad f_0(t) = -2\sqrt{3}(\sin t)^2, \quad f_1(t) = f_2(t) = 8(\sin t)^4, \\ f_3(t) &= -4\sqrt{3}(\sin t)^3(\cos t + \frac{1}{\sqrt{3}} \sin t), \quad f_4(t) = -4\sqrt{3}(\sin t)^3(-\cos t + \frac{1}{\sqrt{3}} \sin t). \end{aligned}$$

The metric  $g_Y$  is represented by the block matrix  $\begin{pmatrix} 4\mathbb{I} & -2\mathbb{I} \\ -2\mathbb{I} & 4\mathbb{I} \end{pmatrix}$ .

(b) We consider the standard NP-structure  $\mathcal{P}_1$  on  $S^7$ , inducing the standard constant curvature metric. Its full automorphism group is  $\text{Aut}(\mathcal{P}_1) = \text{Spin}(7) \subset \text{SO}(8)$ . We consider the octonions  $\mathbb{O} = \{a + be, a, b \in \mathbb{H}\}$  together with the Cayley form  $\Phi \in \Lambda^2 \mathbb{O}$  given by

$$\Phi(x, y, z, w) = \langle x, \frac{1}{2}[y(\bar{z}w) - w(\bar{z}y)] \rangle.$$

It is known that the group  $\text{Sp}(1) \times \text{Sp}(1) \times \text{Sp}(1)$  acting almost faithfully on  $\mathbb{O}$  by

$$(q_1, q_2, q_3) \cdot (a + be) = q_1 a \bar{q}_3 + (q_2 b \bar{q}_3) e$$

preserves the form  $\Phi$  (see [9], p.11) and therefore induces a cohomogeneity one action on the round sphere  $S^7$  preserving the standard NP-structure  $\varphi$  given by  $\varphi_x := \iota_x \Phi$ ,  $x \in S^7$ . If we consider the curve  $\gamma(t) = \cos t + \sin t \cdot e \in \mathbb{O}$ , we see that the corresponding functions  $f_i(t)$  are given by

$$\begin{aligned} \lambda &= 4, \quad f_0(t) = -9 \sin t \cos t, \\ f_1(t) &= 27 \sin^4 t, \quad f_2(t) = 27 \cos^4 t, \quad f_3(t) = f_4(t) = -27 \sin^2 t \cos^2 t. \end{aligned}$$

The metric can be represented by the block matrix  $\begin{pmatrix} 1 & 0 & 0 \\ 0 & a\mathbb{I} & 0 \\ 0 & 0 & b\mathbb{I} \end{pmatrix}$  where  $a = 9 \sin^2 t$ ,  $b = 9 \cos^2 t$ .

(c) We now consider the non standard NP-structure  $\mathcal{P}_2$  on the *squashed*  $S^7$ , with full automorphism group given by  $\text{Aut}(\mathcal{P}_2) = \text{Sp}(2) \cdot \text{Sp}(1) \subset \text{SO}(8)$ . We refer to the exposition in [3], §8.2, where the authors describe this homogeneous structure using the presentation of  $S^7$  as a normal homogeneous space of the group  $\text{Sp}(2) \cdot \text{Sp}(1)$ .

Along the geodesic  $\gamma(t) = (\cos t, \sin t) \in S^7$  we have the following:

$$\begin{aligned} \lambda &= \frac{12}{\sqrt{5}}, \quad f_0(t) = \frac{9}{\sqrt{5}} \sin t \cdot \cos t, \\ f_1(t) &= \frac{27}{\sqrt{5}} (3 \sin^4 t \cdot \cos^2 t - \frac{1}{5} \sin^6 t), \quad f_2(t) = \frac{27}{\sqrt{5}} (3 \cos^4 t \cdot \sin^2 t - \frac{1}{5} \cos^6 t), \\ f_3(t) &= \frac{27}{\sqrt{5}} \sin^2 t \cdot \cos^2 t \cdot \left( \cos^2 t - \frac{11}{5} \sin^2 t \right), \quad f_4(t) = \frac{27}{\sqrt{5}} \sin^2 t \cdot \cos^2 t \cdot \left( \sin^2 t - \frac{11}{5} \cos^2 t \right). \end{aligned}$$

Note that the sign of the constant  $\lambda$  is opposite to that indicated in [3]. The metric can be represented by the block matrix  $\begin{pmatrix} 1 & 0 & 0 \\ 0 & a\mathbb{I} & c\mathbb{I} \\ 0 & c\mathbb{I} & b\mathbb{I} \end{pmatrix}$ , where

$$a := \frac{36}{5} \sin^2 t \cdot \left( \frac{5}{4} - \sin^2 t \right), \quad b := \frac{36}{5} \cos^2 t \cdot \left( \frac{5}{4} - \cos^2 t \right), \quad c := -\frac{36}{5} \sin^2 t \cdot \cos^2 t.$$

**3.2. The functions  $f_i$ 's and the associated nearly parallel structures.** A solution  $(f_0(t), \dots, f_4(t))$  of the system (3.3) defines an NP-structure on the open subset  $J \times G/K$  for some subinterval  $J \subseteq I$ . In this subsection we study the problem when two such NP-structures are (locally) isomorphic. We start with the following proposition, which follows very closely Prop. 4.1 in [25], and characterizes the local homogeneity of a G-manifold with a G-invariant NP-structure.

**Proposition 3.4.** *Let  $X$  be a 7-dimensional manifold endowed with an NP-structure given by a 3-form  $\varphi$ . Assume that the group  $G \cong \mathrm{SU}(2)^3$  acts on  $X$  by automorphisms of the  $G_2$ -structure by cohomogeneity one and assume moreover that the NP-structure is locally homogeneous. Then  $(X, \varphi)$  is locally (isometrically) isomorphic to the standard sphere or to the squashed sphere endowed with their respective NP-structures.*

