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Bed load at low Shields stress on arbitrarily sloping beds: Alternative entrainment formulation

Gary Parker
St. Anthony Falls Laboratory, University of Minnesota, Minneapolis, Minnesota, USA

Giovanni Seminara and Luca Solari[1]
Dipartimento di Ingegneria Ambientale, Università di Genova, Genova, Italy

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[1] According to the Bagnold hypothesis for equilibrium bed load transport, a necessary constraint for the maintenance of equilibrium bed load transport is that the fluid shear stress at the bed must be reduced to the critical, or threshold, value associated with incipient motion of grains. It was shown in a companion paper [*Seminara et al.*, 2002], however, that the Bagnold hypothesis breaks down when applied to equilibrium bed load transport on beds with transverse slopes above a relatively modest value that is well below the angle of repose. An investigation of this failure resulted in a demonstration of its lack of validity even for nearly horizontal beds. The constraint is here replaced with an entrainment formulation, according to which a dynamic equilibrium is maintained by a balance between entrainment of bed grains into the bed load layer and deposition of bed load grains onto the bed. The entrainment function is formulated so that the entrainment rate is an increasing function of the excess of the fluid shear stress at the bed over the threshold value. The formulation is implemented with the aid of a unique set of laboratory data that characterizes equilibrium bed load transport at relatively low shear stresses for streamwise angles of bed inclination varying from nearly 0° to 22°. The formulation is shown to provide a description of bed load transport on nearly horizontal beds that fits the data as well as that resulting from the Bagnold constraint. The entrainment formulation has the added advantage of not requiring the unrealistically high dynamic coefficient of Coulomb friction resulting from the Bagnold constraint. Finally, the entrainment formulation provides reasonable and consistent results on finite streamwise and transverse bed slopes, even those at which the Bagnold formulation breaks down completely. *INDEX TERMS:* 1824 Hydrology: Geomorphology (1625); 1815 Hydrology: Erosion and sedimentation; 3210 Mathematical Geophysics: Modeling; *KEYWORDS:* bedload, sediment transport, entrainment, deposition, saltation, erosion

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1. Introduction

[2] This paper is the successor to *Seminara et al.* [2002]. The general failure of the *Bagnold* [1956] hypothesis when applied to equilibrium bedload transport at even relatively modest transverse slopes was demonstrated by *Seminara et al.* [2002].

[3] This failure motivates the reanalysis of the problem presented here. This reanalysis preserves the overall structure of the Bagnoldean formulation, according to which the bed load layer is described as a thin but finite layer at the base of which the fluid shear stress is reduced due to the transfer of streamwise momentum to the saltating grains. In the present formulation, however, the fluid shear stress at the bed drops to the critical value for particle motion only when there is no motion, i.e., when the granular bed is in static equilibrium. The fluid shear stress at the bed must be above the critical value in order to entrain grains into bed load transport, and the rate of entrainment increases with the excess of fluid shear stress at the bed above the critical value. Dynamic equilibrium is maintained by a balance between entrainment and deposition of grains. The model thus preserves the Bagnoldean structure for near-bed fluid-solid interaction, but abandons the Bagnold constraint in favor of an Einsteinean structure for entrainment.

[4] The new formulation does not in and of itself yield an explicit relation for the sediment entrainment function, which is beyond the scope of the theory presented here. Instead, an example entrainment relation is evaluated empirically using the data of *Fernandez Luque and van Beek* [1976]. This data set is unique in that it characterizes equilibrium bed load transport at relatively low shear stresses for streamwise bed slopes ranging from nearly 0° to 22°. The new formulation is shown to recover the bed load transport relation of Fernandez Luque and van Beek, so agreeing with

[1]Now at Dipartimento di Ingegneria Civile, Università di Firenze, Firenze, Italy.

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the data at all streamwise slopes studied by them. In addition, it generalizes to beds sloping arbitrarily in the transverse as well as streamwise direction without displaying the failure resulting from the Bagnold hypothesis.

[5] As the paper is long and contains numerous equations, a brief summary may be helpful to the reader. (1) The Bagnold condition is replaced with a bed load entrainment formulation, cast in terms of the "inner" variable of fluid shear stress at the bed. The "inner" structure functions are chosen so as to yield relations based on the data of *Fernandez Luque and van Beek* [1976] as their "outer" forms. (2) The analysis is first evaluated for the case of a nearly horizontal bed. (3) The analysis is then extended to the case of a bed with an arbitrary slope in the streamwise or transverse direction. (4) Several limiting cases are considered to test how the analysis compares with previously presented relations for a bed with a substantial streamwise slope and a bed with small but nonzero slope in the transverse direction. (5) Finally, the results of several sample calculations are presented.

2. Entrainment Formulation on a Nearly Horizontal Bed

[6] The scalar formulation for a flow over a nearly horizontal bed is revisited. The Bagnold hypothesis is abandoned in favor of the following entrainment hypothesis. Where $\hat{E}$ denotes the dimensionless entrainment rate (volume pick-up rate per unit bed area) defined in SSP(2d) (equation numbers preceded by SSP throughout refer to equations of *Seminara et al.* [2002]), it is assumed that

$$\hat{E} = F_e(\tau_{*b} - \tau_{*co}) \tag{1}$$

where F_e denotes an entrainment function and

$$\tau_{*b} = \frac{\tau_{fb}}{\rho(s-1)gD} \tag{2}$$

denotes the fluid Shields stress at the bed. The function F_e is hypothesized to have the following characteristics; F_e vanishes when $\tau_{*b} = \tau_{*co}$ and F_e is a nonnegative, monotonically increasing function of $\tau_{*b} - \tau_{*co}$. According to this hypothesis the fluid shear stress at the bed τ_{fb} must exceed the threshold value τ_{fco} if there is to be any bed load transport whatsoever. Equilibrium conditions are reached not when the fluid shear stress at the bed reaches the threshold value, but when the entrainment rate of bed particles into the bed load layer equals the deposition rate of bed load particles onto the bed. That is, the entrainment hypothesis applied to the condition of dynamic equilibrium at the bed yields the result that entrainment equals deposition, rather than a static equilibrium at which no bed particles can be entrained at all.

[7] This entrainment formulation for bed load transport is reminiscent of a more familiar formulation commonly employed to treat the case of suspended sediment. In entrainment formulations for suspended sediment, the entrainment rate into suspension is specified at the bed, rather than the alternative specification of a near-bed concentration of suspended sediment, as described by, e.g., *Garcia and Parker* [1991].

[8] As opposed to the Bagnold hypothesis, which yields a relation for $\hat{\xi}$ as a consequence, the entrainment hypothesis in and of itself does not specify the function F_e. Hence rather than pursuing this issue in detail here an empirical form for F_e which ultimately agrees with FLvB is sought. In order to be able to predict the form of F_e, one should be able to construct a model to describe the dynamic effect of those turbulent events (sweeps, outward interactions), which are known to be responsible for the entrainment of bed load particles [*Nelson et al.*, 1995]. This is a formidable task, still outside the reach of the present modeling capabilities [but see *Schmeeckle*, 1999]. The following empirical form is assumed here;

$$\hat{E} = A_{eo}(\tau_{*b} - \tau_{*co})^{3/2} \tag{3}$$

where A_{eo} is a constant to be evaluated. In fact the exponent in the above relation could have been left arbitrary and then evaluated with the results of FLvB. Instead, however the choice 3/2 is justified a posteriori.

