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SIZE ESTIMATES OF THE INVERSE INCLUSION PROBLEM FOR THE SHALLOW SHELL EQUATION*

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Abstract. In this paper, we consider the problem of estimating the size of an inclusion in the shallow shell. Previously, the same problem was studied in [M. Di Cristo, C. L. Lin, and J. N. Wang, *Ann. Sc. Norm. Super. Pisa Cl. Sci.*] under the assumption of *fatness condition*. We remove this restriction in this work. The main tool is a global doubling estimate for the solution of the shallow shell equation.

Key words. inverse problems, shallow shell, size estimates

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1. Introduction.

In this work we study the inverse problem of estimating the size of an embedded inclusion by one boundary measurement. Let Ω be a bounded domain in $\mathbb{R}^2$. Without loss of generality, we assume $0 \in \Omega$. Let $\bar{\theta} : \overline{\Omega} \to \mathbb{R}$ satisfy appropriate regularity assumption which will be specified later. For a shallow shell, its middle surface is described by $\{(x_1, x_2, \varepsilon\rho_0\bar{\theta}(x_1, x_2)) : (x_1, x_2) \in \overline{\Omega}\}$ for $\varepsilon > 0$, where $\rho_0 > 0$ is the characteristic length of Ω (see section 3.1). From now on, we denote $\theta = \rho_0\bar{\theta}$. Let $\mathbf{u} = (u_1, u_2, u_3) = (u', u_3) : \Omega \to \mathbb{R}^3$ represent the displacement vector of the middle surface. Then $\mathbf{u}$ satisfies the following equations:

$$\begin{cases} \operatorname{div} \mathbf{n}^\theta(\mathbf{u}) = 0 \quad \text{in} \quad \Omega, \\ \operatorname{div}\operatorname{div} \mathbf{m}(u_3) - \operatorname{div}(\mathbf{n}^\theta(\mathbf{u})\nabla\theta) = 0 \quad \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where $\mathbf{m} = (m_{ij})$ and $\mathbf{n}^\theta = (n^\theta_{ij})$ with

$$\begin{aligned} m_{ij}(u_3) &= \rho_0^2 \left\{ \frac{4\lambda\mu}{3(\lambda+2\mu)}(\Delta u_3)\delta_{ij} + \frac{4\mu}{3}\partial^2_{ij}u_3 \right\}, \\ n^\theta_{ij}(\mathbf{u}) &= \frac{4\lambda\mu}{\lambda+2\mu} e^\theta_{kk}(\mathbf{u})\delta_{ij} + 4\mu e^\theta_{ij}(\mathbf{u}), \\ e^\theta_{ij}(\mathbf{u}) &= \frac{1}{2}(\partial_i u_j + \partial_j u_i + (\partial_i\theta)\partial_j u_3 + (\partial_j\theta)\partial_i u_3), \end{aligned} \tag{1.2}$$

and λ, μ are Lamé coefficients. We also denote $\mathbf{e}^\theta = (e^\theta_{ij})$ and $\nabla^2 u_3 = (\partial^2_{ij}u_3)$, the Hessian of u_3. Hereafter, the Roman indices (except n) belong to $\{1, 2\}$ and the Einstein summation is used for repeated indices.

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Assume that D is a measurable subdomain of Ω with $\overline{D} \subset \Omega$. We consider Lamé parameters

$$\widetilde{\lambda} = \lambda + \chi_D \lambda_0 \quad \text{and} \quad \widetilde{\mu} = \mu + \chi_D \mu_0,$$

where χ_D is the characteristic function of D. The domain D represents the inclusion inside Ω. With such parameters $\widetilde{\lambda}, \widetilde{\mu}$, we denote the displacement field $\widetilde{\mathbf{u}} = (\widetilde{u}', \widetilde{u}_3)^t$ satisfying (1.1) and the Neumann boundary conditions on $\partial\Omega$:

$$\begin{cases} \widetilde{\mathbf{n}}^\theta \nu = \rho_0^{-1} \widehat{T}, \\ \nu \cdot \widetilde{\mathbf{m}} \nu = \widehat{M}_\nu, \\ (\operatorname{div} \widetilde{\mathbf{m}} - \widetilde{\mathbf{n}}^\theta \nabla \theta) \cdot \nu + \partial_s (\nu \cdot \widetilde{\mathbf{m}} \tau) = -\partial_s \widehat{M}_\tau, \end{cases} \tag{1.3}$$

where $\widetilde{\mathbf{m}}_{ij} = \widetilde{m}_{ij}(\widetilde{u}_3)$ and $\widetilde{n}^\theta_{ij} = \widetilde{n}^\theta_{ij}(\widetilde{\mathbf{u}})$ are defined in (1.2) with λ, μ, $\mathbf{u}$ being replaced by $\widetilde{\lambda}$, $\widetilde{\mu}$, $\widetilde{\mathbf{u}}$. Hereafter, $\nu = (\nu_1, \nu_2)$, $\tau = (\tau_1, \tau_2)$ are, respectively, the normal and the tangent vectors along $\partial\Omega$, and s is the arclength parameter of $\partial\Omega$. Precisely, the tangent vector τ is obtained by rotating ν counterclockwise of degree $\pi/2$. The boundary field $\widehat{M} = \widehat{M}_\tau \nu + \widehat{M}_\nu \tau$, i.e., $\widehat{M}_\tau = \widehat{M} \cdot \nu$, $\widehat{M}_\nu = \widehat{M} \cdot \tau$. We remark that in the plate theory, $\widehat{M}_\tau$ and $\widehat{M}_\nu$ are the twisting and bending moments applied on $\partial\Omega$. The field $\widehat{T}$ satisfies the compatibility condition which will be specified later. An interesting inverse problem is to determine the geometric information of D from a pair of $\{\widehat{T}, \widehat{M}; \widetilde{u}'|_{\partial\Omega}, (\widetilde{u}_3|_{\partial\Omega}, \partial_\nu \widetilde{u}_3|_{\partial\Omega})\}$, i.e., the Cauchy data of the solution $\widetilde{\mathbf{u}}$. Despite its practical value, the fundamental global uniqueness, even for the scalar equation, is yet to be proved. For the development of the uniqueness issue for this kind of inverse problem, we refer to [2] and references therein for details.