*Proof.* We fix  $p \in X$  and we consider the Lie algebra  $\mathfrak{s}$  of germs of automorphisms of  $(X, \varphi)$  with isotropy subalgebra at  $p$  denoted by  $\mathfrak{u}$ . Then it is known that  $\mathfrak{u}$  is reductive and that it embeds into  $\mathfrak{g}_2$ . Moreover, by local homogeneity, we may suppose that  $p$  is G-regular, hence  $\mathfrak{u}$  contains  $\mathfrak{u}_1 := \mathfrak{u} \cap \mathfrak{g} \cong \mathfrak{su}(2)$ , the isotropy subalgebra  $\mathfrak{g}_p$ . Therefore, looking at the list of possible reductive subalgebras of  $\mathfrak{g}_2$ , we see that  $\mathfrak{u}$  can be isomorphic to  $\mathfrak{su}(2)$ ,  $\mathfrak{su}(3)$ ,  $\mathbb{R} + \mathfrak{su}(2)$ ,  $\mathfrak{su}(2) + \mathfrak{su}(2)$ ,  $\mathfrak{g}_2$ . Let  $S$  be the simply connected Lie group with Lie algebra  $\mathfrak{s}$  and let  $U$  be the connected Lie subgroup of  $S$  with Lie algebra  $\mathfrak{u}$ . We claim that  $U$  is closed in  $S$ , whence  $X$  is locally isomorphic to a globally homogeneous space (see e.g. [26]). Suppose on the contrary that  $U$  is not closed in  $S$ . This implies that  $\mathfrak{u}$  is not semisimple (see [22], p.615), hence  $\mathfrak{u} \cong \mathbb{R} + \mathfrak{su}(2)$  and therefore  $\mathfrak{u} = \mathfrak{u}_1 + \mathbb{R}$ . As  $\mathfrak{u}$  is reductive, we can write  $\mathfrak{s} = \mathfrak{u} + V$ , where  $V \cong \mathbb{R}^7$  is  $\mathrm{ad}(\mathfrak{u})$  invariant and  $\mathrm{ad}(\mathfrak{u})|_V \subset \mathfrak{su}(3)$ . Therefore  $\mathrm{ad}^{\mathfrak{s}}|_{\mathfrak{u}_1} = \rho_1 \oplus \mathrm{ad} \oplus 4\mathbb{R}$ , where  $\rho_1$  is the standard representation of  $\mathfrak{u}_1$  on  $\mathbb{C}^2$ . On the other hand  $\mathfrak{u}_1 \subset \mathfrak{g} \subset \mathfrak{s}$  with  $\mathrm{codim}_{\mathfrak{s}} \mathfrak{g} = 2$  and therefore  $\mathrm{ad}^{\mathfrak{s}}|_{\mathfrak{u}_1} = 3\mathrm{ad} \oplus 2\mathbb{R}$ , a contradiction.

A direct inspection of the globally homogeneous (hence compact) manifolds with G-invariant NP-structure (see [18]) and admitting a cohomogeneity one action of  $G$  proves our claim.  $\square$

Let  $\varphi = \sum_{i=0}^4 f_i \varphi_i$  and  $\tilde{\varphi} = \sum_{i=0}^4 \tilde{f}_i \varphi_i$  be two G-invariant NP-structures, where the 5-tuples  $(f_0, \dots, f_4), (\tilde{f}_0, \dots, \tilde{f}_4)$  satisfy the system (3.3) on some interval  $J \subseteq I$ . Assume then that these two structures are locally isomorphic, i.e. there are two open subsets  $W, \tilde{W} \subseteq J \times G/H$  and a diffeomorphism  $\psi : W \rightarrow \tilde{W}$  with  $\psi^* \tilde{\varphi} = \varphi$  (and therefore inducing an isometry between the induced metrics  $g$  and  $\tilde{g}$  respectively). We will first suppose that  $\psi$  does *not* map G-orbits onto G-orbits. As  $G$  acts with cohomogeneity one, this means that both  $W$  and  $\tilde{W}$  are locally homogeneous and therefore, by Prop.3.4, locally isometric to the standard or squashed sphere. As the full automorphisms of the two homogeneous NP-structures on  $S^7$ , namely  $\mathrm{Spin}(7)$  and  $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)$ , contain precisely one copy of  $\mathrm{SU}(2)^3$  up to conjugation, we can find a local isometry  $\tilde{\psi} : \tilde{W} \rightarrow \tilde{W}$  preserving  $\tilde{\varphi}$  and with  $\tilde{\psi}_*(\psi_*(\mathfrak{g})) = \mathfrak{g}$ .

Therefore we are reduced with the case where  $\psi$  preserves  $G$ -orbits. Up to some traslation by an element of  $G$  and up to some reparameterization  $t \mapsto t+c$ , we can suppose  $\psi(\gamma_t) = \gamma_{\pm t}$ . If  $\psi(\gamma_t) = \gamma_{-t}$ , we can compose  $\psi$  with the transformation  $(t, xH) \mapsto (-t, xH)$  reducing to  $\psi(\gamma_t) = \gamma_t$ ; the corresponding transformation of the functions  $f_i$ 's reads

$$(3.5) \quad \tau_o : (f_0(t), f_1(t), f_2(t), f_3(t), f_4(t)) \mapsto (-f_0(-t), f_1(-t), f_2(-t), f_3(-t), f_4(-t)).$$

The map  $\psi$  induces an automorphism  $\psi_*$  of the Lie algebra  $\mathfrak{g}$  that preserves the regular isotropy  $\mathfrak{h}$ , as  $\psi$  preserves the curve  $\gamma_t$ . If  $\psi_*$  is inner, it is the conjugation by an element  $n \in N_G(\mathbf{H})$ ,  $n = \sigma \cdot h$  with  $h \in \mathbf{H}$  and  $\sigma = (\sigma_1, \sigma_2, \sigma_3)$  where  $\sigma_i = \pm \text{Id} \in \text{SU}(2)$ . It then follows that  $\text{Ad}(g)$  fixes  $\omega$  as well as all  $\varphi_1, \dots, \varphi_4$ , so that the functions  $f_0, f_1, \dots, f_4$  remain unchanged. We now examine the case where  $\psi_*$  is outer, namely it permutes the simple factors  $\mathfrak{f}_i \cong \mathfrak{su}(2)$ ,  $i = 1, 2, 3$  of  $\mathfrak{g}$ . We now describe how the functions  $\varphi_i$ 's transform when  $\psi_*$  induces the generators (12) and (13) of the symmetric group  $S_3$ .

Suppose  $\psi_*$  permutes the factors  $\mathfrak{f}_i$  leaving  $\mathfrak{f}_3$  fixed. Then we easily see that  $\omega \mapsto -\omega$ , while  $\varphi_1, \varphi_2$  are exchanged as well as  $\varphi_3, \varphi_4$ . The functions  $f_i$ 's transform accordingly as follows

$$(3.6) \quad \tau_{(12)} : (f_0, f_1, f_2, f_3, f_4) \mapsto (-f_0, f_2, f_1, f_4, f_3).$$

When  $\psi_*$  induces the permutation (13) on the factors of  $\mathfrak{g}$ , we can compute the corresponding transformation of the invariant forms as follows

$$\begin{aligned} \omega &\mapsto -\omega, \quad \varphi_1 \mapsto \varphi_1 - \varphi_2 - \varphi_3 + \varphi_4, \quad \varphi_2 \mapsto -\varphi_2 \\ \varphi_3 &\mapsto -\varphi_3 - 3\varphi_2 + 2\varphi_4, \quad \varphi_4 \mapsto \varphi_4 - 3\varphi_2 \end{aligned}$$

so that the corresponding trasformation of the functions  $f_i$ 's reads

$$(3.7) \quad \tau_{(13)} : (f_0, f_1, f_2, f_3, f_4) \mapsto (-f_0, f_1, -f_1 - f_2 - 3(f_3 + f_4), -f_1 - f_3, f_1 + 2f_3 + f_4).$$