[9] Note that (1) and (3) are in "inner" form in that they involve the actual fluid stress at the bed rather than the fluid shear stress that would prevail in the absence of a bed load layer. The following "inner" form for the deposition rate $\hat{D}$ is postulated;

$$\hat{D} = A_{do}(\tau_* - \tau_{*b})^{1/2}\hat{\xi} \tag{4}$$

where A_{do} is a constant to be evaluated. This form is assumed for the deposition rate because the factors that control deposition are set within the bed load layer itself and are driven by the shear stress difference across the bed load layer rather than at the bed.

[10] The "inner" forms (3) and (4) merit further justification. They are immediately seen to be similar (but not identical) to the OFLvB relation SSP(4b) and the MFLvB relation SSP(14), respectively, of the companion paper [*Seminara et al.*, 2002]. Both these latter relations are in "outer" form in that they do not involve the fluid Shields stress at the bed τ_{*b}. That is, (3) is the "inner" version of the "outer" formulation SSP(4b), which is obtained directly from the data of *Fernandez Luque and van Beek* [1976]. In addition, (4) is the "inner" version of the "outer" formulation SSP(14), which is derived from the experimental results of *Fernandez Luque and van Beek* [1976] embodied in SSP(4c) and SSP(7c), the conservation relations SSP(7b), SSP(7c) and SSP(9) and the approximation SSP(11a). It is important to note that SSP(4b) and SSP(14) are not assumptions, but are rigorously obtained from data and conservation relations. In point of fact, (3) and (4) are the only "inner" forms that when combined with the framework of the present analysis yield the "outer" forms SSP(4c) and SSP(14) determined from *Fernandez Luque and van Beek* [1976]. Note that the subscript "o" in A_{eo} and A_{do} of (3) and (4) denote values for a nearly horizontal bed.

[11] Applying the equilibrium continuity condition SSP(9) to (3) and (4), it is found that

$$\hat{\xi} = \frac{A_{eo}}{A_{do}} \frac{(\tau_{*be})^{3/2}}{(\tau_{*e} - \tau_{*be})^{1/2}} \tag{5}$$

where

$$\tau_{*e} = \tau_* - \tau_{*co} \tag{6a}$$

$$\tau_{*be} = \tau_{*b} - \tau_{*co} \tag{6b}$$

[12] Two results obtained in section 3 that are independent of the Bagnold hypothesis are used here. The first of these is the relation SSP(32) for particle velocity $\hat{V}_p$. The second of these is SSP(27b), which can be expressed in the dimensionless form

$$\hat{\xi} = \frac{1}{\mu_{do}}\left(\tau_* - \tau_{*b}\right) = \frac{1}{\mu_{do}}\left(\tau_{*e} - \tau_{*be}\right) \tag{7}$$

Between (5) and (7) it is found that

$$\tau_{*be} = \frac{1}{1+K_o}\tau_{*e} \tag{8a}$$

$$K_o = \left(\mu_{do}\frac{A_{eo}}{A_{do}}\right)^{2/3} \tag{8b}$$

where the subscript "o" again refers to a nearly horizontal bed. The "outer" form of the entrainment relation is then found by substituting (8a) into (3);

$$\hat{E} = \frac{A_{eo}}{(1+K_o)^{3/2}}\left(\tau_{*e}\right)^{3/2} = \frac{A_{eo}}{(1+K_o)^{3/2}}\left(\tau_* - \tau_{*co}\right)^{3/2} \tag{9a}$$

Equation (7) can be further reduced with (8) to the "outer" form

$$\hat{\xi} = \frac{1}{\mu_{do}}\frac{K_o}{(1+K_o)}\tau_{*e} = \frac{1}{\mu_{do}}\frac{K_o}{(1+K_o)}\left(\tau_* - \tau_{*co}\right) \tag{9b}$$

Combining (9b) with SSP(32) and SSP(7a) the following relation for bed load transport is obtained;

$$\hat{q} = \hat{V}_P\hat{\xi} = \frac{f}{\mu_{do}}\frac{K_o}{(1+K_o)}\left(\tau_* - \tau_{*co}\right)\left(\sqrt{\tau_*} - \lambda_o\sqrt{\tau_{*co}}\right) \tag{10}$$

[13] Evidently SSP(32), (9b) and (10) correspond precisely in form to the OFLvB relation SSP(4b) and the MFLvB relations SSP(12) and SSP(13a) of Fernandez Luque and Van Beek. Using the coefficients of the last three relations the following evaluations are obtained;

$$K_o = \frac{\frac{1}{f}\mu_{do}\alpha_q}{1 - \frac{1}{f}\mu_{do}\alpha_q} \tag{11a}$$

$$A_{eo} = \alpha_e(1+K_o)^{3/2} \tag{11b}$$

$$A_{do} = \left(\frac{1+K_o}{K_o}\right)^{3/2}\alpha_e\mu_{do} \tag{11c}$$

where $\alpha_q = 7.6$, $\alpha_e = 0.02$ and $f = 11.5$. Further specifying $\lambda_o = 0.7$ allows recovery of the OFLvB form SSP(4c). It is important to note that regardless of the value assumed for μ_{do}, the above specifications recover the appropriate OFLvB or MFLvB relations, i.e., the relation for particle velocity $\hat{V}_P$ of SSP(4c), the relation for areal concentration $\hat{\xi}$ of SSP(12) and the relation for bed load transport $\hat{q}$ of SSP(13a).

[14] The relation (4) for the deposition rate can be rewritten with the aid of (8a) to yield

$$\hat{D} = \alpha_d\left(\tau_* - \tau_{*co}\right)^{1/2}\hat{\xi} \tag{12a}$$

where

$$\alpha_d = \left(\frac{1+K_o}{K_o}\right)^{-1/2} A_{do} = \frac{f\alpha_e}{\alpha_q} = 0.03 \tag{12b}$$

That is, the MFLdV form SSP(14) is again precisely recovered regardless of the value of μ_{do}.

[15] The reduction in fluid shear stress at the bed can be quantified in terms of the following relation determined from (8a);

$$\frac{\tau_{*b} - \tau_{*co}}{\tau_* - \tau_{*co}} = \frac{1}{1+K_o} \tag{13}$$

[16] In contrast, the Bagnold constraint yields the evaluation

$$\tau_{*b} = \tau_{*co} \tag{14a}$$

or alternatively

$$\frac{\tau_{*b} - \tau_{*co}}{\tau_* - \tau_{*co}} = 0 \tag{14b}$$

requiring that

$$K_o = \infty \tag{14c}$$

For the given values of α_q and f, the only value of μ_{do} that satisfies the Bagnold constraint is the previously quoted unrealistically high value $\mu_{do} = 1.515$. A finite value of K_o thus implies that the Bagnold constraint is not satisfied at the bed.

[17] An evaluation of the coefficients given in (11a, 11b, and 11c) are here given for the more reasonable choice of μ_{do} of 0.30, here adopted based on the study of *Nino and Garcia* [1994a, 1994b];

$$K_o = 0.25 \tag{15a}$$

$$A_{eo} = 0.028 \tag{15b}$$

$$A_{do} = 0.068 \tag{15c}$$

A value of K_o of 0.25 implies that the fluid shear stress at the bed is well above the critical value for the onset of motion, i.e., that the Bagnold hypothesis is not satisfied. This is shown in Figure 1. Only the line 0 deg, corresponding to a nearly horizontal bed, should be considered at this point. For a nearly horizontal bed K becomes equal to K_o, which is plotted against μ_{do} for μ_{do} ranging from 0 to 1.51, i.e., the value forced by an application of the Bagnold constraint to the data of FLvB corresponding to a nearly horizontal bed. The plot quantifies the effect of varying μ_{do} on the reduction in fluid shear stress at the bed. For the value of μ_{do} of 0.30 used in the present study, it is seen that only 20 percent of the reduction in the ratio $(\tau_{*b} - \tau_{*co})/(\tau_* - \tau_{*co})$ necessary to satisfy the Bagnold constraint is realized.