In this paper we are interested in estimating the size of the area of D in terms of the Cauchy data of $\widetilde{\mathbf{u}}$. This type of problem has been studied for the scalar equation and the system of equations such as the isotropic elasticity and plate. We refer to the survey article [1] for the early development and [5], [6] for the latest result in the plate equations. Specifically, the size of D is estimated by the following two quantities:

$$\widetilde{W} = \int_{\partial\Omega} \rho_0^{-1} \widehat{T} \cdot \widetilde{u}' + \widehat{M}_\nu \partial_\nu \widetilde{u}_3 + \partial_s \widehat{M}_\tau \widetilde{u}_3$$

and

$$W = \int_{\partial\Omega} \rho_0^{-1} \widehat{T} \cdot u' + \widehat{M}_\nu \partial_\nu u_3 + \partial_s \widehat{M}_\tau u_3,$$

where $\mathbf{u} = (u', u_3)^t$ is the displacement vector that satisfies (1.1) and (1.3) with $D = \emptyset$, i.e., $\widetilde{\lambda} = \lambda$ and $\widetilde{\mu} = \mu$. Here we assume that λ, μ are a priori given, thus, both $\widetilde{W}$ and W are known. To be more precise, in this paper, we will show that under some a priori assumptions, there exist positive constants C_1, C_2 such that

$$C_1 \left| \frac{\widetilde{W} - W}{W} \right| \le \operatorname{area}(D) \le C_2 \left| \frac{\widetilde{W} - W}{W} \right|^{1/p}, \tag{1.4}$$

where C_1, C_2, and $p > 1$ depend on the a priori data.

The derivation of the volume bounds on D relies on the following integral inequalities:

$$\frac{1}{K} \int_D |\mathbf{e}^\theta(\mathbf{u})|^2 + \rho_0^2 |\nabla^2 u_3|^2 \le |W - \widetilde{W}| \le K \int_D |\mathbf{e}^\theta(\mathbf{u})|^2 + \rho_0^2 |\nabla^2 u_3|^2, \tag{1.5}$$

where the constant K depends on the a priori data. The lower bound of area(D) is a consequence of the second inequality of (1.5) and the elliptic regularity estimate for $\mathbf{u}$. To derive the upper bound of area(D), we shall use the first inequality of (1.5). As indicated in all previous related results, we need to estimate $\int_D |\mathbf{e}^\theta(\mathbf{u})|^2 + \rho_0^2|\nabla^2 u_3|^2$ from below. If we have that

$$|D_{h_1\rho_0}| \geq \frac{1}{2}|D|, \tag{1.6}$$

where for $r > 0$ we denote

$$D_r = \{x \in D \,:\, \mathrm{dist}(x, \partial D) > r\}$$

and $|D|$ stands for the Lebesgue measure of D, for a known positive constant h_1, we can derive (1.4) with $p = 1$ [4]. Here the constant C_2 also depends on h_1. The assumption (1.6) is called the *fatness condition*.

The main goal of this paper is to remove the fatness condition (1.6). As in other similar problems, the main tool is a global doubling estimate for $|\mathbf{e}^\theta(\mathbf{u})|^2 + \rho_0^2|\nabla^2 u_3|^2$. In fact, a local doubling estimate for $\mathbf{u}$ was obtained in [4]. One of the difficulties in extending the local doubling estimate for $\mathbf{u}$ to a global doubling estimate for $|\mathbf{e}^\theta(\mathbf{u})|^2 + \rho_0^2|\nabla^2 u_3|^2$ lies in the fact that u' and u_3 have different scalings. To overcome this difficulty, we transform the first equation of (1.1) for u' to a fourth order equation for a scalar function ψ, which will be defined precisely later on (see (4.2)), provided the domain Ω is simply connected. In this case, two scalar functions ψ and u_3 will have the same scaling. This idea comes from the well-known fact that the two-dimensional elasticity equation and the thin plate equation are equivalent.

We wish to mention here that an analogous problem has been considered in [3], where it is assumed that both medium properties inside and outside the inclusion are given. The arguments in [3] are based on the method of translation and the bounds given there depend on nonlinear combinations of the boundary data. For the result here, the material properties inside the inclusion are unknown and the bounds are expressed in terms of the normalized power gap.

This paper is organized as follows. In section 2, we state some needed results obtained in [4]. The investigation of the inverse problem is given in section 3. Section 4 is devoted to the proof of our main theorem, Theorem 3.1.

Notation.