Using these, we see that the remaining non-trivial permutations induce the following trasformations

$$(3.8) \quad \begin{aligned} \tau_{(23)} &:(f_0, f_1, f_2, f_3, f_4) \mapsto (-f_0, -f_1 - f_2 - 3(f_3 + f_4), f_2, f_2 + f_3 + 2f_4, -f_2 - f_4). \\ \tau_{(123)} &:(f_0, f_1, f_2, f_3, f_4) \mapsto (f_0, -f_1 - f_2 - 3(f_3 + f_4), f_1, f_1 + 2f_3 + f_4, -f_1 - f_3). \\ \tau_{(132)} &:(f_0, f_1, f_2, f_3, f_4) \mapsto (f_0, f_2, -f_1 - f_2 - 3(f_3 + f_4), -f_2 - f_4, f_2 + f_3 + 2f_4). \end{aligned}$$

**3.3. Local Existence.** We consider the regular ODE system given by equations (1)-(5) (for a fixed  $\lambda$ ) in (3.3). Any solution of such a system is a curve in  $\mathbb{R}^5$  lying in the subset

$$C := \{(a_0, a_1, \dots, a_4) \in \mathbb{R}^5 \mid a_0 \neq 0, 0 > a_1 a_4 - a_3^2, 0 > a_2 a_3 - a_4^2, R_1 = 0, R_2 = 0\}$$

where

$$(3.9) \quad \begin{aligned} R_1(a_0, \dots, a_4) &:= a_3 + a_4 + \frac{1}{6} \lambda a_0^2, \\ R_2(a_0, \dots, a_4) &:= (a_1 a_4 - a_3^2) (a_2 a_3 - a_4^2) - \frac{1}{4} (a_1 a_2 - a_3 a_4)^2 - a_0^6. \end{aligned}$$

We fix  $t_o = \frac{\pi}{4}$  and the initial condition

$$x_o := \left( -\frac{9}{2}, \frac{27}{4}, \frac{27}{4}, -\frac{27}{4}, -\frac{27}{4} \right)$$

that corresponds to the initial values at  $t_o$  of the homogeneous structure  $\mathcal{P}_1$  on the sphere  $S^7$  with constant  $\lambda = 4$ . In a suitable neighborhood  $W$  of  $x_o$  the set  $C \cap W$  is a 3-dimensional submanifold, as it can be easily verified. Moreover if  $\Sigma$  is the group of transformations in  $\mathbb{R}^5$  generated by  $\tau_{(12)}, \tau_{(13)}$ , we can shrink  $W$  so that  $\tau(W) \cap W = \emptyset$  for every  $\tau \in \Sigma$ . Now if  $F(t)$  is the (homogeneous) solution starting from  $x_o$ , we can fix a 2-dimensional submanifold  $\mathcal{S}$  in  $C \cap W$  that is transversal to the trace of  $F$ ; solutions starting from points in  $\mathcal{S}$  are all mutually non equivalent and not locally homogeneous. Therefore we have proved the following

**Proposition 3.5.** *There exists a 2-dimensional family of mutually non equivalent and not locally homogeneous  $G$ -invariant NP-structures on the space  $J \times G/H$  for some open interval  $J \subset \mathbb{R}$ .*

**3.4. Extendability over the singular orbit  $G/K^+ \cong S^3$ .** We aim at finding necessary and sufficient conditions on the functions  $f_i$  so that the 3-form  $\varphi$  and the corresponding metric  $g_\varphi$  extend smoothly over one singular orbit  $G \cdot p = G/K^+ \cong S^3$ , where  $p = \gamma(0)$ .

We first remark that when  $\varphi$  extends smoothly over the singular orbit, then  $\varphi_{\gamma(t)}(\hat{e}_i, \hat{e}_j, \hat{e}_k)$  is smooth in a neighborhood of the origin; therefore, as all Killing vector fields in  $V^+$  vanish at  $p$ , we have

$$f_0(0) = f_1(0) = f_3(0) = f_4(0) = 0.$$

Moreover the element  $h = \left( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, 1 \right) \in K^+$  reverses the curve  $\gamma$ , i.e.  $h(\gamma(t)) = \gamma(-t)$  and its adjoint  $\text{Ad}_G(h) = \text{Id}$ , so that the functions  $f_i$  extend as follows:

$$f_0 \text{ odd; } \quad f_i \text{ even } \quad i = 1, 2, 3, 4.$$

We now consider the slice representation  $\rho$  of  $K^+$  at  $p$ . If we write the  $\text{ad}(\mathfrak{g})$ -invariant decomposition  $\mathfrak{u} = \mathfrak{g} + \mathfrak{p}$ , where  $\mathfrak{u} = \mathfrak{sp}(2) + \mathfrak{su}(2)$  and  $\mathfrak{p} \cong \mathbb{H}$ , then  $\rho$  can be identified with  $\text{Ad}^U|_{K^+}$  restricted to the invariant module  $\mathfrak{p}$ . If we choose standard coordinates  $\{t = x_1, x_2, x_3, x_4\}$  on  $\mathfrak{p} \cong \mathbb{H}$ , we see that

$$\hat{e}_2|_{(t,0,0,0)} = 3t \frac{\partial}{\partial x_2}, \quad \hat{e}_3|_{(t,0,0,0)} = 3t \frac{\partial}{\partial x_3}, \quad \hat{e}_4|_{(t,0,0,0)} = 3t \frac{\partial}{\partial x_4}.$$

We also need an  $\text{ad}(\mathfrak{k}^+)$ -stable complement  $\mathfrak{s}$  in  $\mathfrak{g}$ , namely  $\mathfrak{s} := \{(X, 0, -X) \mid X \in \mathfrak{su}(2)\}$  and we fix the basis

$$w_1 := (h, 0, -h), \quad w_2 := (e, 0, -e), \quad w_3 := (v, 0, -v)$$

so that

$$w_i = -\frac{1}{3}e_{i+1} - \frac{2}{3}e_{4+i}, \quad i = 1, 2, 3.$$

We consider the local frame along the curve  $t \mapsto (t, 0, 0, 0) \in \mathfrak{p}$  given by

$$e_1, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \hat{w}_1|_{(t,0,0,0)}, \hat{w}_2|_{(t,0,0,0)}, \hat{w}_3|_{(t,0,0,0)}$$

with corresponding coframe  $dt = x_0, dx_1, dx_2, dx_3, w^1, w^2, w^3$  satisfying

$$\begin{aligned} e^1 &= dt, \quad e^2 = \frac{1}{3t}dx_2 - \frac{1}{3}w^1, \quad e^3 = \frac{1}{3t}dx_3 - \frac{1}{3}w^2, \quad e^4 = \frac{1}{3t}dx_4 - \frac{1}{3}w^3, \\ e^5 &= -\frac{2}{3}w^1, \quad e^6 = -\frac{2}{3}w^2, \quad e^7 = -\frac{2}{3}w^3. \end{aligned}$$