3. Entrainment Formulation on an Arbitrarily Sloping Bed

[18] The case of an arbitrarily sloping bed is considered. It is assumed here that the parameter τ_{*c} has already been

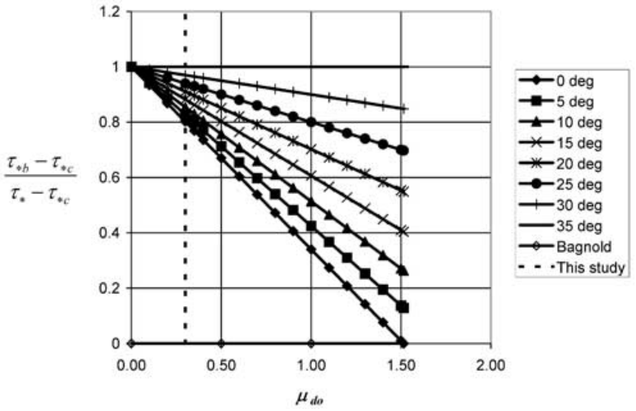

Figure 1. Plot of $(\tau_{*b} - \tau_{*c})/(\tau_* - \tau_{*c})$ versus μ_{do}, with streamwise angle α as a secondary parameter for the case of streamwise slope only. Note that $(\tau_{*b} - \tau_{*c})/(\tau_* - \tau_{*c})$ must be vanishing in order to satisfy the Bagnold constraint. The choice of μ_{do} of 0.3 used in the present study is highlighted in the diagram.

obtained from SSP(49) and the parameter $\hat{\boldsymbol{V}}_P$ (or equivalently $\hat{V}_P$ and $\hat{\mathbf{s}}_p$) has already been obtained from SSP(62). The generalization of SSP(27b) to equilibrium bed load transport on arbitrarily sloping beds is SSP(69a), which can be expressed in dimensionless form as

$$\hat{\xi}\left(|\boldsymbol{k}_n|\hat{\mathbf{s}}_p - \frac{\boldsymbol{k}_t}{\mu_d}\right) = \frac{1}{\mu_d}\left(\tau_* - \tau_{*b}\right) \tag{16}$$

[19] In order to make further progress it is necessary to generalize the forms for entrainment and deposition rates to the case of an arbitrarily sloping bed. The forms assumed here are extensions of the inner forms (3) and (4); but with τ_{*co} replaced with τ_{*c};

$$\hat{E} = A_e\left(\tau_{*b} - \tau_{*c}\right)^{3/2} \tag{17a}$$

$$\hat{D} = A_d\left(\tau_* - \tau_{*b}\right)^{1/2}\hat{\xi} \tag{17b}$$

Note that in the above relations A_{eo}, A_{do} and τ_{*co} have been replaced by A_e, A_d and τ_{*c}, parameters that may take values on an arbitrarily sloping bed that differ from their limiting values for a nearly horizontal bed. The issue of determining forms for A_e and A_d, and, for that matter, λ on an arbitrarily sloping bed is approached in the next section. For now it is simply assumed that generalization to an arbitrarily sloping bed is possible.

[20] Between (17a, 17b) it is found that

$$\hat{\xi} = \frac{A_e}{A_d}\frac{\left(\tau_{*b} - \tau_{*c}\right)^{3/2}}{\left(\tau_* - \tau_{*b}\right)^{1/2}} \tag{18}$$

Substituting (18) into (16) leads to a predictive relation for τ_{*b};

$$\frac{A_e}{A_d}\frac{\left(\tau_{*b} - \tau_{*c}\right)^{3/2}}{\left(\tau_* - \tau_{*b}\right)^{1/2}}\left(|\boldsymbol{k}_n|\hat{\mathbf{s}}_p - \frac{\boldsymbol{k}_t}{\mu_d}\right) = \frac{1}{\mu_d}\left(\tau_*\hat{\boldsymbol{s}} - \tau_{*b}\hat{\mathbf{s}}_b\right) \tag{19a}$$

where

$$\hat{\mathbf{s}}_b = \frac{\boldsymbol{\tau}_{*b}}{\left|\boldsymbol{\tau}_{*b}\right|} \tag{19b}$$

The magnitude τ_{*b} is thus determined as follows;

$$\tau_{*b} = \left|\tau_*\hat{\boldsymbol{s}} - \mu_d\frac{A_e}{A_d}\frac{\left(\tau_{*b} - \tau_{*c}\right)^{3/2}}{\left(\tau_* - \tau_{*b}\right)^{1/2}}\left(|\boldsymbol{k}_n|\hat{\mathbf{s}}_p - \frac{\boldsymbol{k}_t}{\mu_d}\right)\right| \tag{19c}$$

[21] In principle the above relations can be solved iteratively for τ_{*b} and $\hat{\mathbf{s}}_b$ once the geometric parameters $\boldsymbol{k}_t$, and $\boldsymbol{k}_n$, and functional forms for the dynamic parameters A_e, A_d and μ_d are specified. Note that τ_{*c} is computed from SSP(49) and $\hat{\mathbf{s}}_p$ is computed from SSP(62). A first guess of τ_{*b} may be obtained from (13). This value may be inserted into (19c) to obtain a better estimate, and the process continued until convergence. The direction $\hat{\mathbf{s}}_b$ is then obtained from (19a). Once this is done τ_{*b} can be substituted into (18) to determine the areal concentration $\hat{\xi}$. The vectorial bed load transport rate $\hat{\boldsymbol{q}}$ is then given as

$$\hat{\boldsymbol{q}} = \hat{\xi}\hat{\boldsymbol{V}}_P \tag{20}$$

or using SSP(65),

$$\hat{\boldsymbol{q}} = \hat{q}[cos\psi\hat{s} + \sin\psi(\hat{\boldsymbol{n}}\,x\,\hat{\boldsymbol{s}})] \tag{21a}$$

where

$$\hat{q} = \hat{\xi}\hat{V}_P \tag{21b}$$

[22] Equation (20) and the relations on which it is based embody the essential result of the present analysis. It provides a generalized bed load transport equation on arbitrarily sloping beds that does not employ the Bagnold constraint. The issue of functional forms for A_e, A_d and λ on an arbitrarily sloping bed is considered in the next section. That the formulation suffers from none of the defects resulting from the use of the Bagnold constraint is demonstrated in the section after next.

[23] Before proceeding, however, it is of value to introduce and define three scalar parameters for arbitrarily sloping beds. The first of these is the magnitude S_P of the bed slope in the direction of particle saltation, given by the expression

$$S_P = -\frac{\hat{\boldsymbol{k}}\cdot\hat{\mathbf{s}}_p}{\sqrt{1 - \left(\hat{\boldsymbol{k}}\cdot\hat{\mathbf{s}}_p\right)^2}} \tag{22}$$

The second and third of these, r_{sc} and r_{dc}, denote the respective fraction reduction in the magnitude of the static and dynamic Coulomb resistance force acting on saltating particles due to bed slope, given by the expressions

$$r_{sc} = \sqrt{1 - \left(\hat{\mathbf{k}}\cdot\hat{\mathbf{s}}_p\right)^2}\left(1 + \frac{\hat{\mathbf{k}}\cdot\hat{\mathbf{s}}_p}{\mu\sqrt{1 - \left(\hat{\mathbf{k}}\cdot\hat{\mathbf{s}}_p\right)^2}}\right) \tag{23a}$$

$$r_{dc} = \sqrt{1 - \left(\hat{\mathbf{k}}\cdot\hat{\mathbf{s}}_p\right)^2}\left(1 + \frac{\hat{\mathbf{k}}\cdot\hat{\mathbf{s}}_p}{\mu_d\sqrt{1 - \left(\hat{\mathbf{k}}\cdot\hat{\mathbf{s}}_p\right)^2}}\right) \tag{23b}$$

Note that $S_p \rightarrow 0$, $r_{sc} \rightarrow 1$ and $r_{dc} \rightarrow 1$ as the bed approaches horizontal.

