Bed-load transport equation on arbitrarily sloping beds

Bed-Load Transport Equation on Arbitrarily Sloping Beds

Simona Francalanci[1] and Luca Solari[2]

Abstract: This work presents a simple tool to evaluate bed-load transport intensity and direction on arbitrarily sloping beds with local longitudinal and transversal inclinations up to 25° and in the case of uniform sediments and low values of the applied Shields stress. The tool is composed of a set of equations which fit the results obtained via an iterative procedure by the semiempirical model recently proposed in 2003 by Parker, Seminara, and Solari. The tool provides a fully nonlinear description of bed-load transport which overcomes the limitations of linear formulations developed in the case of negligible local bed inclinations. The proposed tool can be easily implemented in any morphodynamic model to describe the evolution of the bottom topography and to capture the dynamics of relevant sloping beds.

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Introduction

The evaluation of bed-load transport on an arbitrarily sloping bed is one of the crucial aspects in any morphodynamic model which considers the evolution of the bed: when the bed is tilted in both the longitudinal and the transversal direction, gravity plays an important role in particle dynamics and on bed-load transport.

This problem has been recently tackled from a theoretical point of view by Seminara et al. (2002) by means of a vectorial formulation of bed-load transport showing the failure of Bagnold's hypothesis; based on this failure, a semiempirical nonlinear model of bed-load transport on arbitrarily sloping beds at low Shields stress has been proposed by Parker et al. (2003). The results of this model were experimentally verified by Francalanci and Solari (2007), showing that sediments may experience a large deviation from the direction of the applied Shields stress thus suggesting that the lateral component of bed-load transport can attain relatively large values. The recent experimental work by Francalanci et al. (2006) which devoted its study to the evolution of a mobile bed from the initial, laterally sloping configuration, to the final, laterally flat equilibrium configuration, clarifies this point. In particular, the measured bed profiles are compared with the results obtained from a three-dimensional numerical model employing both a linear (Ikeda 1982) and the nonlinear formulation (Parker et al. 2003) to evaluate the bed-load transport over an arbitrarily sloping bed. Results suggest that the linear formulations devised for beds with small inclinations (e.g., Ashida and Michiue 1972; Engelund and Fredsoe 1976), lead to a great underestimation of the lateral bed-load transport which prevents the model from adequately describing the evolution of the bed, whereas in the case of the nonlinear formulation of Parker et al. (2003) the time and spatial morphodynamic evolution of the bed are well reproduced. These results suggest the adoption of Parker et al.'s model in any morphological modeling based on the evaluation of bed-load transport. However, in this model bed-load transport intensity and direction are obtained by means of an iterative method, which can largely increase the numerical complexity of a morphodynamic model that considers local arbitrarily sloping beds. Hence, in this work simplified equations that approximate the Parker et al. (2003) model are derived. The proposed equations can be easily implemented in any morphodynamic model to describe the evolution of the bottom topography subject to the action of a free surface flow in the case of dominant bed-load transport.

Theoretical Framework

The main physical aspects and equations of the Seminara et al. (2002) and Parker et al. (2003) models are reported here, while the reader is referred to the latter papers for details of the analysis. Let us consider a free surface flow on a nonplanar cohesionless bed (see Fig. 1 for notations). Let P be a point lying on the bed and let z be the coordinate of a vertical axis with origin in P. Let the flow exert a tangential stress on the bed $\boldsymbol{\tau}$. The direction of such stress and the vertical one provide the only two externally imposed directions on the problem. Let α be the longitudinal and φ the lateral inclinations of the bed in P: α is defined as the inclination of the line obtained intersecting the bed surface with a vertical plane aligned with the direction of bed shear stress; similarly φ is the inclination of the line obtained by intersecting the bed surface with a vertical plane orthogonal to the bed-shear stress.