On the tubular neighborhood of the singular orbit  $G/K^+$  given by  $G \times_{K^+} \mathfrak{p}$ , the 3-form  $\varphi$  defined by (2.1) on the complement of the zero section is completely determined by its restriction to  $\mathfrak{p}$ . Therefore we can see  $\varphi$  as a  $K^+$ -equivariant map  $\hat{\varphi} : \mathfrak{p} \setminus \{0\} \rightarrow \wedge^3(\mathfrak{p}^* + \mathfrak{s}^*)$ , which is fully determined by its restriction to the curve  $\gamma(t) = (t, 0, 0, 0) \in \mathfrak{p}$ . We can write down the components of  $\hat{\varphi}|_{\gamma(t)}$  ( $t \neq 0$ ) along each of the four  $K^+$ -summands in the decomposition

$$\wedge^3(\mathfrak{p}^* + \mathfrak{s}^*) = \wedge^3\mathfrak{p}^* \oplus \wedge^3\mathfrak{s}^* \oplus (\wedge^2\mathfrak{p}^* \otimes \mathfrak{s}^*) \oplus (\mathfrak{p}^* \otimes \wedge^2\mathfrak{s}^*).$$

We will denote by  $\pi_i$  the  $K^+$ -equivariant projection of  $\wedge^3(\mathfrak{p}^* + \mathfrak{s}^*)$  onto its  $i$ -th summand,  $i = 1, \dots, 4$ .

(a) Along  $\wedge^3\mathfrak{p}^* \cong \mathfrak{p}$  we have

$$\pi_1 \circ \hat{\varphi}|_{\gamma(t)} = \frac{1}{27t^3} f_1(t) dx_2 \wedge dx_3 \wedge dx_4$$

and therefore we may consider the  $K^+$ -equivariant map  $\mathfrak{p} \rightarrow \mathfrak{p}^*$  whose restriction to  $\gamma$  is given by  $\frac{1}{27t^3} f_1(t) dt$  for  $t \neq 0$ . This extends smoothly on the whole  $\mathfrak{p}$  if and only if  $f_1(t)$  is an even smooth function of  $t$  with  $f_1(0) = f_1'(0) = 0$  (indeed  $\frac{f_1(t)}{t^4}$  must extend smoothly and we already know that  $f_1$  is even and vanishes at  $t = 0$ ).

(b) Along the trivial  $K^+$ -module  $\wedge^3\mathfrak{s}^*$  we obtain the component

$$\pi_2 \circ \hat{\varphi}|_{\gamma(t)} = -\frac{1}{27}(f_1 + 8f_2 + 6f_3 + 12f_4) w^1 \wedge w^2 \wedge w^3.$$

Since  $w^1 \wedge w^2 \wedge w^3$  is  $K^+$ -invariant, the extendability condition in this case boils down to the condition that  $f_1 + 8f_2 + 6f_3 + 12f_4$  must be even. This follows from the fact each  $f_i$  is even for  $i = 1, \dots, 4$ .

(c) Along  $\mathfrak{p}^* \otimes \wedge^2\mathfrak{s}^*$  we obtain the component

$$\pi_3 \circ \hat{\varphi}|_{\gamma(t)} = \frac{1}{27t}(f_1 + 4f_3 + 4f_4) (dx_2 \wedge w^2 \wedge w^3 - dx_3 \wedge w^1 \wedge w^3 + dx_4 \wedge w^1 \wedge w^2).$$

This case can be handled in two different ways. First we note that  $K^+$  contains the normal subgroup  $I = \{(1, q, 1) \in K^+ \mid q \in \mathrm{Sp}(1)\}$  which still acts transitively on the unit sphere in  $\mathfrak{p}$ , but trivially on the orbit  $G/K^+$ . Using the subgroup  $I$  we can determine the full expression of  $\hat{\varphi}$  and check that its component along  $\mathfrak{p}^* \otimes \wedge^2\mathfrak{s}^*$  extends smoothly over the whole  $\mathfrak{p}$  if and only if  $\frac{1}{t^2}(f_1 + 4f_3 + 4f_4)$  extends smoothly over  $t = 0$ . This last condition is automatic as we are supposing  $f_1, f_3, f_4$  to be even functions vanishing at  $t = 0$ . The second approach considers the  $K^+$ -module  $\mathfrak{p}^* \otimes \wedge^2\mathfrak{s}^* \cong \mathfrak{p}^* \otimes \mathfrak{s} \cong Q_1 \oplus Q_2$ , where  $Q_1 \cong \mathfrak{p}$  and  $Q_2 \cong \mathbb{R}^8$  is the real part of the complex  $K^+$ -irreducible representation  $\mathbb{C}^2 \otimes S^3(\mathbb{C}^2)$  (here  $K^+ \cong \mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2$  acts on  $\mathbb{C}^2$  via  $\mathrm{SU}(2)_1$  and on  $S^3(\mathbb{C}^2)$  via  $\mathrm{SU}(2)_2$ ). Since the space  $Q_2$  does not contain any non-zero fixed point vector under the action of the subgroup  $\mathrm{Sp}(1) \cong \mathrm{H} \subset K^+$ , the element  $dx_2 \wedge w^2 \wedge w^3 - dx_3 \wedge w^1 \wedge w^3 + dx_4 \wedge w^1 \wedge w^2$  belongs to the submodule  $Q_1 \cong \mathfrak{p}$  and therefore we can consider this component of  $\hat{\varphi}$  as a  $K^+$ -equivariant map into  $\mathfrak{p}$ , leading to the same conclusion as above.

(d) Along the summand  $\bigwedge^2 \mathfrak{p}^* \otimes \mathfrak{s}^*$  we have

$$\begin{aligned} \pi_4 \circ \hat{\varphi}|_{\gamma(t)} &= -\frac{2}{9t} f_0 \cdot (dx_0 \wedge dx_1 \wedge w^1 + dt \wedge dx_2 \wedge w^2 + dt \wedge dx_3 \wedge w^3) + \\ &\quad -\frac{f_1 + 2f_3}{27t^2} \cdot (dx_1 \wedge dx_2 \wedge w^3 + dx_2 \wedge dx_3 \wedge w^1 - dx_1 \wedge dx_3 \wedge w^2). \end{aligned}$$

We may use the subgroup I to determine the full expression of  $\pi_4 \circ \hat{\varphi}$  on  $\mathfrak{p} \setminus \{0\}$ .