4. Bed Load Transport on a Finite Streamwise Slope

[24] The data of FLvB is the only set known to the authors to cover bed load transport on streamwise slopes up to 22°. The authors of that study are able to collapse all the data at any slope into common relations which contain an explicit slope dependence only in the parameter $\tau*_c$. The power of this unique data set will become apparent in this section.

[25] The formulation of the previous section reduces to a relatively simple form for the case of a bed sloping only in the streamwise direction ($\alpha > 0$, $\varphi = 0$). Setting $\varphi = 0$ in SSP(45) and SSP(46) and reducing (22) and (23a, 23b), it is found that

$$S_P = \tan\alpha \tag{24}$$

$$r_{sc} = \cos\alpha\left(1 - \frac{\tan\alpha}{\mu}\right) \tag{25a}$$

$$r_{dc} = \cos\alpha\left(1 - \frac{\tan\alpha}{\mu_d}\right) \tag{25b}$$

[26] As shown in SSP(52b), the relation for critical Shields stress on a bed sloping only in the streamwise direction reduces to the form

$$\tau*_c = \tau*_{co} r_{sc} \tag{26}$$

Equation SSP(62) similarly yields the following relation for bed load velocity;

$$\hat{V}_P = f\left(\sqrt{\tau*} - \lambda\sqrt{r_{dc}\tau*_{co}}\right) \tag{27}$$

It is seen that (27) takes a form very similar but not identical to the OFLvB form SSP(4c), a form that was found to fit the data over a wide range of streamwise slopes up to 22°.

[27] A problem, however, immediately arises in the form for critical shear stress in (27). The value of μ_{do} of 0.30 quoted earlier for a nearly horizontal bed would imply that the critical condition for incipient motion of the bed would be realized at vanishing applied shear stress for a streamwise slope α of 16.7°. If this were true the bed would begin avalanching sediment under the direct influence of gravity at a slope lower than either of the values of 18° and of 22° at which equilibrium bed load transport was observed by FLvB. Evidently the value of μ_d at finite slope must become larger than the value μ_{do} on a nearly horizontal bed. In particular, as the bed slope steepens to the angle of repose $\phi_r = \tan^{-1}(\mu)$ bed load transport grades into to a continuous avalanching corresponding to, e.g., the progradation of a Gilbert delta. That is, as $\alpha \rightarrow \phi_r$ it can be expected that $\mu_d \rightarrow \mu$. The precise nature of this functionality is beyond the scope of the present paper. The problem is approached herein with a simple linear bridge, i.e.

$$\mu_d = \mu_{do} r_\mu \tag{28a}$$

$$r_\mu = \left(1 - \frac{\tan\alpha}{\mu}\right) + \frac{\tan\alpha}{\mu_{do}} \tag{28b}$$

Thus (25b) can be rewritten as

$$r_{dc} = \cos\alpha\left(1 - \frac{\tan\alpha}{\mu_{do} r_\mu}\right) \tag{29}$$

[28] It is now possible to reduce (27) directly to the OFLvB form SSP(4c). Comparing SSP(4c) and (27) with the aid of (26), it is seen that the only way this reduction is possible is if

$$\lambda\sqrt{r_{dc}\tau*_{co}} = \lambda_o\sqrt{r_{sc}\tau*_{co}} \tag{30a}$$

or thus

$$r_\lambda \equiv \frac{\lambda}{\lambda_o} = \sqrt{\frac{r_{sc}}{r_{dc}}} \tag{30b}$$

That is, (30b) and the choice $\lambda_o = 0.7$ provide an explicit expression as to how the parameter λ must depend on streamwise slope if the formulation is to yield the OFLvB relation SSP(4c) at any streamwise slope angle α.

[29] A similar analysis can be performed for the parameters A_e and A_d. In the case of a streamwise slope only (16) reduces to the form

$$\hat{\xi} r_{dc} = \frac{1}{\mu_d}\left(\tau* - \tau*_b\right) \tag{31}$$

Between (31) and (18) the following two relations are obtained;

$$\tau*_b - \tau*_c = \frac{1}{1+K}\left(\tau* - \tau*_c\right) \tag{32a}$$

$$\tau* - \tau*_b = \frac{K}{1+K}\left(\tau* - \tau*_c\right) \tag{32b}$$

where

$$K = \left(\mu_d r_{dc}\frac{A_e}{A_d}\right)^{2/3} \tag{32c}$$

[30] Relations (32a) and (32b) allow the translation of (17a) and (17b) from "inner" to "outer" form;

$$\hat{E} = \frac{A_e}{(1+K)^{3/2}}\left(\tau* - \tau*_c\right)^{3/2} \tag{33a}$$

$$\hat{D} = A_d\left(\frac{K}{1+K}\right)^{1/2}\left(\tau* - \tau*_c\right)^{1/2}\hat{\xi} \tag{33b}$$

Matching (33a) to the OFLvB form SSP(4b) and (33b) to the MFLvB form SSP(14), respectively, it is found that

$$\frac{A_e}{(1+K)^{3/2}} = \alpha_e \tag{34a}$$

$$A_d\left(\frac{K}{1+K}\right)^{1/2} = \alpha_d \tag{34b}$$

where α_e and α_d take the respective values 0.02 and 0.03 regardless of the magnitude of the streamwise slope angle α.

[31] Substituting the definitions

$$r_e = \frac{A_e}{A_{eo}} \tag{35a}$$

$$r_d = \frac{A_d}{A_{do}} \tag{35b}$$

into (34a) and (34b) and reducing with (35a) and (35b), (28) and (32c), it is found after some algebra that

$$r_e = \left[1 + \left(1 - r_\mu r_{dc}\right)K_o\right]^{-3/2} \tag{36a}$$

$$r_d = \left(r_\mu r_{dc}\right)^{-1/2} \tag{36b}$$

[32] The choices (30b) for r_λ and (36a) and (36b) for r_e and r_d, respectively ensure that the present analysis yields the OFLvB relation SSP(4b) for bed load entrainment rate and the MFLvB relation SSP(14) (but with $\tau_{*co} \rightarrow \tau_{*c}$ therein) for bed load deposition rate at any streamwise slope angle α. Further substituting (36a) and (36b) and (32b) into (31) yields the MFLvB relation SSP(12) (but with $\tau_{*co} \rightarrow \tau_{*c}$ therein) for bed load areal concentration at any streamwise slope angle α. Finally, between this expression for bed load areal concentration, (27) for bed load particle velocity and the continuity relation SSP(7a) the MFLvB relation SSP(13a) (but with $\tau_{*co} \rightarrow \tau_{*c}$ therein) for bed load transport rate valid for any streamwise slope angle α is obtained.