Definition 1.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ with $n \geq 2$. Given $k \in \mathbb{Z}^+$, we say that $\partial\Omega$ is of class $C^{k,1}$ with constants ρ_0, A_0 if for any point $z \in \partial\Omega$ there exists a rigid coordinates transform under which $z = 0$ and*

$$\Omega \cap B_{\rho_0}(0) = \{x = (x_1, \ldots, x_{n-1}, x_n) = (x', x_n) \in B_{\rho_0}(0) : x_n > \varphi(x')\},$$

where $\varphi(x')$ is a $C^{k,1}$ function on $B'_{\rho_0}(0) = B_{\rho_0}(0) \cap \{x_n = 0\}$ satisfying $\varphi(0) = 0$ and $\nabla\varphi(0) = 0$ if $k \geq 1$ and

$$\|\varphi\|_{C^{k,1}(B'_{\rho_0}(0))} \leq A_0\rho_0.$$

Throughout the paper, we will normalize all norms such that they are dimensionally homogeneous and coincide with the standard definitions when the dimensional parameter is one. With this in mind, we define

$$\|\varphi\|_{C^{k,1}(B'_{\rho_0}(0))} = \sum_{j=0}^{k} \rho_0^j \|\nabla^j \varphi\|_{L^\infty(B'_{\rho_0}(0))} + \rho_0^{k+1}\|\nabla^{k+1}\varphi\|_{L^\infty(B'_{\rho_0}(0))}.$$

Similarly, considering $w : \Omega \to \mathbb{R}$, with Ω defined as above, we define

$$\|w\|_{C^{k,1}(\Omega)} = \sum_{j=0}^{k} \rho_0^j \|\nabla^j w\|_{L^\infty(\Omega)} + \rho_0^{k+1} \|\nabla^{k+1} w\|_{L^\infty(\Omega)},$$
$$\|w\|^2_{L^2(\Omega)} = \rho_0^{-n} \int_\Omega w^2,$$
$$\|w\|^2_{H^k(\Omega)} = \rho_0^{-n} \sum_{j=0}^{k} \rho_0^{2j} \int_\Omega |\nabla^j w|^2, \quad k \geq 1.$$

In particular, if $\Omega = B_\rho(0)$, then Ω satisfies Definition 1.1 with $\rho_0 = \rho$.

Let $\mathcal{A}$ be an open connected component of $\partial\Omega$. For any given point $z_0 \in \mathcal{A}$, we define the positive orientation of $\mathcal{A}$ associated with an arclength parametrization $\zeta(s) = (x_1(s), x_2(s))$, $s \in [0, \text{length}(\mathcal{A})]$ such that $\zeta(0) = z_0$ and $\zeta'(s) = \tau(\zeta(s))$. Finally, we define for any $h > 0$

$$\Omega_h = \{x \in \Omega \mid \text{dist}(x, \partial\Omega) > h\}.$$

2. The forward problem and known estimates. At this moment, we assume $\partial\Omega \in C^{1,1}$ with constants A_0, ρ_0. Also, let Ω satisfy

$$|\Omega| \leq A_1 \rho_0^2 \tag{2.1}$$

throughout the article, and

$$\|\nabla\theta\|_{L^\infty(\Omega)} = \rho_0 \|\nabla\bar{\theta}\|_{L^\infty(\Omega)} \leq A_2 \tag{2.2}$$

for some positive constants A_1 and A_2. We will investigate the Neumann boundary value problem, the forward problem, for the shallow shell system. To begin, let us assume that Lamé coefficients $\lambda, \mu \in L^\infty(\Omega)$ satisfying

$$\mu(x) \geq \delta_0 > 0, \quad 3\lambda(x) + 2\mu(x) \geq \delta_0 \quad \forall\, x \in \Omega \text{ a.e.} \tag{2.3}$$

We aim to find $\mathbf{u} = (u', u_3)$ satisfying

$$\begin{cases} \operatorname{div} \mathbf{n}^\theta(\mathbf{u}) = 0 \quad \text{in} \quad \Omega, \\ \operatorname{div}\operatorname{div} \mathbf{m}^\theta(u_3) - \operatorname{div}(\mathbf{n}^\theta(\mathbf{u})\nabla\theta) = 0 \quad \text{in} \quad \Omega \end{cases} \tag{2.4}$$

with boundary conditions

$$\begin{cases} \mathbf{n}^\theta(\mathbf{u})\nu = \rho_0^{-1}\widehat{T}, \\ \nu \cdot \mathbf{m}\nu = \widehat{M}_\nu, \\ (\operatorname{div}\mathbf{m} - \mathbf{n}^\theta\nabla\theta) \cdot \nu + \partial_s(\nu \cdot \mathbf{m}\tau) = -\partial_s \widehat{M}_\tau. \end{cases} \tag{2.5}$$

To solve (2.4)–(2.5), $(\widehat{T}, \widehat{M})$ must satisfy the following compatibility condition:

$$\int_{\partial\Omega} \rho_0^{-1}\widehat{T} \cdot (a + W \cdot x + b\theta) - b_1\widehat{M}_1 + b_2\widehat{M}_2 = 0. \tag{2.6}$$

Note that taking $b = 0$, we have the usual compatibility condition for the traction of the elasticity equation, i.e.,

$$\int_{\partial\Omega} \widehat{T} \cdot (a + W \cdot x) = 0.$$

On the other hand, to guarantee the uniqueness of the forward problem, we impose the following normalization conditions:

$$\int_\Omega u = 0, \quad \int_\Omega \nabla u_3 = 0, \tag{2.7}$$

and

$$\int_\Omega (\partial_1 u_2 - \partial_2 u_1) + (\partial_1\theta\partial_2 u_3 - \partial_2\theta\partial_1 u_3) = 0. \tag{2.8}$$

The boundary value problem (2.4)–(2.5) can be solved by the standard variational method. The detailed proof was given in [4].