The bed-load transport vector per unit width, which is made dimensionless with Einstein's scale, is evaluated as follows

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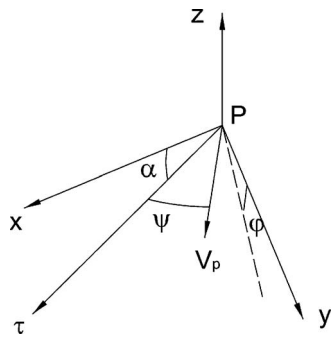

Fig. 1. Sketch and notations

$$\hat{\mathbf{q}} = \hat{\xi} \cdot \hat{\mathbf{V}}_p \tag{1}$$

where $\hat{\xi}$=volume areal concentration of bed load made dimensionless with the sediment diameter D and $\hat{\mathbf{V}}_p$=layer-averaged mean velocity vector of bed-load particles made dimensionless with $\sqrt{(s-1)gD}$, where g=gravity and s=relative density of sediment to water.

The average particle velocity vector in terms of intensity $|\hat{\mathbf{V}}_p|$ and direction $\hat{\mathbf{s}}_p$ is obtained by imposing a vectorial balance between averaged active and resistive forces, where the former is represented by the drag force evaluated by means of a drag coefficient C_d, and the tangential component along the bed of the submerged particle weight; the latter interprets the effect of continuous or intermittent hydrodynamic interactions of particles with the bed and is estimated through a coefficient of dynamic friction μ_d. Note that in general the direction of particle motion $\hat{\mathbf{s}}_p$ does not coincide with the direction of the applied shear stress at the bed $\boldsymbol{\tau}$ unless the bed is longitudinally tilted only; the angle between the two latter vectors is called the *deviation angle* and denoted by ψ.

In order to find the average areal concentration of sediment in motion many models rely on Bagnold's assumption. The theoretical work of Seminara et al. (2002) shows that the Bagnold constraint is not valid; based on this finding Parker et al. (2003) introduce an entrainment formulation, according to which a dynamic equilibrium is maintained by a balance between entrainment of bed grains into the bed-load layer and deposition of bed-load grains onto the bed. The bed shear stress in the presence of a saltating bed-lod layer will be reduced below $\boldsymbol{\tau}$ to a value $\boldsymbol{\tau_b}$ due to the transfer of fluid phase momentum to solid phase momentum via drag. The entrainment flux of bed particles into the bed-load layer $\hat{E}$ is determined with an empirical function that ultimately agrees with the experimental results of Fernandez Luque and Van Beek (1976). In particular it can be expressed as an increasing function of the excess of the fluid shear stress at the bed over the critical shear stress for the onset of particle motion with the expression $\hat{E}=A_e \cdot (\tau_{*b}-\tau_{*c})^{3/2}$, where A_e=parameter to be evaluated; τ_{*b}=dimensionless Shield stress defined as $\boldsymbol{\tau}_b/(\rho(s-1)gD)$; and τ_{*c}=critical Shields stress for the onset of sediment motion on an arbitrarily sloping bed. Note that the entrainment flux is a non-negative, monotonically increasing function of $\tau_{*b}-\tau_{*c}$, which may be taken as a measure of the residual turbulent activity at the bed which appears to be responsible for the sediment motion. According to this model equilibrium bed-load transport conditions are reached not when the fluid shear stress at the bed reaches the threshold value, but when the entrainment rate of bed particles into the bed-load layer equals the deposition rate of bed-load particles onto the bed. The deposition rate

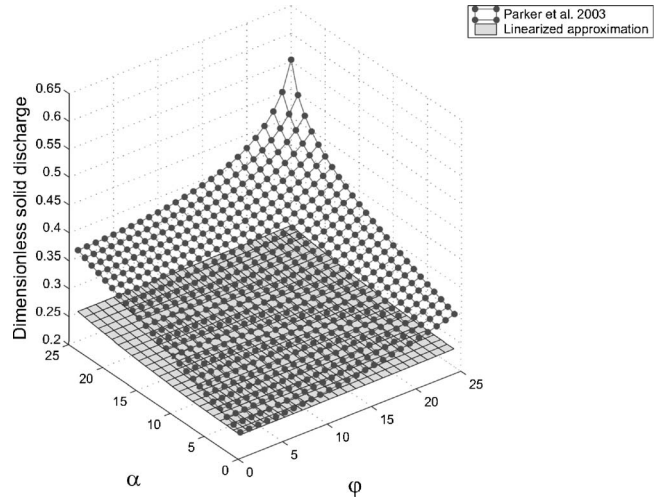

Fig. 2. Parker et al. (2003) model versus its linearized approximation as function of longitudinal inclination α and of transversal inclination φ of bed, for $\tau_*/\tau_{*co}=5$