We only write here the component of  $\pi_4 \circ \hat{\varphi}$  along the 3-form  $dx_0 \wedge dx_1 \wedge w^1$ , namely

$$-\frac{2}{9t} f_0 \cdot \frac{x_0^2 + x_1^2}{t^2} - \frac{f_1 + 2f_3}{27t^2} \cdot \frac{x_2^2 + x_3^2}{t^2},$$

where  $t = \sum_{i=1}^4 x_i^2$ . This can be clearly rewritten as

$$-\frac{2}{9t} f_0 + \left( \frac{2f_0}{9t} - \frac{f_1 + 2f_3}{27t^2} \right) \cdot \frac{x_2^2 + x_3^2}{t^2}.$$

Therefore we see that the extendibility condition reduces to  $\frac{2f_0}{9t} - \frac{f_1 + 2f_3}{27t^2} = O(t^2)$ . Since  $f_0$  is odd and  $f_1, f_3$  are even, the condition can be written as  $\lim_{t \rightarrow 0} \frac{2f_0}{9t} - \frac{f_1 + 2f_3}{27t^2} = 0$  or equivalently, using the fact that  $f_1 = O(t^4)$  by (a),

$$(3.10) \quad 6f_0'(0) = f_3''(0).$$

It can be easily checked that the conditions on the extendability of the other components of  $\pi_4 \circ \hat{\varphi}$  are all equivalent to (3.10).

Summing up, the 3-form  $\varphi$  on the regular part  $M_o$  extends on the whole tube  $G \times_{K^+} \mathfrak{p}$  if and only if the functions  $f_i$  extend smoothly around  $t = 0$  with the following properties:

$$(3.11) \quad f_0 \text{ is odd, } f_i \text{ are even, } i = 1, 2, 3, 4.$$

$$(3.12) \quad f_i(0) = 0, \quad i = 1, 3, 4;$$

$$(3.13) \quad f_1''(0) = 0; \quad 6f_0'(0) = f_3''(0).$$

When these conditions hold, the 3-form  $\varphi$  extends smoothly at the singular point with the expression

$$\begin{aligned} \varphi_p &= A w^1 \wedge w^2 \wedge w^3 + B (dx_0 \wedge dx_1 \wedge w^1 + dt \wedge dx_2 \wedge w^2 + dt \wedge dx_3 \wedge w^3 \\ &\quad + dx_1 \wedge dx_2 \wedge w^3 + dx_2 \wedge dx_3 \wedge w^1 - dx_1 \wedge dx_3 \wedge w^2), \end{aligned}$$

where  $A := -\frac{8}{27} f_2(0)$  and  $B := -\frac{2}{9} f_0'(0)$ . Now, it is not difficult to check that the 3-form  $\varphi_p$  is stable (i.e. the orbit  $\text{GL}(\text{T}_p M) \cdot \varphi_p$  is open in  $\bigwedge^3 \text{T}_p M^*$ ) if and only if  $A \cdot B < 0$  and in this case the induced metric is positive definite, coinciding with the limit metric  $g_\varphi$  at  $p$ . Therefore we need to consider the non-degeneracy condition

$$(3.14) \quad f_2(0) \cdot f_0'(0) < 0.$$

Actually, if we use (3.11), (3.12), (3.13) and (7)-(8) in (3.3), we see that  $f_2(0) \cdot f_0'(0) \leq 0$ , so that the only condition we need to add is

$$(3.15) \quad f_2(0) \neq 0, \quad f_0'(0) \neq 0.$$

Therefore we have proved the following

**Proposition 3.6.** *Let  $\varphi$  be a  $G$ -invariant 3-form on  $M_o$  whose restriction to the curve  $\gamma(t)$  ( $t \neq 0$ ) has the expression (2.1). Assume that the form  $\varphi$  defines an NP-structure, so that the functions  $f_i$ 's satisfy the algebro-differential system (3.3). Then the form  $\varphi$  extends smoothly to a  $G$ -invariant 3-form on  $M$  defining an NP-structure on  $M$  if and only if the functions  $f_i$ 's extend smoothly around  $t = 0$  fulfilling the conditions (3.11), (3.12), (3.13), (3.15).*

#### 4. THE MAIN THEOREM

In this section we will prove the existence of a one-parameter family of NP-structures on  $M$ . In particular we will prove our main theorem, namely

**Theorem 4.1.** *There exists a one parameter family  $\mathcal{F}_a$ , ( $a \in \mathbb{R}^+$ ) of NP-structures on  $M$ . These structures are mutually non isomorphic and not locally homogeneous, with the exception of two of them, which are locally isomorphic to the structures  $\mathcal{P}_1$  and  $\mathcal{P}_2$  on  $S^7$ .*

**Remark 4.2.** The parameter  $a \in \mathbb{R}^+$  measures the size of the singular orbit  $S^3$ .

The proof of Theorem 4.1 will be achieved through several steps in this section. We know how to describe  $G$ -invariant NP-structures on the open dense subset  $M_o$  of  $G$ -regular points, which identifies with the complement of the zero section in the bundle  $G \times_{K^+} \mathbb{H}$ . Given a  $G$ -invariant 3-form  $\varphi$  on  $M_o$  which defines an NP-structure, we considered its expression (2.1) along a transversal curve  $\gamma$  and we could derive the algebro-differential system of equations (3.3) the functions  $f_i$ 's in (2.1) have to satisfy. Moreover we found necessary and sufficient conditions in terms of the functions  $f_i$ 's so that the NP-structure on  $M_o$  extends smoothly to a global  $G_2$ -structure on  $M$ . Now instead of solving for the functions  $f_i$ 's, in view of Proposition 3.6 we may look for smooth *even* functions  $h_i$  defined on some interval  $(-\varepsilon, \varepsilon)$ ,  $\varepsilon \in \mathbb{R}^+$ , so that

$$(4.1) \quad f_0 = t \cdot h_0, \quad f_1 = t^4 \cdot h_1, \quad f_2 = h_2, \quad f_3 = t^2 \cdot h_3, \quad f_4 = t^2 \cdot h_4,$$

Note that

$$h_4 = -h_3 - \frac{\lambda}{6} h_0^2.$$

If we set  $a_i := h_i(0)$ ,  $i = 0, \dots, 4$ , then the extendability conditions (3.11),(3.12),(3.13), (3.15) are then simply written as

$$(4.2) \quad a_3 = 3a_0, \quad a_0, a_2 \neq 0,$$

We now rewrite the system (3.3) using the above defined functions  $h_i$  as follows.

$$(4.3) \quad h_0' = \frac{1}{t}(f_0' - h_0) = \\ = -\frac{1}{t} \left( h_0 + \frac{3h_2h_3^2}{h_0^4} \right) - \frac{3}{2h_0^4} (t(h_3 - h_4)(h_1h_2 + h_3h_4) - 2t^3h_1h_4^2);$$

$$(4.4) \quad h_1' = -\frac{4}{t}h_1 + \frac{1}{t^4}f_1' =$$

$$\begin{aligned}
&= -\frac{4}{t}h_1 + \lambda \frac{1}{t^7 h_0^3} \left( t^8 h_1 \frac{h_1 h_2 - h_3 h_4}{2} - t^2 h_3 (t^6 h_1 h_4 - t^4 h_3^2) \right) = \\
&= \frac{1}{t} \left( -4h_1 + \frac{\lambda h_3^3}{h_0^3} \right) + \frac{\lambda t}{2h_0^3} (h_1^2 h_2 - h_1 h_3 h_4 - 2h_1 h_3 h_4);
\end{aligned}$$

$$(4.5) \quad h'_2 = f'_2 = \frac{\lambda t}{h_0^3} \left( h_4 (h_2 h_3 - t^2 h_4^2) - \frac{1}{2} h_2 (h_1 h_2 - h_3 h_4) \right);$$

$$(4.6) \quad h'_3 = -\frac{2}{t}h_3 + \frac{1}{t^2}f'_3 = \frac{1}{t}(-2h_3 + 6h_0) + \lambda \frac{t}{2h_0^3} (h_1 h_2 h_3 + h_3^2 h_4 - t^2 (h_1 h_4^2 + h_1 h_4^2))$$