PROPOSITION 2.1 (see [4, Theorem 3.3]). *Assume that θ satisfies* (2.2) *and $\lambda, \mu \in L^\infty(\Omega)$ satisfy* (2.3). *Given any boundary field $(\widehat{T}, \widehat{M}) \in H^{-1/2}(\partial\Omega)$ and the compatibility condition* (2.6) *holds. Then* (2.4)–(2.5) *admits a unique weak solution $\mathbf{u} = (u', u_3)^t$ satisfying the conditions* (2.7)–(2.8) *and*

$$\|u'\|_{H^1(\Omega)} + \|u_3\|_{H^2(\Omega)} \le C\|(\widehat{T}, \widehat{M})\|_{(H^{-1/2}(\partial\Omega))^3}, \tag{2.9}$$

where C depends on A_0, A_1, A_2, δ_0.

To study the inverse problem, we also need a global regularity theorem for the shallow shell equations. To simplify our presentation, we impose a technical assumption on $\bar\theta$ (or θ) in this section. Assume that $\bar\theta$ satisfies

$$\bar\theta = \nabla\bar\theta = 0 \quad \text{on} \quad \partial\Omega. \tag{2.10}$$

We proved the following theorem.

THEOREM 2.2 (see [4, Theorem 3.4]). *Assume that Ω is a bounded domain in $\mathbb{R}^2$ satisfying* (2.1) *whose boundary $\partial\Omega$ is of class $C^{4,1}$ with constants A_0 and ρ_0. Let $\lambda, \mu \in C^{1,1}(\bar\Omega)$ satisfy* (2.3) *and $\bar\theta \in C^{2,1}(\bar\Omega)$ satisfy* (2.10) *and*

$$\|\lambda\|_{C^{1,1}(\bar\Omega)} + \|\mu\|_{C^{1,1}(\bar\Omega)} + \|\bar\theta\|_{C^{2,1}(\bar\Omega)} \le A_2. \tag{2.11}$$

Let $u \in (H^1(\Omega))^2 \times H^2(\Omega)$ be the weak solution of (2.4), (2.5) *with Neumann boundary condition $(\widehat{T}, \widehat{M}) \in (H^{1/2}(\partial\Omega))^2 \times H^{3/2}(\partial\Omega)$ satisfying* (2.6). *Assume that u satisfies the normalization conditions* (2.7). *Then there exists a constant $C > 0$, depending on A_0, A_1, A_2, δ_0 such that*

$$\|u'\|_{H^2(\Omega)} + \|u_3\|_{H^4(\Omega)} \le C\|(\widehat{T}, \widehat{M})\|_{(H^{1/2}(\partial\Omega))^2 \times H^{3/2}(\partial\Omega)}. \tag{2.12}$$

A key ingredient in solving our inverse problem is a continuation estimate from the interior for the solution u of (2.4), (2.5). To do this, we need some assumptions on the coupled field $(\widehat{T}, \widehat{M})$. We assume that $(\widehat{T}, \widehat{M})$ satisfies

$$\operatorname{supp}(\widehat{T}, \widehat{M}) \subset \Gamma_0, \tag{2.13}$$

where Γ_0 is an open subarc of $\partial\Omega$ with

$$|\Gamma_0| \le (1 - \gamma_0)|\partial\Omega| \tag{2.14}$$

for some $\gamma_0 > 0$. We obtained the following estimate in [4, Theorem 4.15]. The proof of this theorem relies on the three-ball inequalities for $|\mathbf{e}^\theta(\mathbf{u})|^2 + \rho_0^2|\nabla^2 u_3|^2$.

THEOREM 2.3 (Lipschitz propagation of smallness). *Assume that Ω is a bounded domain having boundary $\partial\Omega \in C^{4,1}$ with constants A_0, ρ_0. Let $\lambda, \mu \in C^{1,1}(\bar{\Omega})$ satisfy* (2.3) *and $\bar{\theta} \in C^{2,1}(\bar{\Omega})$ satisfy* (2.10) *and let* (2.11) *hold. Let $u \in (H^1(\Omega))^2 \times H^2(\Omega)$ be the weak solution of* (2.4), (2.5) *satisfying* (2.7) *with Neumann boundary condition $(\widehat{T}, \widehat{M}) \in (H^{1/2}(\partial\Omega))^2 \times H^{3/2}(\partial\Omega)$ satisfying* (2.6), (2.13), (2.14).

There exists a positive number $\chi > 1$, depending on δ_0, A_2, such that for every $\rho > 0$ and every $x \in \Omega_{\chi\rho\rho_0}$, we have

$$\int_{B_{\rho\rho_0}(x)} |\mathbf{e}^\theta(u)|^2 + \rho_0^2 |\nabla^2 u_3|^2 \geq C_\rho \int_\Omega |\mathbf{e}^\theta(u)|^2 + \rho_0^2 |\nabla^2 u_3|^2, \tag{2.15}$$

where C_ρ depends on $A_0, A_1, A_2, \delta_0, \gamma_0, \rho$, and

$$\|(\widehat{T}, \widehat{M})\|^2_{(L^2(\partial\Omega))^2 \times H^{1/2}(\partial\Omega)} / \|(\widehat{T}, \widehat{M})\|_{(H^{-1/2}(\partial\Omega))^3}.$$