$\hat{D}$ is set equal to $A_d \cdot (\tau_*-\tau_{*b})^{1/2}\hat{\xi}$; where A_d=parameter to be evaluated and τ_*=Shields stress based on $\boldsymbol{\tau}$. It is useful to compare the fully nonlinear results of the entrainment formulation for bed-load transport on a finite, arbitrary slope against its linear approximation valid for small slopes. Note that the linearized approximation is formally equivalent to bed-load transport formulations devised for a modestly sloping bed (e.g. Parker 1984; Struiksma et al. 1985). In Fig. 2 the sediment transport rate from the fully nonlinear model of Parker et al. (2003) as a two-dimensional function of the longitudinal and transversal bed inclinations is compared with its linearized approximation valid for low values of the inclination of the bed: for a given value of τ_*/τ_{*co} the dimensionless bed-load transport per unit width rate increases more as the longitudinal and transversal inclinations increase; and the percentual residuals of dimensionless solid discharge become higher for larger values of the bed inclinations showing great underestimation of the linearized approximation of the model (Fig. 3). Moreover further results, not reported here,

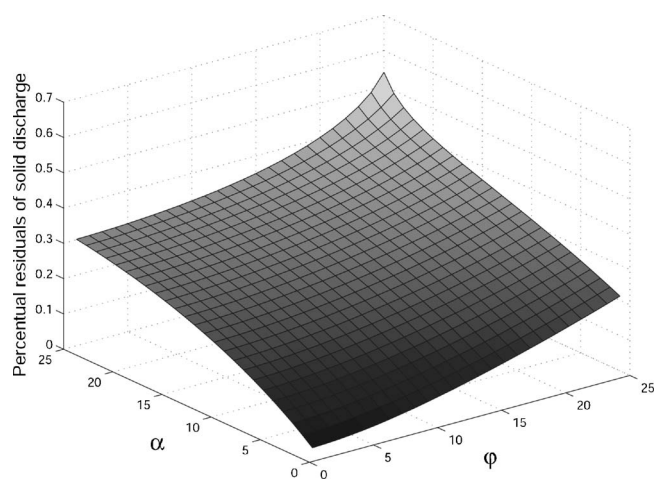

Fig. 3. Percentual residuals of dimensionless solid discharge between Parker et al. (2003) model and its linearized approximation, as function of longitudinal inclination α and of transversal inclination φ of bed, for $\tau_*/\tau_{*co}=5$

Table 1. Coefficients for A_q, B_q, C_q

a_q	b_q	c_q	d_q	e_q
$8.677849 \cdot 10^{-3}$	$-2.521125 \cdot 10^{-3}$	$-6.420916 \cdot 10^{-6}$	-0.2897854	$2.281153 \cdot 10^{-4}$
f_q	g_q	h_q	i_q	j_q
$-7.952039 \cdot 10^{-5}$	$9.543719 \cdot 10^{-3}$	$1.068214 \cdot 10^{-3}$	$7.940476 \cdot 10^{-6}$	$-2.1673804 \cdot 10^{-2}$
k_q	l_q	m_q	n_q	
$8.2769811 \cdot 10^{-4}$	$2.1321179 \cdot 10^{-5}$	$3.1805551 \cdot 10^{-6}$	$8.8845287 \cdot 10^{-6}$	

show that the residuals also increase with higher values of the Shields stress ratio.