Therefore if we put  $h := (h_0, h_1, h_2, h_3)$  the system takes the form

$$(4.7) \quad h'(t) = \frac{1}{t}A(h) + B(h, t), \quad h(0) = \bar{h} = (a_0, a_1, a_2, a_3)$$

where  $A : \mathbb{R}^4 \rightarrow \mathbb{R}^4$  is given by

$$(4.8) \quad A(h) = \left( -h_0 - \frac{3h_2 h_3^2}{h_0^4}, -4h_1 + \frac{\lambda h_3^3}{h_0^3}, 0, -2h_3 + 6h_0 \right)$$

and  $B : \mathbb{R}^4 \times \mathbb{R} \rightarrow \mathbb{R}^4$  is defined by

$$\begin{aligned}
B(h, t) = & \left( -\frac{3}{2h_0^4} (t(h_3 - h_4)(h_1 h_2 + h_3 h_4) - 2t^3 h_1 h_4^2), \right. \\
& \frac{\lambda t}{2h_0^3} (h_1^2 h_2 - h_1 h_3 h_4 - 2h_1 h_3 h_4), \frac{\lambda t}{h_0^3} (h_4 (h_2 h_3 - t^2 h_4^2) - \frac{1}{2} h_2 (h_1 h_2 - h_3 h_4)), \\
& \left. \frac{\lambda t}{2h_0^3} (h_1 h_2 h_3 + h_3^2 h_4 - t^2 (h_1 h_4^2 + h_1 h_4^2)) \right).
\end{aligned}$$

Clearly we must have  $A(\bar{h}) = 0$ , hence

$$a_3 = 3a_0, \quad a_2 = -\frac{1}{27}a_0^3, \quad a_1 = \frac{27}{4}\lambda.$$

**Proposition 4.3.** *The functions  $f_i$ 's as in (2.1) define a  $G$ -invariant NP-structure on  $M = G \times_{K^+} \mathbb{H}$  (with fixed constant  $\lambda$ ) if and only if there exist smooth even functions  $h_0, h_1, h_2, h_3$  defined on some interval  $(-\varepsilon, \varepsilon)$ , ( $\varepsilon \in \mathbb{R}^+$ ), satisfying the equation (4.7) for  $t \neq 0$  with initial condition at  $t = 0$  given by  $\bar{h} = (h_0(0), \dots, h_3(0))$  with*

$$(4.9) \quad h_0(0) = a, \quad h_1(0) = \frac{27}{4}\lambda, \quad h_2(0) = -\frac{1}{27}a^3, \quad h_3(0) = 3a,$$

for some  $a \in \mathbb{R}$ ,  $a \neq 0$ .

*Proof.* It is enough to prove the “if” part. We clearly define the  $f_i$ 's in terms of the  $h_i$ 's using (4.1). Then the  $f_i$ 's satisfy the differential system (3.3) (1)-(5) on the open set  $M_\circ$ ,

while we have to prove that the algebraic conditions (3.3) (6)-(7) are also satisfied. By Lemma 3.1 the two quantities

$$F_1 := f_4 + f_3 + \frac{1}{6} \lambda f_0^2,$$

$$F_2 := f_0^6 - (f_1 f_4 - f_3^2)(f_2 f_3 - f_4^2) + \frac{1}{4}(f_1 f_2 - f_3 f_4)^2$$

are actually constant on the open set  $t \neq 0$  and both vanish at  $t = 0$ , hence they vanish everywhere. Moreover condition (4.2) are also satisfied, so that the functions  $f_i$ 's define an NP-structure which extends to a  $G_2$ -structure on the whole M.  $\square$

We now prove the existence in the following

**Lemma 4.4.** *The equation (4.7) admits a unique solution  $h = (h_0, \dots, h_3)$  which is smooth in an interval  $(-\varepsilon, \varepsilon)$  for  $(\varepsilon \in \mathbb{R}^+)$  and satisfies the initial conditions (4.9). Moreover, the functions  $h_0, \dots, h_3$  are even.*

*Proof.* We use Theorem 4.7 in [16] (see also [7]), which asserts that the singular initial value problem we are considering has a unique smooth solution provided the following conditions are fulfilled:

- a)  $A(\bar{h}) = 0$ ;
- b)  $dA|_{\bar{h}} - l \cdot \text{Id}$  is invertible for all  $l \in \mathbb{N}$ ,  $l \geq 1$ .

Condition (a) has been already fixed, while we can easily compute

$$dA|_{\bar{h}} = \begin{bmatrix} -5 & 0 & -27a^{-2} & 2/3 \\ -81\lambda a^{-1} & -4 & 0 & 27\lambda a^{-1} \\ 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & -2 \end{bmatrix}$$

whence

$$\det(dA|_{\bar{h}} - l \cdot \text{Id}) = l(l+4)(l^2 + 7l + 6) > 0, \quad l \geq 1.$$

Therefore for any  $a \neq 0$ , we obtain a unique solution  $(h_0(t), \dots, h_3(t))$  of the singular system (4.7) with initial data  $\bar{h}$ . The fact that the solutions  $h_i$  are even follows from uniqueness and the fact that  $B(h, -t) = -B(h, t)$  for all  $(h, t) \in \mathbb{R}^4 \times \mathbb{R}$ .  $\square$

We now investigate the question when two NP-structures determined by two solutions  $h, \bar{h}$  are isomorphic. By the results obtained in §3.2, we see that the associated functions  $f := (f_i)_{i=0, \dots, 4}$  and  $\bar{f} := (\bar{f}_i)_{i=0, \dots, 4}$  are related by a transformation  $\tau$  in the group generated by  $\tau_{12}, \tau_{13}$ . A simple check shows that  $\tau(f) = \bar{f}$  satisfies the extendability conditions (3.11), (3.12), (3.13), if and only if  $\tau = \tau_{13}$ . This transformation is equivalent to reversing  $a \mapsto -a$  in (4.9) and therefore we can restrict to  $a > 0$ . This concludes the proof of the main Theorem 4.1.

**Remark 4.5.** The two (locally) homogeneous solutions (with  $\lambda = 1$ ) correspond to the values  $a = 36$  (round sphere) and  $a = \frac{108}{5}$  (squashed sphere), as it can be easily seen using §3.1 and the remark 3.2 (recall that the opposite value of  $a$  gives the same NP-structure).