3. Inverse problem and statement of the main theorem. In this section, we would like to study the problem of estimating the size of a general inclusion embedded in a shallow shell by one boundary measurement. Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain with boundary $\partial\Omega$, which is of class $C^{4,1}$ with constants A_0, ρ_0. Assume that (2.1) holds. Now let D be a measurable subdomain of Ω possibly disconnected satisfying

$$\operatorname{dist}(D, \partial\Omega) \geq d_0 \rho_0 \tag{3.1}$$

for some given constant d_0. Let $\lambda, \mu \in C^{1,1}(\bar{\Omega})$ satisfy (2.3) and $\bar{\theta} \in C^{2,1}(\bar{\Omega})$ satisfy (2.10). As well, assume that the estimate (2.11) holds. For measurable functions λ_0, μ_0, we define

$$\tilde{\lambda} = \lambda + \chi_D \lambda_0 \qquad \text{and} \qquad \tilde{\mu} = \mu + \chi_D \mu_0,$$

where χ_D is the characteristic function of D, and we assume

$$0 < \tilde{\delta}_0 \leq \tilde{\mu} \quad \text{and} \quad \tilde{\delta}_0 \leq 3\tilde{\lambda} + 2\tilde{\mu} \quad \forall\, x \in \Omega \text{ a.e.}$$

To describe the jump condition, we introduce some shorthand notation. We set

$$a = \frac{4\lambda\mu}{\lambda + 2\mu},\ b = 4\mu,\ c = \frac{4\lambda\mu}{3(\lambda + 2\mu)},\ d = \frac{4\mu}{3}, \tag{3.2}$$

and the corresponding $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ replacing λ, μ with $\tilde{\lambda}, \tilde{\mu}$, respectively. We assume the following condition on the jump at the interface ∂D. There exists a constant $k_0 > 0$ such that

$$\frac{1}{k_0} f \leq (\tilde{f} - f) \leq k_0 f \qquad \text{a.e. in } \Omega, \tag{3.3}$$

where $f = a, b, c, d$ and $\tilde{f} = \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$. Similarly, we can also treat the case where $-\frac{1}{k_0} f \leq (\tilde{f} - f) \leq -k_0 f$. For the sake of simplicity, we only consider (3.3) in the paper. On the prescribed boundary field $(\widehat{T}, \widehat{M})$, we assume that

$$(\widehat{T}, \widehat{M}) \in (H^{1/2}(\partial\Omega))^2 \times H^{3/2}(\partial\Omega) \quad \text{and} \quad \operatorname{supp}(\widehat{T}, \widehat{M}) \subset \Gamma_0 \tag{3.4}$$

satisfy the compatibility condition (2.6), where Γ_0 is an open subarc of $\partial\Omega$ satisfying (2.14). Let $\mathbf{u} = (u', u_3)$ solve (2.4)–(2.5) and satisfy the normalization conditions (2.7). Next we consider the perturbed system. Let $\widetilde{\mathbf{u}} = (\widetilde{u}', \widetilde{u}_3)$ solve

$$\begin{cases} \operatorname{div} \widetilde{\mathbf{n}}^\theta(\widetilde{\mathbf{u}}) = 0 \quad \text{in} \quad \Omega, \\ \operatorname{div}\operatorname{div} \widetilde{\mathbf{m}}(\widetilde{u}_3) - \operatorname{div}(\widetilde{\mathbf{n}}^\theta(\widetilde{\mathbf{u}})\nabla\theta) = 0 \quad \text{in} \quad \Omega \end{cases} \tag{3.5}$$

with the Neumann boundary condition (2.5). Likewise, $\widetilde{\mathbf{u}}$ satisfies the normalization conditions (2.7). Denote

$$\begin{aligned} \widetilde{W} &= \int_{\partial\Omega} \rho_0^{-1}\widehat{T}\cdot\widetilde{u}' + \widehat{M}_\nu\partial_\nu\widetilde{u}_3 + \partial_s\widehat{M}_\tau\widetilde{u}_3, \\ W &= \int_{\partial\Omega} \rho_0^{-1}\widehat{T}\cdot u' + \widehat{M}_\nu\partial_\nu u_3 + \partial_s\widehat{M}_\tau u_3 \\ &= \int_\Omega \mathbf{n}^\theta(\mathbf{u})\cdot\mathbf{e}^\theta(\mathbf{u}) + \mathbf{m}(u_3)\cdot\nabla^2 u_3, \end{aligned}$$

which represent the work exerted by the boundary field when the inclusion is present or absent, respectively. We can now state the main result.

THEOREM 3.1. *Suppose that all the hypotheses stated in this section are satisfied. Furthermore, assume that Ω is simply connected. Then the estimate*

$$C_1\rho_0^2\left|\frac{W-\widetilde{W}}{W}\right| \le |D| \le C_2\rho_0^2\left|\frac{W-\widetilde{W}}{W}\right|^{1/p} \tag{3.6}$$

holds, where C_1 depends on A_0, A_1, A_2, d_0, k_0, and δ_0, C_2 depends on $A_0, A_1, A_2, \delta_0, k_0$, γ_0, and d_0, and p depends on $A_0, A_1, A_2, \delta_0, \gamma_0, d_0$, and the ratio

$$\|(\widehat{M},\widehat{T})\|_{(L^2(\partial\Omega))^2\times H^{1/2}(\partial\Omega)}/\|(\widehat{M},\widehat{T})\|_{(H^{-1/2}(\partial\Omega))^3}.$$

4. Proof of Theorem 3.1. We prove the main theorem in this section. Using integration by parts, it is not hard to show that there exist positive constants $\tilde{C}_1, \tilde{C}_2$ depending on $A_0, A_1, A_2, \delta_0, k_0$ such that

$$\tilde{C}_1\int_D |\mathbf{e}^\theta(\mathbf{u})|^2 + \rho_0^2|\nabla^2 u_3|^2 \le W - \widetilde{W} \le \tilde{C}_2\int_D |\mathbf{e}^\theta(\mathbf{u})|^2 + \rho_0^2|\nabla^2 u_3|^2 \tag{4.1}$$

(see [4, Lemma 5.2]). The derivation of the lower bound of $|D|$ can be found in [4]. It is an easy consequence of the interior estimate and the Sobolev embedding theorem. We will not repeat the arguments here.