Bed-Load Transport Equations

The interpolated model has been evaluated on a discrete grid of values from the complete analytical solution given by Parker et al. (2003), for each combination of the longitudinal and transversal bed inclinations up to 25°. The semiempirical model of Parker et al. (2003) requires as input various parameters, namely, A_e, A_d, C_d, the ratio of the lift and drag coefficients C_l/C_d, μ_d, the angle of repose of the sediments ϕ, and a dimensionless coefficient λ that is a function of the ratio between the critical Shields stress for the cessation of the bed-load motion and the critical Shields stress. The parameters A_e, A_d, and λ were based on the experimental results of Fernandez Luque and Van Beek (1976), while the others were fitted in order to obtain the best agreement of the theoretical model with the experimental values of Francalanci and Solari (2007): to this aim several simulations were carried out with different values of the drag coefficient, the dynamic friction coefficient, and the ratio of lift to drag coefficient. The best agreement with the semiempirical model was found for the following set of values: $\mu_d=0.3$, $C_d=0.4$, $C_l/C_d=1.25$, and $\phi=35°$; note that these values are very much reasonable and in agreement with the literature. In particular with regard to the dynamic friction coefficient, Bagnold (1956) experimentally obtained a value of μ_d of about 0.32 for smooth spheres; the results of Abbott and Francis (1977) and Francis (1973) showed a mean value of about 0.4, while Abbott and Francis (1977) also suggest an increase of the average value of μ_d from 0.38 to 0.71 for increasing Shields stress. On the contrary Nino and Garcia (1994) obtained a fairly constant value of μ_d of about 0.3, for larger particle Reynolds numbers than Abbott and Francis (1977). A higher value of μ_d of about 0.8 has been predicted numerically by Sekine and Kikkaua (1992). The relative uncertainty of the parameter μ_d is due to the fact that the dynamic friction coefficient is related to a physical property characterizing the contact between different solid materials and the hydrodynamic process of the particle rebound as well; such a process presents some aspects not satisfactorily clarified yet. The critical Shields stress for the onset of sediment motion in the case of vanishing bed slope τ_{*co} was experimentally estimated by Francalanci and Solari (2007) to be around the mean value of 0.03.

The dimensionless intensity of bed-load transport rate per unit width $|\hat{\mathbf{q}}|$, the dimensionless particle velocity intensity $|\hat{\mathbf{V}}_p|$, and the deviation angle ψ of particle velocity from the bed shear stress direction are expressed as functions of the ratio between the applied Shields stress τ_* and the critical Shields stress τ_{*co}, for given values of the longitudinal α and transversal φ inclinations, while the dimensionless average areal concentration is evaluated from the previous quantities. The dimensionless bed-load transport rate and the dimensionless particle velocity were found to be second-order polynomial functions of the Shields stress ratio, and the deviation angle was found to be a power function of the Shields stress ratio, as shown in the following equations

$$|\hat{\mathbf{q}}| = A_q(\alpha,\varphi) \cdot \left(\frac{\tau_*}{\tau_{*co}}\right)^2 + B_q(\alpha,\varphi) \cdot \left(\frac{\tau_*}{\tau_{*co}}\right) + C_q(\alpha,\varphi) \tag{2}$$

$$|\hat{\mathbf{V}}_p| = A_{Vp}(\alpha,\varphi) \cdot \left(\frac{\tau_*}{\tau_{*co}}\right)^2 + B_{Vp}(\alpha,\varphi) \cdot \left(\frac{\tau_*}{\tau_{*co}}\right) + C_{Vp}(\alpha,\varphi) \tag{3}$$

$$\psi = A_\psi(\alpha,\varphi) \cdot \left(\frac{\tau_*}{\tau_{*co}}\right)^{B_\psi(\alpha,\varphi)} \tag{4}$$

For given values of τ_*, α, φ, Eqs. (2)–(4) lead to unique values of the estimated quantities; as the shape of the curves is always the same, all the outputs of the model for $\alpha=0-25°$, and $\varphi=0-25°$ have been fitted to the same parametric equations, in order to obtain the values of the various coefficients A, B, C in Eqs. (2)–(4). At this point, the problem reduces to the estimation of the relationships between the coefficients $A(\alpha,\varphi), B(\alpha,\varphi), C(\alpha,\varphi)$, and the parameters (α,φ). The best expressions that fit the results of the model of Parker et al. (2003) have been evaluated through a commercial software (TableCurve3D), in order to maximize the fitting and minimize the total number of coefficients in the expression.