[33] FLvB do not provide values of the angles of repose $\phi_r = \tan^{-1}(\mu)$ of the five sediment types used in their experiments. Natural sediments, however, typically have values of ϕ_r between 30° and 40° [*Stevens and Simons*, 1971]. With this in mind a value of ϕ_r of 35°, and thus a value of μ of 0.700 is adopted for the present study in addition to the value of μ_{do} of 0.30. In Figure 2 r_μ, r_{dc}, r_λ, r_e and r_d are plotted against streamwise slope angle α using the adopted values of μ and μ_{do}. It is of value to note that all five of these parameters can be approximated as unity for small streamwise slope angle α. Note also that $r_d \rightarrow \infty$, $r_{dc} \rightarrow 0$ and the product $r_{dc}r_d \rightarrow 0$ as $\alpha \rightarrow \phi_r$.

[34] The Bagnold constraint is reconsidered in light of the above analysis. After some manipulation (32a) can be reduced to the form

$$\frac{\tau_{*b} - \tau_{*c}}{\tau_* - \tau_{*c}} = \frac{1 + \left[\left(1 - r_\mu r_{dc}\right)K_o\right]}{1 + K_o} \tag{37}$$

Note that (37) is a generalization of (13) to the case of an arbitrary streamwise bed slope. The Bagnold constraint is realized as the right-hand side of (37) approaches 0. In Figure 1 the fractional decrease in bed Shields stress $(\tau_{*b} - \tau_{*c})/(\tau_* - \tau_{*c})$ is plotted as a function of the horizontal-bed dynamic coefficient of friction μ_{do} and the streamwise bed slope angle α. In the calculation $\phi_r = \tan^{-1}(\mu)$ has been set equal to 35°. The value of μ_{do} adopted in the present analysis is highlighted in the diagram. Note that in Figure 1 the Bagnold constraint is not satisfied on a horizontal bed for $\mu_{do} < 1.51$. In addition, once $\alpha > 0$ the Bagnold constraint is not satisfied even for $\mu_{do} = 1.51$.

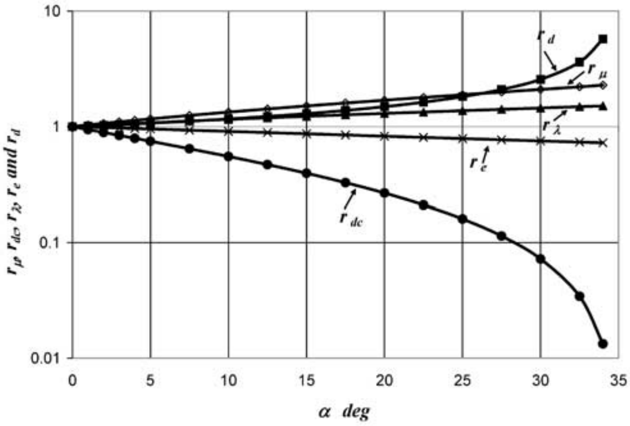

Figure 2. Plot of r_μ, r_λ, r_{dc}, r_d, and r_e as functions of α for the case of streamwise slope only.

5. Method of Calculation for a Bed With a Finite Slope in an Arbitrary Direction

[35] It is now possible to outline a method for calculating bed load transport on a bed with a finite transverse as well as streamwise slope. The expression for r_μ of (28b) is generalized with the aid of (22) to

$$r_\mu = \left(1 - \frac{S_P}{\mu}\right) + \frac{S_P}{\mu_{d0}} \tag{38}$$

The expression for r_{sc} of (25a) is generalized to (23a), and the expression for r_{dc} of (25b) is generalized with the aid of (25b) and (28a) to

$$r_{dc} = \sqrt{1 - \left(\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{s}}_p\right)^2} \cdot \left(1 + \frac{\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{s}}_p}{\mu_{do} r_\mu \sqrt{1 - \left(\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{s}}_p\right)^2}}\right) \tag{39}$$

The generalized expressions for r_λ, r_e and r_d are still given by (30b), (35a) and (35b) respectively, but using the generalized expressions for r_μ, r_{sc} and r_{dc}.

[36] With this in mind SSP(62) now takes the form

$$\left|\sqrt{\tau_*}\,\hat{\boldsymbol{s}} - \frac{\hat{V}_P}{f}\hat{\boldsymbol{s}}_p\right| \left(\sqrt{\tau_*}\,\hat{\boldsymbol{s}} - \frac{\hat{V}_P}{f}\hat{\boldsymbol{s}}_p\right) = \lambda_o^2 r_\lambda^2 \tau_{*c} \left(|\boldsymbol{k}_n| \hat{\boldsymbol{s}}_p - \frac{\boldsymbol{k}_t}{\mu_{do} r_\mu}\right) \tag{40a}$$

where

$$\tau_{*c} = \tau_{*co} r_{sc} \tag{40b}$$

and (19a) generalizes to

$$r_{ed} \frac{A_{eo}}{A_{do}} \frac{\left(\tau_{*b} - \tau_{*c}\right)^{3/2}}{\left(\tau_* - \tau_{*b}\right)^{1/2}} \left(|\boldsymbol{k}_n| \hat{\boldsymbol{s}}_p - \frac{\boldsymbol{k}_t}{\mu_{do} r_\mu}\right) = \frac{1}{\mu_{do} r_\mu} \left(\tau_* \hat{\boldsymbol{s}} - \tau_{*b} \hat{\boldsymbol{s}}_b\right) \tag{41a}$$

where

$$r_{ed} = \frac{r_e}{r_d} = \frac{\left(r_\mu r_{dc}\right)^{1/2}}{\left[1 + \left(1 - r_\mu r_{dc}\right) K_o\right]^{3/2}} \tag{41b}$$

[37] The general method of solution is as follows. Equation (40) may be solved iteratively for $\hat{V}_P$ and $\hat{\boldsymbol{s}}_p$. Once $\hat{\boldsymbol{s}}_p$ is known the parameters r_μ, r_{dc}, r_λ, r_e, r_d, and r_{ed} can be evaluated. In analogy to (19c), the following form can be deduced from (14a);

$$\tau *_b = \left| \tau * \hat{\boldsymbol{s}} - \mu_{do} r_\mu r_{ed} \frac{A_{eo}}{A_{do}} \frac{\left(\tau *_b - \tau *_c\right)^{3/2}}{\left(\tau * - \tau *_b\right)^{1/2}} \left(|\boldsymbol{k}_n| \hat{\boldsymbol{s}}_p - \frac{\boldsymbol{k}_t}{\mu_{do} r_\mu} \right) \right| \tag{42}$$

The above equation can be solved iteratively for $\tau *_b$, after which $\hat{\boldsymbol{s}}_b$ can be evaluated from (41a). Once $\tau *_b$ is known then $\hat{\xi}$ can be evaluated from the relation

$$\hat{\xi} = r_{ed} \frac{A_{eo}}{A_{do}} \frac{\left(\tau *_b - \tau *_c\right)^{3/2}}{\left(\tau * - \tau *_b\right)^{1/2}} \tag{43}$$

obtained from (18). Substituting the solution for $\hat{\boldsymbol{V}}_P$ and $\hat{\boldsymbol{s}}_p$ from (40) with the solution for $\hat{\boldsymbol{\xi}}$ from (43) into (20) yields the solution for the vectorial bed load transport rate $\hat{q}$.