To estimate the upper bound for $|D|$, we will transform the first equation of (1.1) into a plate-like equation. Denote

$$\mathbf{n}^\theta(\mathbf{u}) = \begin{pmatrix} n_{11}^\theta(\mathbf{u}) & n_{12}^\theta(\mathbf{u}) \\ n_{12}^\theta(\mathbf{u}) & n_{22}^\theta(\mathbf{u}) \end{pmatrix}.$$

The first equation of (1.1) is written as

$$\begin{cases} \partial_1 n_{11}^\theta(\mathbf{u}) + \partial_2 n_{12}^\theta(\mathbf{u}) = 0, \\ \partial_1 n_{12}^\theta(\mathbf{u}) + \partial_2 n_{22}^\theta(\mathbf{u}) = 0. \end{cases}$$

Since Ω is simply connected, there exists a scalar function ψ such that

$$\mathbf{n}^\theta(\mathbf{u}) = \begin{pmatrix} n_{11}^\theta & n_{12}^\theta \\ n_{21}^\theta & n_{22}^\theta \end{pmatrix} = \rho_0 \begin{pmatrix} \partial_{22}^2\psi & -\partial_{12}^2\psi \\ -\partial_{21}^2\psi & \partial_{11}^2\psi \end{pmatrix} = \rho_0 \mathbb{R}\nabla^2\psi, \tag{4.2}$$

where $\partial_{12}\psi = \partial_{21}\psi$ and the fourth order tensor $\mathbb{R}$ is defined by

$$\mathbb{R}\mathbf{M} = \mathbf{R}_\perp^T \mathbf{M}\mathbf{R}_\perp$$

for any 2×2 matrix $\mathbf{M}$, where

$$\mathbf{R}_\perp = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

To guarantee the uniqueness of ψ in (4.2), it suffices to impose the normalization condition

$$\int_\Omega \psi = \int_\Omega \nabla\psi = 0. \tag{4.3}$$

The second equation of (1.2) can be written as

$$\mathbf{e}^\theta(\mathbf{u}) = \mathbb{S}\mathbf{n}^\theta(\mathbf{u}), \tag{4.4}$$

where the fourth order tensor $\mathbb{S}$ is defined by

$$\mathbb{S}\mathbf{A} = \frac{1}{4\mu}\mathbf{A}^{\text{sym}} - \frac{\lambda}{4\mu(3\lambda + 2\mu)}(\text{Tr}\mathbf{A})I_2 \tag{4.5}$$

for any matrix $\mathbf{A}$, where I_2 is the 2×2 identity matrix. On the other hand, from the form of $\mathbf{e}^\theta(\mathbf{u})$ (the third equation of (1.2)), we have

$$\mathbf{e}^\theta(\mathbf{u}) - \frac{1}{2}\left(\nabla\theta \otimes \nabla u_3 + (\nabla\theta \otimes \nabla u_3)^T\right) = \frac{1}{2}\left(\nabla u' + (\nabla u')^T\right)$$

and it follows that

$$\text{div}\,\text{div}\left(\mathbb{R}\mathbf{e}^\theta(\mathbf{u})\right) - \frac{1}{2}\text{div}\,\text{div}\left(\mathbb{R}(\nabla\theta \otimes \nabla u_3) + \mathbb{R}(\nabla\theta \otimes \nabla u_3)^T\right) = 0. \tag{4.6}$$

Substituting (4.2) and (4.4) into (4.6) yields

$$\rho_0 \text{div}\,\text{div}(\mathbb{L}\nabla^2\psi) - \frac{1}{2}\text{div}\,\text{div}\left(\mathbb{R}(\nabla\theta \otimes \nabla u_3) + \mathbb{R}(\nabla\theta \otimes \nabla u_3)^T\right) = 0, \tag{4.7}$$

where $\mathbb{L} = \mathbb{R}\mathbb{S}\mathbb{R}$. It is easily seen that $\mathbb{L} = \mathbb{S}$. Replacing the first equation of (1.1) by (4.7), (1.1) is then transformed into

$$\begin{cases} \rho_0 \text{div}\,\text{div}(\mathbb{L}\nabla^2\psi) - \frac{1}{2}\text{div}\,\text{div}\left(\mathbb{R}(\nabla\theta \otimes \nabla u_3) + \mathbb{R}(\nabla\theta \otimes \nabla u_3)^T\right) = 0, \\ \rho_0 \text{div}\,\text{div}(\mathbb{M}\nabla^2 u_3) - \text{div}(\mathbb{R}\nabla^2\psi\nabla\theta) = 0, \end{cases} \tag{4.8}$$

where

$$\mathbb{M}\nabla^2 u_3 = \mathbf{m}(u_3) = \frac{4\mu}{3}\nabla^2 u_3 + \frac{4\lambda\mu}{3(\lambda + 2\mu)}(\text{Tr}\nabla^2 u_3)I_2.$$

We refer the reader to [3], where the reduce system was first derived.