Coefficients of Bed-Load Transport Equations

The coefficients A_q, B_q, C_q of the interpolated expression of the dimensionless solid discharge per unit width are expressed by the following equations

$$A_q(\alpha,\varphi) = \frac{a_q + b_q \ln \alpha + c_q \cdot \varphi}{1 + d_q \ln \alpha + e_q \cdot \varphi + f_q \cdot \varphi^2} \tag{5a}$$

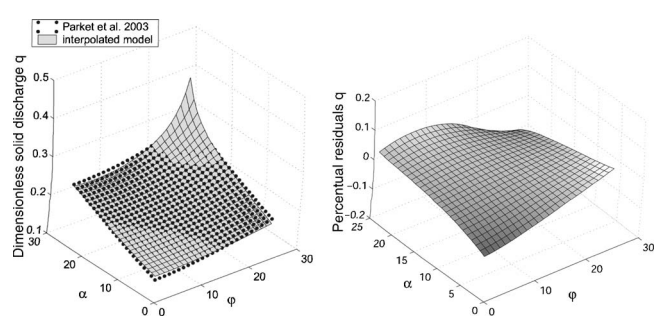

Fig. 4. Comparison between Parker et al. (2003) model and interpolated model for $\tau_*/\tau_{*co}=4$: dimensionless solid discharge per unit width and percentual residuals

Table 2. Coefficients for A_{Vp}, B_{Vp}, and C_{Vp}

a_{Vp}	b_{Vp}	c_{Vp}	d_{Vp}	e_{Vp}
$-4.613664\cdot10^{-2}$	$4.7361025\cdot10^{-6}$	$-5.3148316\cdot10^{-6}$	0.85679823	$-5.3598537\cdot10^{-5}$
f_{Vp}	g_{Vp}	h_{Vp}	i_{Vp}	j_{Vp}
$6.0047913\cdot10^{-5}$	−0.18761477	$4.4260106\cdot10^{-2}$	$2.7469067\cdot10^{-2}$	$7.2199853\cdot10^{-4}$

$$B_q(\alpha,\varphi) = g_q + h_q\cdot\alpha + i_q\cdot\varphi^2 \tag{5b}$$

$$C_q(\alpha,\varphi) = j_q + k_q\cdot\alpha + l_q\cdot\alpha^2 + m_q\cdot\varphi^2 + n_q\cdot\alpha\varphi \tag{5c}$$

The coefficients of Eqs. (5*a*)–(5*c*) are given in Table 1. For low values of the longitudinal inclination, $\alpha < 1.1°$, the function $\ln\alpha$ needs to be substituted with the function $0.079\cdot\alpha^2$ in Eq. (5*a*).

The interpolated dimensionless solid discharge shows a good fit with the fully nonlinear model (R^2=0.975), and the percentual residuals present low values (Fig. 4). Moreover Fig. 4 shows that the bed-load transport intensity increases with the local bed inclination even more so as both α and φ attain large values. Note also that the effect of the lateral bed inclination becomes more pronounced as α increases.

The coefficients of the dimensionless velocity are evaluated as follows

$$A_{Vp}(\alpha,\varphi) = a_{Vp} + b_{Vp}\cdot\varphi^2 + c_{Vp}\cdot\alpha\varphi \tag{6a}$$

$$B_{Vp}(\alpha,\varphi) = d_{Vp} + e_{Vp}\cdot\varphi^2 + f_{Vp}\cdot\alpha\varphi \tag{6b}$$

$$C_{Vp}(\alpha,\varphi) = g_{Vp} + h_{Vp}\cdot\alpha + i_{Vp}\cdot\varphi + j_{Vp}\cdot\alpha\varphi \tag{6c}$$

The coefficients of Eqs. (6*a*)–(6*c*) are given in Table 2. The dimensionless particle velocity intensity is very well fitted by the interpolated model, as shown in Fig. 5. The low values of the percentual residuals and R^2=0.99 confirm the good fit. Moreover Fig. 5 shows that the intensity of particle velocity increases with the local bed inclination; in particular it appears that this quantity is largely affected by the longitudinal bed inclination while the lateral bed inclination seems to plays a relatively minor role.