We conclude this section pointing out some features of a possible  $G$ -invariant NP-structure in the family  $\mathcal{F}_a$  on  $M$  which extends to a global one on some  $G$ -equivariant compactification  $\overline{M}$ . Note that  $\overline{M}$  is diffeomorphic to either  $S^7$  or to  $S^3 \times S^4$ .

**Proposition 4.6.** *Any NP-structure in the family  $\mathcal{F}_a$ ,  $a > 0$ , which extends to a global NP-structure on some  $G$ -equivariant compactification  $\overline{M}$  and has not constant curvature is proper, namely the relative cone metric has full holonomy  $\text{Spin}(7)$ .*

*Proof.* Let  $(\overline{M}, \overline{g}, \overline{\varphi})$  be a  $G$ -invariant NP-structure extending some element in  $\mathcal{F}_a$ , say for  $a = \overline{a}$ , and with  $\overline{g}$  of non-constant curvature. We will denote by  $\overline{G}$  the connected component of the full isometry group of  $(\overline{M}, \overline{g})$ . We can also suppose that  $\overline{G}$  does not act transitively on  $\overline{M}$ . Indeed, if  $\overline{M} \cong S^7$ , it is well known that there are precisely two homogeneous Einstein metrics, which correspond to the round and the squashed sphere (see e.g. [29]). If  $\overline{M} \cong S^3 \times S^4$ , using a result by Kamberich (see [23], p. 274),  $\overline{G}$  contains a transitive subgroup  $N$  locally isomorphic to  $SU(2) \times SO(5)$ , acting on  $\overline{M}$  in a standard way. Now any  $N$ -invariant Riemannian metric on  $\overline{M}$  is reducible, while  $\overline{g}$  is irreducible.

Now, as  $\overline{M}$  is compact and simply connected, our claim follows if we show that the holonomy  $\mathcal{H}$  of the cone metric on  $\overline{M} \times \mathbb{R}^+$  is not  $SU(4)$  nor  $Sp(2)$ .

(a) We prove that  $\mathcal{H} \neq SU(4)$ , i.e. that  $(\overline{M}, \overline{g})$  does not carry any Sasakian structure which is not part of a 3-Sasakian structure. If  $\xi$  is the unit length Killing vector field giving the Sasakian structure, a classical theorem by Tanno ([28]) states that  $\xi$  belongs to the center of  $\overline{\mathfrak{g}}$ , since  $\overline{g}$  has non-constant curvature. On the other hand the isotropy representation of  $K^+$  at  $p$  has no non-trivial fixed vector, forcing  $\xi_p = 0$ , a contradiction.

(b) We now suppose that  $(\overline{M}, \overline{g})$  carries a 3 Sasakian structure given by a Lie algebra  $\mathfrak{s} \cong \mathfrak{sp}(1)$  generated by three unit length Killing vector fields  $\xi_1, \xi_2, \xi_3$ . Again by a result due to Tanno ([28]) we know that the Lie algebra  $\overline{\mathfrak{g}}$  splits as a sum of ideals  $\overline{\mathfrak{g}} = \overline{\mathfrak{g}}_o \oplus \mathfrak{s}$ , where  $\overline{\mathfrak{g}}_o$  is the centralizer of  $\mathfrak{s}$  in  $\overline{\mathfrak{g}}$ . As  $\overline{G}$  is supposed to act non-transitively on  $\overline{M}$ , it has the same orbits as its subgroup  $G$ . In particular,  $\overline{G}$  acts transitively on  $S^3 \times S^3$ .

**Lemma 4.7.** *The semisimple part  $\overline{\mathfrak{g}}_s$  of  $\overline{\mathfrak{g}}$  is isomorphic to  $3\mathfrak{su}(2)$  or to  $4\mathfrak{su}(2)$ .*

*Proof.* The isotropy subalgebra  $\mathfrak{f}$  of  $\overline{\mathfrak{g}}_s$  at a  $G$ -regular point  $q$  embeds as a compact subalgebra of  $\mathfrak{so}(6)$ . Looking at the list of maximal subalgebras in  $\mathfrak{so}(6)$ , we see that  $\dim \mathfrak{f} \leq 7$ , unless  $\mathfrak{f} \cong \mathfrak{so}(5)$  or  $\mathfrak{su}(3), \mathfrak{u}(3)$ . If  $\mathfrak{f}$  contains a copy of  $\mathfrak{su}(3)$ , it acts transitively on the unit sphere of  $T_q(Gq)$ , hence  $Gq \cong S^3 \times S^3$  has constant curvature, a contradiction; the case  $\mathfrak{f} \cong \mathfrak{so}(5)$  can be ruled out using a result about the gaps in the dimension of the isometry group of a Riemannian manifold (Thm. 3.3. in [21]). Therefore  $\dim \mathfrak{f} \leq 7$  and  $\dim \overline{\mathfrak{g}}_s \leq 13$ . On the other hand,  $\overline{G}$  contains a subgroup isomorphic to  $SU(2)^2$  that acts on  $Gq$  (almost) freely; therefore by a result in [23], Cor.3, p. 237, the algebra  $\overline{\mathfrak{g}}_s$  contains an ideal  $\mathfrak{a}$  isomorphic to  $2\mathfrak{su}(2)$ . Then  $\overline{\mathfrak{g}}_s = \mathfrak{a} \oplus \mathfrak{b}$  for some other semisimple ideal  $\mathfrak{b}$ . As  $\mathfrak{g} \subseteq \overline{\mathfrak{g}}_s$  and  $1 \leq \dim \mathfrak{b} \leq 7$ , we see that  $\mathfrak{b}$  is isomorphic to  $2\mathfrak{su}(2)$  or to  $\mathfrak{su}(2)$  and our claim follows.  $\square$

Suppose now  $\overline{\mathfrak{g}}_s = 4\mathfrak{su}(2)$ . This implies that the semisimple part of  $\overline{\mathfrak{g}}_o$ , say  $\mathfrak{l}$ , is isomorphic to  $3\mathfrak{su}(2)$ . As  $[\mathfrak{l}, \mathfrak{s}] = 0$ , the isotropy  $\mathfrak{l}_q$  leaves a 3-dimensional subspace fixed, hence  $\dim \mathfrak{l}_q \leq 3$ . This implies that  $\dim Lq = 6$ , where  $L$  is the subgroup with Lie algebra  $\mathfrak{l}$ . Then  $L$  has the same orbits as  $G$  and leaves each  $\xi_i$  fixed, a contradiction by the same arguments used

in (a). Therefore we are left with  $\bar{\mathfrak{g}}_{\mathfrak{s}} = 3\mathfrak{su}(2) = \mathfrak{g}$ , i.e. the ideal  $\mathfrak{s}$  is one of the three ideals, say  $\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3$  of  $\mathfrak{g}$ . If we denote by  $g_i(t)$ ,  $i = 1, 2, 3$  the functions

$$g_1(t) := \|\hat{e}_2\|_{\gamma(t)}^2, \quad g_2(t) := \|\hat{e}_5\|_{\gamma(t)}^2, \quad g_3(t) := \|\hat{e}_2 + \hat{e}_5\|_{\gamma(t)}^2,$$

we are reduced to considering the three possibilities when  $g_i$  are constant functions.