[38] Before continuing it is of value to reduce (19a) with the aid of SSP(62) to obtain the form

$$\frac{1}{\lambda^2 \tau *_{co}} \frac{A_e}{A_d} \frac{\left(\tau *_b - \tau *_c\right)^{3/2}}{\left(\tau * - \tau *_b\right)^{1/2}} \left| \sqrt{\tau *}\, \hat{\boldsymbol{s}} - \frac{\hat{V}_P}{f} \hat{\boldsymbol{s}}_p \right| \cdot \left(\sqrt{\tau *}\, \hat{\boldsymbol{s}} - \frac{\hat{V}_P}{f} \hat{\boldsymbol{s}}_p \right) = \frac{1}{\mu_d} \left(\boldsymbol{\tau} * - \boldsymbol{\tau} *_b \right) \tag{44}$$

The above equation requires that the vector of Shields stress difference across the bed load layer $\boldsymbol{\tau}* - \boldsymbol{\tau}*_b$ be collinear with the drag force acting on the bed load particles.

6. Limiting Case of Small Bed Slope in an Arbitrary Direction

[39] Many researchers have developed linearized formulations for estimating bed load transport on beds that are only modestly sloping in an arbitrary direction. Examples include *Hasegawa* [1981], *Parker* [1984], and *Struiksma et al.* [1985]. The present analysis can be reduced to an equivalent form under the small-angle assumptions

$$(\sin\alpha, \sin\varphi, \sin\psi) \cong (\tan\alpha, \tan\varphi, \tan\psi) \tag{45a}$$

$$(\cos\alpha, \cos\varphi, \cos\psi) \cong (1, 1, 1) \tag{45b}$$

These same assumptions allow approximation of the parameters r_μ, r_{dc}, r_λ, r_e, r_d and r_{ed} as unity. Under these conditions it is found that SSP(67) and SSP(68) readily yield the following linear dependence between $\tan\psi$ and $\tan\varphi$;

$$\tan\psi = \lambda_o \sqrt{\frac{\tau *_{co}}{\tau *}} \tan\varphi \tag{46}$$

Using the same approximations it is found that at lowest order the relation for $\hat{\xi}$ remains unchanged from that for a flat bed, and is still given by (9b). As a result the linearized form of the bed load transport relation for modest bed slopes in an arbitrary direction takes the form

$$\hat{\boldsymbol{q}} = \hat{\boldsymbol{q}}_o \left[\hat{\boldsymbol{s}} + \frac{r_t}{\sqrt{\tau *}} \tan\varphi (\hat{\boldsymbol{n}} x \hat{\boldsymbol{s}}) \right] \tag{47a}$$

where

$$r_t = \lambda_o \sqrt{\tau *_{co}} \tag{47b}$$

and $\hat{q}_o$ now denotes the magnitude of the dimensionless Einstein bed load transport rate obtained from the entrainment formulation on a nearly horizontal bed;

$$q_o = \frac{f}{\mu_{do}} \frac{K_o}{(1 + K_o)} \left(\tau * - \tau *_{co}\right) \left(\sqrt{\tau *} - 0.7 \sqrt{\tau *_{co}}\right) \tag{48}$$

7. Sample Calculations for a Bed With Finite Slope in an Arbitrary Direction

[40] It is now possible to compare the results of the fully nonlinear results of the entrainment formulation for bed load transport on a finite, arbitrary slope against its linear approximation for small slope. This is done in Figure 3, where $\hat{q}$ is plotted against τ_*/τ_{*co} for various combinations with $\alpha = 0°$ and $25°$ as well as $\varphi = 0°$ and $25°$. The input parameters into the calculation are the previously introduced values of τ_{*co}, μ and μ_{do} of 0.03, 0.70 and 0.30, respectively. The linear and nonlinear formulations obviously yield the same result for the case $\alpha = 0°$ and $\varphi = 0°$. The linearized formulation somewhat underpredicts the transport rate when $\alpha = 0°$ and $\varphi = 25°$. The linearized formulation grossly underpredicts the transport rate when $\alpha = 25°$ and $\varphi = 0°$. Note that the nonlinear bed load relation based on the entrainment formulation performs reasonably at high angles, and does not suffer from the failure resulting from the Bagnold constraint.

[41] A program that implements the fully nonlinear formulation presented here for equilibrium bed load transport over an arbitrarily sloping bed can be downloaded from http://www.dicea.unifi.it/luca.solari/.

8. Discussion

[42] It should be pointed out that the Bagnold constraint suffers from other limitations in addition to the crucial failure on arbitrarily sloped bed. Consider, for example, the transport of mixtures of sediment sizes. It has been shown that in a sediment mixture exposed on the surface of a bed the coarser grains are somewhat less mobile than the finer grains [e.g., *Parker*, 1991; *Wilcock and McArdell*, 1993]. As a result the individual size classes have different critical shear stresses. There is no obvious way that the

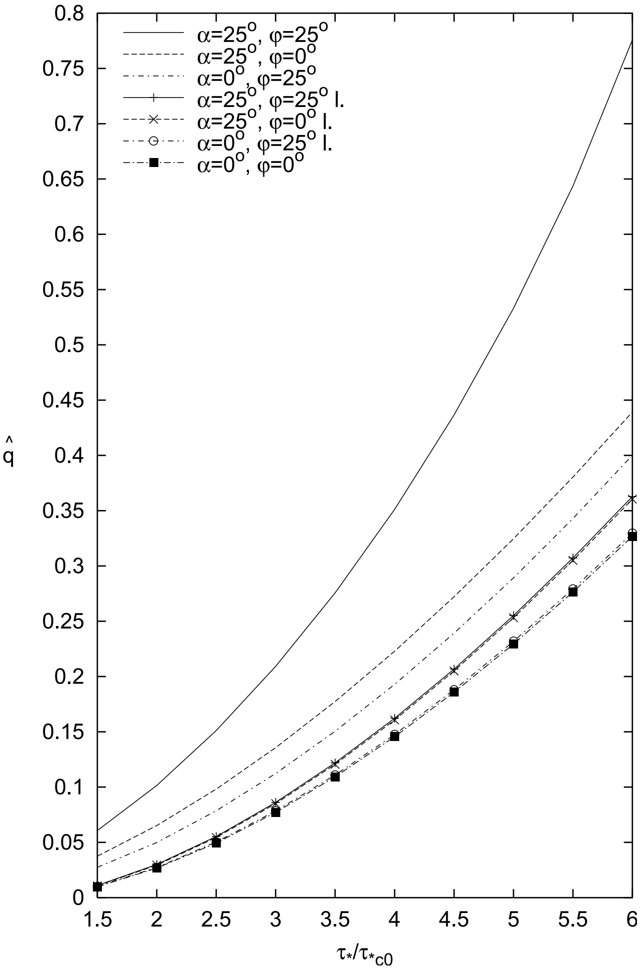

Figure 3. Plot of $\hat{q}$ predicted from the theory presented here as a function τ_*/τ_{*co} for various combinations of streamwise and transverse slope. The notation "l." denotes a prediction of the linearized form of theory; otherwise the prediction is of the fully nonlinear theory.

Bagnold constraint could be applied to such a case. *Bridge and Bennett* [1992] have attempted to do so, but implying that the fluid shear stress is reduced to one critical value above the big grains and another, different value over the small grains places a strain on what turbulent flows are capable of doing.

[43] The entrainment formulation does not suffer from this defect. In particular, it is relatively easy to install a hiding function into the entrainment function to account for the differential transport of different sizes in a mixture. *Tsujimoto* [1991] has already implemented such a formulation, though not in the context of the "inner" formulation presented here.