The coefficients of the particle deviation angle are evaluated as follows

$$A_\psi(\alpha,\varphi) = a_\psi\cdot\alpha + b_\psi\cdot\varphi + c_\psi\cdot\alpha^2 + d_\psi\cdot\varphi^2 + e_\psi\cdot\alpha\varphi + f_\psi\cdot\alpha^3 + g_\psi\cdot\alpha\varphi^2 \tag{7a}$$

$$B_\psi(\alpha,\varphi) = h_\psi + i_\psi\cdot\alpha + l_\psi\cdot\alpha^2 + m_\psi\cdot\varphi^2 + n_\psi\cdot\alpha\varphi \tag{7b}$$

The coefficients of Eqs. (7*a*) and (7*b*) are given in Table 3. The particle deviation angle is well fitted by the power law with a R^2=0.999. The relatively high values of the percentual residuals shown in Fig. 6 are due to the low value of ψ, and the effective difference is minimum as shown in the plot of the particle deviation angle. Moreover Fig. 6 shows that the angle of deviation increases with the local bed inclination, in particular the effect of an increasing lateral bed inclination produces larger ψ when the longitudinal bed inclination is relatively low.

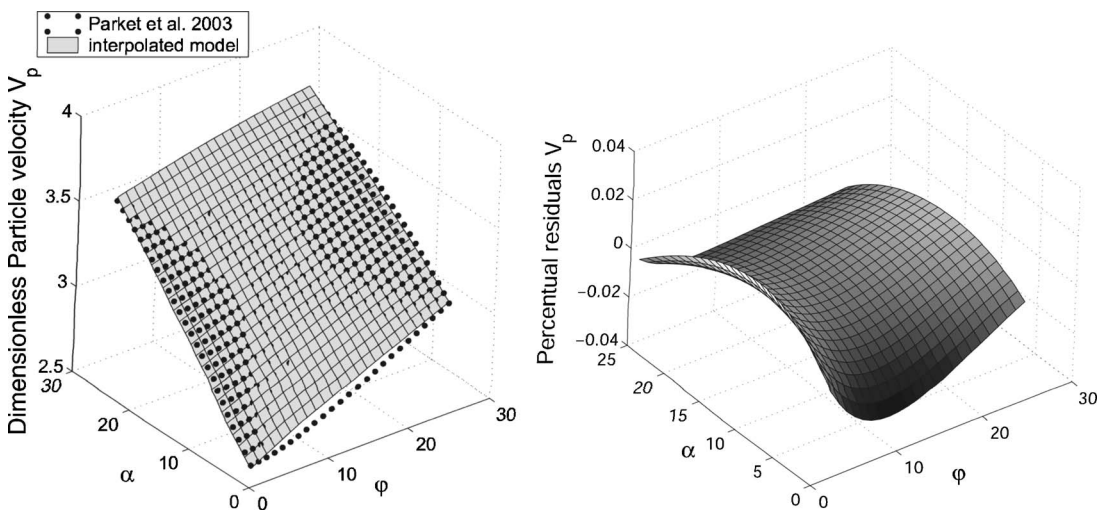

Fig. 5. Comparison between Parker et al. (2003) model and interpolated model for $\tau*/\tau*_{co}$=4: dimensionless particle velocity and percentual residuals

Table 3. Coefficients for A_ψ and B_ψ

a_ψ	b_ψ	c_ψ	d_ψ	e_ψ	f_ψ
−0.14685725	2.3786164	$9.0826885\cdot10^{-3}$	$-4.5780649\cdot10^{-2}$	$-6.2852556\cdot10^{-2}$	$-1.0321542\cdot10^{-4}$
g_ψ	h_ψ	i_ψ	l_ψ	m_ψ	n_ψ
$1.275393\cdot10^{-3}$	−0.48075694	$8.4818038\cdot10^{-3}$	$-1.1241921\cdot10^{-4}$	$1.339106\cdot10^{-4}$	$-2.22984992\cdot10^{-4}$