Let us denote $U = (\psi, u_3)^T$. The system (4.8) can be written into a nondivergence form in which the leading operator is the bi-Laplacian. Using the same arguments as in [6, Proposition 4.2], we can derive a global doubling inequality for U.

PROPOSITION 4.1 (global doubling inequality). *Assume that Ω is a bounded domain having boundary $\partial\Omega \in C^{0,1}$ with constants A_0, ρ_0. Let $U \in H^4_{loc}(\Omega)$ be a nonzero solution to* (4.8) *in Ω. Then there exists a positive constant $\vartheta < 1$, depending on A_2, δ_0, such that for every $\bar{r} > 0$ and for every $x_0 \in \Omega_{\bar{r}\rho_0}$, we have for every $r \le \frac{\vartheta}{2}\bar{r}\rho_0$*

$$\int_{B_{2r}(x_0)} |U|^2 dx \le K \int_{B_r(x_0)} |U|^2 dx, \tag{4.9}$$

where K depends on $A_0, A_1, A_2, \delta_0, \bar{r}$, and the ratio $\frac{||U||_{H^{\frac{1}{2}}(\Omega)}}{||U||_{L^2(\Omega)}}$.

To study our inverse problem, we need a global doubling inequality for $|\mathbf{e}^\theta(\mathbf{u})|^2 + \rho_0|\nabla^2 u_3|^2$.

PROPOSITION 4.2 (doubling inequality in terms of the boundary data). *Assume that Ω is a bounded domain having boundary $\partial\Omega \in C^{4,1}$ with constants A_0, ρ_0. Let $\lambda, \mu \in C^{1,1}(\bar{\Omega})$ satisfy* (2.3) *and $\bar{\theta} \in C^{2,1}(\bar{\Omega})$ satisfy* (2.10) *and let* (2.11) *hold. Let $\mathbf{u} \in (H^1(\Omega))^2 \times H^2(\Omega)$ be the weak solution of* (2.4), (2.5) *satisfying* (2.7) *with Neumann boundary condition $(\widehat{T}, \widehat{M}) \in (H^{1/2}(\partial\Omega))^2 \times H^{3/2}(\partial\Omega)$ satisfying* (2.6), (2.13), (2.14). *Then there exists a positive constant $\vartheta < 1$, depending on A_2, δ_0, such that for every $\bar{r} > 0$ and for every $x_0 \in \Omega_{\bar{r}\rho_0}$, we have for every $r \le \frac{\vartheta}{2}\bar{r}\rho_0$*

$$\int_{B_{2r}(x_0)} (|\mathbf{e}^\theta(\mathbf{u})|^2 + \rho_0|\nabla^2 u_3|^2) dx \le K \int_{B_r(x_0)} (|\mathbf{e}^\theta(\mathbf{u})|^2 + \rho_0|\nabla^2 u_3|^2) dx, \tag{4.10}$$

where K depends on $A_0, A_1, A_2, \delta_0, \gamma_0, \bar{r}$, and

$$\|(\widehat{T}, \widehat{M})\|^2_{(L^2(\partial\Omega))^2 \times H^{1/2}(\partial\Omega)} / \|(\widehat{T}, \widehat{M})\|_{(H^{-1/2}(\partial\Omega))^3}.$$

The proof of Proposition 4.2 will be given later. Having the doubling inequality at hand, we can get the following A_p property as in [6, Proposition 4.6].

PROPOSITION 4.3 (A_p property). *Let the assumptions of Proposition* 4.2 *be satisfied. For every $\bar{r} > 0$, there exist positive constants B and p such that for every $x_0 \in \Omega_{\bar{r}\rho_0}$, we have for every $r \le \frac{\vartheta}{2}\bar{r}\rho_0$*

$$\begin{aligned} &\left(\frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} (|\mathbf{e}^\theta(\mathbf{u})|^2 + \rho_0|\nabla^2 u_3|^2) dx\right) \\ &\qquad \cdot \left(\frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} (|\mathbf{e}^\theta(\mathbf{u})|^2 + \rho_0|\nabla^2 u_3|^2)^{-1/(p-1)} dx\right)^{p-1} \le B, \end{aligned} \tag{4.11}$$

where B depends on $A_0, A_1, A_2, \delta_0, \gamma_0, \bar{r}$, and

$$\|(\widehat{T}, \widehat{M})\|^2_{(L^2(\partial\Omega))^2 \times H^{1/2}(\partial\Omega)} / \|(\widehat{T}, \widehat{M})\|_{(H^{-1/2}(\partial\Omega))^3}.$$

Finally, the derivation of the upper bound of $|D|$ in Theorem 3.1 follows standard arguments based on Proposition 4.3. We refer to the proof in [6, Theorem 3.1] for more details.

We now prove Proposition 4.2. To this end, we first observe that the following holds.

LEMMA 4.4. *There exist positive constants $\tilde{C}_1, \tilde{C}_2$ depending on A_2, δ_0 such that*

$$\tilde{C}_1(|\mathbf{e}^\theta(\mathbf{u})|^2 + \rho_0^2|\nabla^2 u_3|^2) \le \rho_0^2|\nabla^2 U|^2 \le \tilde{C}_2(|\mathbf{e}^\theta(\mathbf{u})|^2 + \rho_0^2|\nabla^2 u_3|^2). \tag{4.12}$$

We also need the following Caccioppoli type estimate for U.