The condition  $g_1(t) = f_3^2 - f_1 f_4 = \text{const}$  can be easily ruled out using Prop.4.3 and (4.1). Moreover under the admissible transformation  $\tau_{13}$  the function  $g_2$  goes over to  $g_3$ , so that we can confine ourselves to the case  $g_2(t) = \text{const}$ . Using Maple we can write the series expansion of the solutions  $f_i$ 's as well as of the function  $g_2(t)$  for  $\lambda = 1$  obtaining

$$g_2(t) = \frac{1}{9}a^2 + \left(-\frac{5}{576}a^2 + \frac{1}{8}a + \frac{27}{4}\right)t^2 + o(t^2).$$

Therefore if  $g_2$  is constant we immediately get  $a = 36, -\frac{108}{5}$ , which correspond to the known homogenous solutions. This concludes the proof.  $\square$

**Remark 4.8.** By the previous result, we can also show that none of the non-homogeneous Einstein metrics in the family  $\mathcal{F}$ , whenever extended to  $S^7$ , is one of the metrics found by Böhm in [7]. Indeed assume  $g$  is a metric in the family  $\mathcal{F}$  which is also in the Böhm's family. The NP-structure associated to  $g$  is proper, hence its full isometry group preserves the NP-structure and therefore has dimension less or equal to 9 by Thm. 7.1 in [18]. On the other hand the metrics found in [7] are invariant under a bigger group, namely  $SO(4) \times SO(4)$ .

#### REFERENCES

- [1] A.V. Alekseevsky, D.V. Alekseevsky,  $G$ -manifolds with one dimensional orbit space. *Adv. in Sov. Math.*, **8**, 1–31, 1992.
- [2] M. Alexandrino, R. Bettiol, Lie Groups and geometric aspects of isometric actions. Springer Verlag 2015.
- [3] B. Alexandrov, U. Semmelmann, Deformations of nearly parallel  $G_2$ -structures. *Asian J. Math.*, **16**, 713–744, 2012.
- [4] C. Bär, Real Killing spinors and holonomy. *Comm. Math. Phys.* **154**, 509–521, 1993.
- [5] A. Bilal, S. Metzger, Compact weak  $G_2$ -manifolds with conical singularities. *Nuclear Phys. B* **663**, 343–364, 2003.
- [6] C.P. Boyer, K. Galicki, Sasakian geometry, holonomy, and supersymmetry. in *Handbook of Pseudo-Riemannian Geometry and Supersymmetry*, IRMA Lectures in Math. and Theor. Physics, **16**, Eur.Math.Soc., 39–83, 2010.
- [7] C. Böhm, Inhomogeneous Einstein metrics on low-dimensional spheres and other low-dimensional spaces. *Invent. Math.* **134** 145–176, 1998.
- [8] R. Bryant, Metrics with exceptional holonomy. *Annals of Math.* **126** 525–576, 1987.
- [9] R. Bryant, R. Harvey, Submanifolds in hyper-Kähler geometry. *J. Amer. Math. Soc.* **2**, 1–31, 1989.
- [10] R. Bryant, S. Salamon, On the construction of some complete metrics with exceptional holonomy. *Duke Math. J.* **58**, 829–850, 1989.
- [11] J.-B. Butruille, Classification des variétés approximativement kähleriennes homogènes. *Ann. Glob. Anal. Geom.* **27** 201–225, 2005.
- [12] R. Cleyton and A. Swann, Cohomogeneity-one  $G_2$ -structures. *J. Geom. Phys.*, **44**, 202–220, 2002.
- [13] V. Cortés, T. Leistner, L. Schäfer, F. Schulte-Hegensbach, Half-flat structures and special holonomy, *J. London Math. Soc.*, **102**, 113–158, 2011.
- [14] M. Fernández, A. Gray, Riemannian manifolds with structure group  $G_2$ . *Ann. Mat. Pura Appl.*, **32**, 19–45, 1982.
- [15] M. Fernández, S. Ivanov, V. Muñoz, L. Ugarte, Nearly hypo structures and compact nearly Kähler 6-manifolds with conical singularities, *J. Lond. Math. Soc.*, **78**, 580–604, 2008.

- [16] F. Foscolo, M. Haskins, New  $G_2$ -holonomy cones and exotic nearly Kähler structures on  $S^6$  and  $S^3 \times S^3$ . *Ann. of Math. (2)* **185**, 59-130, 2017.
- [17] F. Foscolo, M. Haskins, J. Nordström, Infinitely many new families of complete cohomogeneity one  $G_2$ -manifolds:  $G_2$  analogues of the Taub-NUT and Eguchi-Hanson spaces. arXiv:1805.02612v2, 2018.
- [18] Th. Friedrich, I. Kath, A. Moroianu, U. Semmelmann, Nearly parallel  $G_2$ -structures. *J. Geom. Phys.* **23**, 259–286, 1997.
- [19] A. Gray, Weak holonomy groups. *Math. Z.* **123**, 290–300, 1971.
- [20] N. Hitchin, Stable forms and special metrics. In *Global differential geometry: the mathematical legacy of Alfred Gray (Bilbao, 2000)*, v. 288 of *Contemp. Math.*, pp. 70–89. Amer. Math. Soc., 2001.
- [21] S. Kobayashi, Transformation groups in Differential Geometry. Springer-Verlag 1972.
- [22] G.D. Mostow, The extensibility of local Lie groups of transformations and groups on surfaces. *Ann. of Math.*, **52**, 606–636, 1950.
- [23] A.L. Onishchik, Topology of transitive transformation groups. Johann Ambrosius Barth, 1994
- [24] F. Podestà, A. Spiro, Six-dimensional nearly Kähler manifolds of cohomogeneity one. *J. Geom. Phys.* **60**, 156-164, 2010.
- [25] F. Podestà, A. Spiro, Six-dimensional nearly Kähler manifolds of cohomogeneity one (II). *Comm. Math. Phys.*, **312**, 477-500, 2012.
- [26] A. Spiro, Lie pseudogroups and locally homogeneous Riemannian spaces. *Boll. U.M.I.*, **6B**, 843–872, 1992.
- [27] S. Stock, Lifting  $SU(3)$ -structures to nearly parallel  $G_2$ -structures, *J. Geom. Phys.*, **59**, 1–7, 2009.
- [28] S. Tanno, On the isometry groups of Sasakian manifolds. *J. Math. Soc. Japan*, **22**, 579–590, 1970.
- [29] W. Ziller, Homogeneous Einstein metrics on spheres and projective spaces. *Math. Ann.* **259**, 351–358, 1982.

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