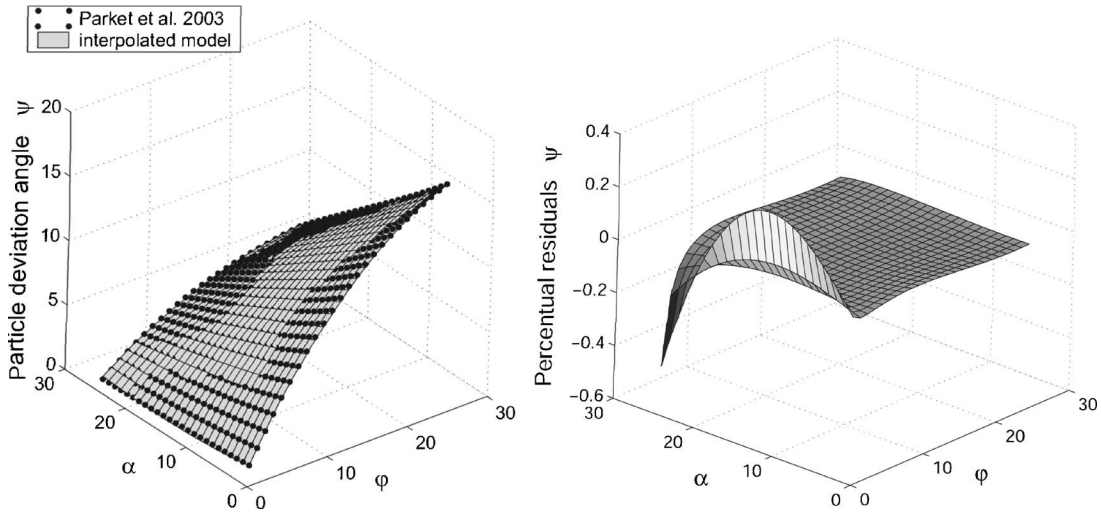

Fig. 6. Comparison between Parker et al. (2003) model and interpolated model for $\tau_*/\tau_{*co}=4$: particle deviation angle and percentual residuals

Conclusions

In this work a simplified set of equations to evaluate bed-load transport intensity and direction obtained via interpolation of the nonlinear model of Parker et al. (2003) is derived. The proposed equations are valid in the following range of the main variables: values of the ratio between the applied Shields stress and the critical Shields stress for the onset of motion on a nearly horizontal bed up to 5°; and values of longitudinal and transversal inclinations between 0 and 25°.

Despite the large number of coefficients, the interpolated model proposed here presents simple relationships and can be easily implemented in a numerical morphodynamic model to study the evolution of the bed.

Notation

The following symbols are used in this technical note:

A_d = coefficient of sediment deposition flux;
A_e = coefficient of sediment entrainment (pickup) flux;
A_q–C_q = coefficients to calculate dimensionless bed-load transport intensity per unit width [Eqs. (5*a*)–(5*c*)];
A_{Vp}–C_{Vp} = coefficients to calculate dimensionless particle velocity intensity [Eqs. (6*a*)–(6*c*)];
A_ψ–B_ψ = coefficients to calculate deviation angle [Eqs. (7*a*) and (7*b*)];
a_q–f_q = coefficients to calculate A_q;
a_{Vp}–c_{Vp} = coefficients to calculate A_{V_P};
a_ψ–g_ψ = coefficients to calculate A_ψ;
C_d = particle drag coefficient;
C_l = particle lift coefficient;
D = particle grain size;
$\hat{D}$ = dimensionless volume rate of particle deposition per unit bed area;
d_{Vp}–f_{Vp} = coefficients to calculate B_{V_P};
$\hat{E}$ = dimensionless volume rate of particle erosion (pickup) per unit bed area;
g = acceleration of gravity;
g_q–i_q = coefficients to calculate B_q;
g_{Vp}–j_{Vp} = coefficients to calculate C_{V_P};
h_ψ–n_ψ = coefficients to calculate B_ψ;
j_q–n_q = coefficients to calculate C_q;
$\hat{\mathbf{q}}$ = dimensionless volume bed-load sediment transport rate vector per unit width;
s = specific gravity of sediment;
$\hat{\mathbf{s}}_p$ = unit vector in direction of V_p;
$\hat{\mathbf{V}}p$ = dimensionless velocity vector of bed-load particles;
z = vertical distance upward from bed;
α = streamwise bed angle;
λ = dimensionless coefficient relating Shields stress necessary for bed-load transport to stop to critical Shields stress for initiation of bed-load transport;
μ_d = coefficient of dynamic friction;
$\hat{\xi}$ = dimensionless volume areal concentration of bed load;
ρ = density of water;
$\boldsymbol{\tau}$ = mean fluid bed shear stress that would prevail in absence of bed-load transport;
$\boldsymbol{\tau}_b$ = mean fluid shear stress at bed in presence of bed-load layer;
τ_* = Shields stress based on $\boldsymbol{\tau}$;
τ_{*b} = Shields stress based on $\boldsymbol{\tau}_b$;
τ_{*c} = critical Shields stress on arbitrarily sloping bed;
τ_{*co} = critical Shields stress for onset of sediment motion on nearly horizontal bed;
ϕ = angle of repose of sediments;
φ = transverse bed angle; and
ψ = deviation angle: angle between direction of applied bed shear stress and direction of particle velocity.

References

Abbott, J. E., and Francis, J. R. D. (1977). "Saltation and suspension trajectories of solid grains in a water streams." *Philos. Trans. R. Soc. London, Ser. A*, 284, 225–254.

Ashida, K., and Michiue, M. (1972). "Study on hydraulic resistance and bedload transport rate in alluvial streams." *Trans. Jpn. Soc. Civ. Eng.*, 206, 59–69.

Bagnold, R. A. (1956). "The flow of cohesionless grains in fluids." *Philos. Trans. R. Soc. London, Ser. A*, 249, 235–297.

Engelund, F., and Fredsoe, J. (1976). "A sediment transport model for straight alluvial channels." *Nord. Hydrol.*, 7, 293–306.

Fernandez Luque, R., and Van Beek, R. (1976). "Erosion and transport of bedload sediment." *J. Hydraul. Res.*, 14(2), 127–144.

Francalanci, S., and Solari, L. (2007). "Gravitational effects on bed load transport at low shields stress: Experimental observations." *Water Resour. Res.*, 43, W03424.

Francalanci, S., Solari, L., and Vignoli, G. (2006). "Gravitational effects on river morphodynamics." *Proc., River Flow, Conf. on River Hydraulics*, Lisbon, Portugal, 1129–1136.

Francis, J. R. D. (1973). "Experiments on the motion of solitary grains along the bed of a water stream." *Proc. R. Soc. London, Ser. A*, 332, 443–471.

Ikeda, S. (1982). "Lateral bed load transport on side slopes." *J. Hydr. Div.*, 108(11), 1369–1373.

Nino, Y., and Garcia, M. (1994). "Gravel saltation. 1. Experiments." *Water Resour. Res.*, 30(6), 1907–1914.

Parker, G. (1984). "Discussion of lateral bed load transport on side slopes." *J. Hydraul. Eng.*, 110(2), 197–199.

Parker, G., Seminara, G., and Solari, L. (2003). "Bed load at low Shields stress on arbitrarily sloping beds: Alternative entrainment formulation." *Water Resour. Res.*, 39(7), 1183.

Sekine, M., and Kikkaua, H. (1992). "Mechanics of saltating grains. II." *J. Hydraul. Eng.*, 118(4), 536–558.

Seminara, G., Solari, L., and Parker, G. (2002). "Bed load at low Shields stress on arbitrarily sloping beds: Failure of the Bagnold hypothesis." *Water Resour. Res.*, 38(11), 1249.

Struiksma, N., Olesen, K., Flokstra, C., and Vriend, H. de. (1985). "Bed deformation in a curved alluvial channel." *J. Hydraul. Res.*, 23(1), 57–59.