[44] A second limitation pertains to relaxation effects. It is known that bed load transport does not respond immediately to a change in imposed shear stress, but instead has a characteristic time of relaxation. There is no obvious way to generalize the Bagnold constraint to cover such disequilibrium conditions. According to the entrainment formulation, however, disequilibrium results when the entrainment rate is not equal to the deposition rate. *Nakagawa and Tsujimoto* [1980] and *Tsujimoto* [1987], for example, have used an entrainment formulation to describe the role of this disequilibrium in regard to the development of bed forms.

[45] The present layer-averaged formulation could be greatly improved by implementing the entrainment formulation in the context of a model that computes saltation trajectories [e.g., *Wiberg and Smith*, 1985, 1989]. Such a model could provide a more realistic alternative to the many models that employ the Bagnold constraint. In a saltation model it would be necessary to specify the form of only the entrainment function (in "inner" form). The calculation of the rate of deposition onto the bed would follow as a consequence of the dynamics of saltation and the assumptions concerning the topography of a granular bed [*Sekine and Kikkawa*, 1992]. *Schmeeckle* [1999] has presented an analysis that represents an important first step in this direction.

[46] While the present analysis provides a framework for the implementation of an entrainment relation in "inner" form, the analysis does not specify the relation. The empirical entrainment relation introduced here has been verified only at relatively low transport rates. A general form of the relation must be determined by means of a combination of theoretical and experimental approaches [e.g., *Nino and Garcia*, 1994a, 1994b].

[47] The analysis presented here uses a plausible but unverified "bridge" function to determine the variation of the dynamic coefficient of friction as a function of bed inclination. In addition, the functional forms of the parameters r_λ, r_e, and r_d have been determined using the extremely simple data collapse obtained by *Fernandez Luque and van Beek* [1976]. An analytical derivation of these or similar forms remains to be performed.

[48] The analysis presented here confirms the qualitative conclusions of *Fernandez Luque and van Beek* [1976], according to which the Bagnold constraint is not in general satisfied at the bed. Their specific results concerning this point are not employed, however, in this analysis. This is because their result concerning the fluid shear stress at the bed was inferred from a model rather than directly measured. The latter authors calculated the reduction in fluid shear stress due to the bed load by evaluating the average transfer of momentum by the particle to the bed surface, a calculation which relied on an estimate of the average time required for a particle to cover the average saltation length. This length was taken by the above authors to be a constant multiple of particle diameter. On the contrary, measurements of *Nino et al.* [1994a, 1994b] suggest that the saltation length increases linearly with the excess applied stress.

[49] It is now shown that the estimate of the reduction of fluid shear stress given by *Fernandez Luque and van Beek* [1976] contradicts the results obtained by the same authors for the dependence of the average particle speed on the excess external stress. In fact, if the reduction of fluid stress is calculated using equation (11) of *Fernandez Luque and van Beek* [1976] and substituted into the right-hand side of SSP(62) of the present analysis, one readily derives the following relationship for the average particle speed;

$$\hat{V}_p = \Omega f \sqrt{\tau_*} \tag{49}$$

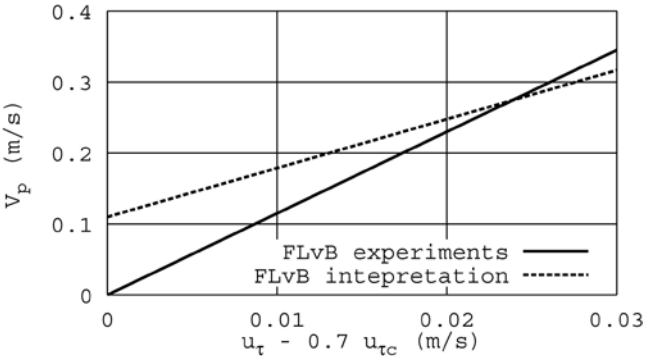

Figure 4. Plot of V_P versus $u_\tau - 0.7u_{\tau c}$ showing the relation originally inferred by *Fernandez Luque and van Beek* [1976] (L B interpretation) and the one obtained from (141) (L B experiments).

where Ω is given as

$$\Omega = 0.238 \frac{(s + c_M)^{1/2}}{\sqrt{c_D}} \tag{50}$$

and where in addition c_M denotes the added mass coefficient for a sphere, here assumed to be equal to 0.5. The relationship (49) is compared in Figure 4 with that proposed by *Fernandez Luque and van Beek* [1976] as appropriate to interpret their experimental observations for the average particle speed. To allow for comparability, in the plot the dimensioned value V_P is plotted against the dimensioned parameter $u_\tau - 0.7u_{\tau c}$. The discrepancy between the two relationships can be easily resolved provided the average saltation length is allowed to depend linearly on the excess applied stress.

9. Conclusion

[50] The Bagnold constraint pertaining to bed load transport fails when applied to arbitrarily sloping beds. In particular, there exists a range of bed slopes well below the angle of repose for which no areal concentration of bed load is sufficient to satisfy the Bagnold constraint. The constraint is thus abandoned.

[51] An entrainment formulation is offered in its place. The entrainment formulation is used to describe a dynamic equilibrium for which the rate of entrainment of bed particles into the bed load layer equals the rate of deposition onto the bed from the bed load layer. This differs markedly from the conceptual framework of the Bagnold constraint, which describes a static equilibrium at which the fluid shear stress at the bed is reduced to the critical value. In the entrainment formulation the fluid shear stress at the bed must be above the critical value if entrainment is to balance deposition and an equilibrium bed load transport is to be maintained. The entrainment formulation provides a consistent explanation of the experimental data of *Fernandez Luque and van Beek* [1976]. It generalizes to arbitrarily sloping beds without the failure that characterizes the results that stem from the Bagnold constraint. In addition, the formulation can be used to explain the data of *Fernandez Luque and van Beek* [1976] on arbitrary streamwise slopes up to 22°.

[52] The entrainment formulation offers a consistent basis for extension to the cases of (1) disequilibrium bed load transport and (2) bed load transport of mixed sizes. The application of the Bagnold constraint to either of these cases poses serious problems that are not easily overcome.

Notation

A, A' parameters defined by SSP(68a) and SSP(68b), respectively.
A_d coefficient in the "inner" form of the sediment deposition relation (17b).
A_{do} value of A_d on a nearly horizontal bed.
A_e coefficient in the "inner" form of the sediment entrainment (pick-up) relation (17a).
A_{eo} value of A_e for a nearly horizontal bed.
B dimensionless parameter in SSP(43a) which takes the value 30 for fully rough turbulent flow.
c local mean volume concentration of particles participating in bed load transport.
C layer-averaged mean volume concentration of particles participating in bed load transport.
c_D particle drag coefficient.
c_L particle lift coefficient.
D grain size.
D_b volume rate of particle deposition per unit bed area.
$\hat{D}$ dimensionless version of D_b defined by SSP(2e).
E_b volume rate of particle erosion (pickup) per unit bed area.
$\hat{E}$ dimensionless version of E_b defined by SSP(2d).
$\hat{E}_a$ rescaled version of $\hat{E}$ defined by SSP(5d).
f $= U/u_\tau$ in accordance with SSP(29); = 11.5 based on SSP(4c).
F_D magnitude of drag force on a particle.
$\boldsymbol{F}_D$ vectorial generalization of F_D.
$\boldsymbol{F}_{Db}$ effective drag acting on a bed load particle defined by SSP(A3).
F_L magnitude of lift force on a particle.
$\boldsymbol{F}_D$ vectorial generalization of F_L.
g acceleration of gravity.
G_s magnitude of the submerged weight of a particle, defined by SSP(41f).
$\boldsymbol{G}_s$ vectorial generalization of G_s.
h_s mean height of the bed load layer.
K parameter defined by (32c).
K_o value of K on a nearly horizontal bed.
$\hat{\boldsymbol{k}}$ upward vertical unit vector.
$\boldsymbol{k}_n$ normal component of $(-\hat{\boldsymbol{k}})$.
k_s roughness height of the bed.
$\boldsymbol{k}_t$ transverse component of $(-\hat{\boldsymbol{k}})$.
L_s mean saltation length.
$\hat{L}_s$ dimensionless version of L_s defined by SSP(2g).
L_{step} mean step length.
$\hat{L}_{step}$ dimensionless version of L_{step} defined by SSP(2h).
M parameter defined by SSP(A12a).
N parameter defined by SSP(A15).
N' parameter defined by SSP(A16).
$\hat{\boldsymbol{n}}$ unit vector upward normal to the bed.
n_k dimensionless parameter relating roughness height k_s to grain size D defined in SSP(43b).
P parameter defined by SSP(A12).

p_{ls} local mean pressure (or the negative of the normal stress) of the solid phase at elevation z above the bed.

p_{sb} value of p_{lb} at the bed.

Q parameter defined by SSP(A14).

q volume sediment transport rate per unit width.

$\hat{q}$ dimensionless Einstein bed load transport rate defined by SSP(2a).

$\hat{\mathbf{q}}$ vectorial generalization of $\hat{q}$.

$\hat{q}_a$ rescaled version of $\hat{q}$ defined by SSP(5a).

R parameter defined by SSP(A13).

r_L parameter describing the effect of lift defined in SSP(51b).

r_d coefficient defined by (36b).

r_{dc} coefficient defined by (23b) in general and (25b) for the case of streamwise slope only.

r_e coefficient defined by (36a).

r_{ed} coefficient defined by (41b).

r_t parameter defined by (47b).

r_{sc} coefficient defined by (23a) in general and (25a) for the case of streamwise slope only.

r_λ coefficient defined by (30b).

r_μ coefficient defined by (28b).

S magnitude of bed slope.

S_P magnitude of bed slope in direction of particle motion, defined by (22).

s specific gravity of sediment.

$\hat{\mathbf{s}}$ unit vector in the direction of τ_{fB}.

$\hat{\mathbf{s}}_b$ unit vector in the direction of τ_{fb}.

$\hat{\mathbf{s}}_p$ unit vector in the direction of $\mathbf{V}_P$.

T_{fsx} value of t_{fsx} resulting from averaging over the bed load layer.

$\mathbf{t}_{fs}$ vectorial rate of transfer of momentum per unit volume per unit time from the fluid phase to the solid phase.

t_{fsx} streamwise component of $\mathbf{t}_{fs}$.

t_{fsz} upward normal component of $\mathbf{t}_{fs}$.

U streamwise component of $\mathbf{u}$ averaged over the bed load layer.

$\mathbf{u}$ local vectorial mean velocity of the fluid phase at elevation z in the bed load layer.

u_τ shear velocity defined by SSP(28).

$u_{\tau c}$ critical shear velocity.

V_P layer-averaged mean velocity of bed load particles.

$\hat{V}_P$ dimensionless version of V_P defined by SSP(2c).

$\mathbf{v}$ local vectorial mean velocity of the solid phase at elevation z in the bed load layer.

$\hat{\mathbf{x}}_b$ unit vector defined by SSP(A8a).

x_p horizontal axis lying orthogonal to the vertical plane $(\boldsymbol{\tau}_{fB}, \hat{\mathbf{k}})$.

$\hat{\mathbf{x}}_p$ unit vector in the direction of x_p.

$\hat{\mathbf{x}}_p{}'$ unit vector defined by SSP(A8b).

x_τ horizontal axis lying in the vertical plane $(\hat{\mathbf{x}}_\tau, \hat{\mathbf{k}})$.

$\hat{\mathbf{x}}_\tau$ unit vector in the direction of x_τ.

z vertical distance upward from the bed.

α streamwise bed angle.

α' streamwise angle of the vector τ_{fb}.

α_d = 0.0301; coefficient in the "outer" deposition relation SSP(14).

α_e = 0.0199; coefficient in the "outer" entrainment relation SSP(4b).

α_q = 7.59; coefficient in the bed load transport relation SSP(13a).

β angle between the vectors τ_{fB} and $\mathbf{F}_{Db}$.

χ = τ_*/τ_{*c} or τ_*/τ_{*co}.

Δ parameter describing the effect of lift defined by SSP(51a).

ϕ_r angle of repose, = $tan^{-1}(\mu)$.

η bed elevation.

φ transverse bed angle.

φ' transverse angle of the vector τ_{fb}.

κ Karman constant of 0.40.

λ dimensionless coefficient relating the Shields stress necessary for bed load transport to stop to the critical Shields stress for the initiation of bed load transport.

λ_o value of λ on a nearly horizontal bed; = 0.7 based on SSP(4c).

μ coefficient of static friction.

μ_d coefficient of dynamic friction.

μ_{do} coefficient of dynamic friction on a nearly horizontal bed.

ν kinematic viscosity of water.

ρ density of water.

τ_D effective "drag stress" defined by SSP(69c).

τ_{fB} mean fluid bed shear stress that would prevail in the absence of bed load transport.

$\boldsymbol{\tau}_{fB}$ vectorial generalization of τ_{fB}.

τ_{fb} magnitude of mean fluid shear stress at the bed in the presence of a bed load layer.

τ_{fco} critical fluid shear stress at the bed for the initiation of bed load.

τ_{fI} magnitude of mean fluid shear stress at the interface between the bed load layer and the fluid above.

τ_{lf} local mean streamwise fluid shear stress at distance z above the bed.

τ_{ls} local mean streamwise shear stress of the solid phase at distance z above the bed.

τ_{sb} magnitude of mean shear stress exerted on the bed by the solid phase (particle collision).

$\boldsymbol{\tau}_{sb}$ vectorial generalization of τ_{sb}.

τ_* Shields stress based on τ_{fB}.

$\boldsymbol{\tau}_*$ vectorial generalization of τ_*.

τ_{*e} = $\tau_* - \tau_{*co}$.

τ_{*b} Shields stress based on τ_{fb} defined in (2).

τ_{*be} = $\tau_{*b} - \tau_{*co}$.

τ_{*c} critical Shields stress on an arbitrarily sloping bed.

τ_{*co} critical Shields stress on a nearly horizontal bed.

τ_{*cso} critical Shields stress for the cessation of bed load motion, defined by SSP(30).

τ_* dimensionless form of $\tau_{f\mathbf{B}}$ defined by SSP(2b).

ω vorticity of the flow acting on a "dangerously placed" bed particle.

ξ volume areal concentration of bed load such that $\boldsymbol{\xi} = ch_s$.

$\hat{\xi}$ dimensionless version of $\boldsymbol{\xi}$ defined by SSP(2f).

ψ angle between the direction of the applied shear stress and the direction of particle velocity.

ζ elevation above the bed used for the evaluation of the lift force.

$\forall$ particle volume defined by SSP(A2).

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G. Parker, St. Anthony Falls Laboratory, University of Minnesota, Minneapolis, MN 55414, USA. (parke002@tc.umn.edu)

G. Seminara, Dipartamento di Ingegneria Ambientale, Università degli Studi di Genova, Via Montallegro 1, 16145, Genova, Italy. (sem@diam.unige.it)

L. Solari, Dipartamento di Ingegneria Civile, Università degli Studi di Firenze, Via S. Marta 3, 50139, Firenze, Italy. (luca.solari@dicea.unifi.it)