LEMMA 4.5. *Let $\rho_0 = 1$ in (4.8). Assume that $\lambda(x), \mu(x) \in L^\infty(B_\rho)$ satisfying (2.3) and there exists $K_3 > 0$ such that*

$$\|\lambda\|_{L^\infty(B_\rho)} + \|\mu\|_{L^\infty(B_\rho)} + \|\nabla\theta\|_{L^\infty(B_\rho)} \le K_3.$$

Let $U = (u_3, \psi) \in (H^1(B_\rho))^2 \times H^2(B_\rho)$ be a solution of (4.12) in B_ρ. Then there exists a constant $C > 0$ depending on δ_0, K_3 such that

$$\int_{B_{\rho/2}} |\nabla^2 U|^2 \le \frac{C}{\rho^4}\int_{B_\rho} |U|^2. \tag{4.13}$$

Proof. The proof of this lemma is adopted from [6]. Let $\eta \in C_0^4(B_\rho)$ with $0 \le \eta \le 1$ and $\eta \equiv 1$ on $B_{\rho/2}$ satisfying

$$\sum_{|\alpha|\le 3} \rho^{|\alpha|}|\partial^\alpha \eta| \le C_1 \quad \text{in} \quad B_\rho \tag{4.14}$$

for some positive constant C_1. Multiplying the first equation of (4.12) by $\eta^4\psi$ and the second equation of (4.12) by $\eta^4 u_3$ and performing integration by parts, we can obtain that

$$\begin{aligned}
&\int_{B_\rho} (\mathbb{L}\nabla^2\psi)\cdot\partial^2(\eta^4\psi) + \int_{B_\rho} (\mathbb{M}\nabla^2 u_3)\cdot\partial^2(\eta^4 u_3)\\
&\quad - \frac{1}{2}\int_{B_\rho} \big(\mathbb{R}(\nabla\theta\otimes\nabla u_3) + \mathbb{R}(\nabla\theta\otimes\nabla u_3)^T\big)\cdot\partial^2(\eta^4\psi)\\
&\quad + \int_{B_\rho} (\mathbb{R}\nabla^2\psi\nabla\theta)\cdot\nabla(\eta^4 u_3) = 0.
\end{aligned} \tag{4.15}$$

It is easy to see that

$$\begin{aligned}
&\int_{B_\rho} (\mathbb{L}\nabla^2\psi)\cdot\partial^2(\eta^4\psi)\\
&\quad = \int_{B_\rho} (\mathbb{L}\nabla^2\psi)\cdot(\eta^4\partial^2\psi + 2\nabla\psi\otimes\nabla\eta^4 + \psi\partial^2\eta^4)\\
&\quad = \frac{2\lambda+\mu}{4\mu(3\lambda+2\mu)}\int_{B_\rho} |\nabla^2\psi|^2 + \int_{B_\rho} (\mathbb{L}\nabla^2\psi)\cdot(2\nabla\psi\otimes\nabla\eta^4 + \psi\nabla^2\eta^4)\\
&\quad \ge C_2\int_{B_\rho} |\nabla^2\psi|^2 - \varepsilon_1\int_{B_\rho} |\nabla^2\psi|^2 - \frac{C_3}{\varepsilon_1\rho^4}\int_{B_\rho} |\psi|^2\\
&\qquad - \frac{C_3}{\varepsilon_1}\int_{B_\rho} \eta^2|\nabla\eta|^2|\nabla\psi|^2,
\end{aligned} \tag{4.16}$$

where ε_1 is a positive constant to be determined later. Similarly, we have

$$\begin{aligned}\int_{B_\rho} &(\mathbb{M}\nabla^2 u_3)\cdot\partial^2(\eta^4 u_3)\\ &\geq C_4\int_{B_\rho}|\nabla^2 u_3|^2-\varepsilon_1\int_{B_\rho}|\nabla^2\psi|^2-\frac{C_5}{\varepsilon_2\rho^4}\int_{B_\rho}|u_3|^2\\ &\quad-\frac{C_5}{\varepsilon_2}\int_{B_\rho}\eta^2|\nabla\eta|^2|\nabla u_3|^2. \end{aligned}\tag{4.17}$$

For the third term of (4.15), we can derive

$$\begin{aligned}&\left|-\frac{1}{2}\int_{B_\rho}\left(\mathbb{R}(\nabla\theta\otimes\nabla u_3)+\mathbb{R}(\nabla\theta\otimes\nabla u_3)^T\right)\cdot\nabla^2(\eta^4\psi)\right|\\ &\leq C_6\int_{B_\rho}\eta^2|\nabla\eta|^2|\nabla U|^2+\frac{C_6}{\rho^4}\int_{B_\rho}|\psi|^2.\end{aligned}\tag{4.18}$$

Likewise, it is not hard to see that

$$\left|\int_{B_\rho}(\mathbb{R}\nabla^2\psi\nabla\theta)\cdot\nabla(\eta^4 u_3)\right|\leq\varepsilon_3\int_{B_\rho}|\nabla^2\psi|^2+\frac{C_7}{\varepsilon_3\rho^4}\int_{B_\rho}|\psi|^2+\frac{C_7}{\varepsilon_3}\int_{B_\rho}\eta^2|\nabla\eta|^2|\nabla\psi|^2.\tag{4.19}$$

Using the same computations as on pp. 10–11 of [6], we obtain that

$$\int_{B_\rho}\eta^2|\nabla\eta|^2|\nabla U|^2\leq\frac{C_8}{\rho^4}\left(1+\frac{1}{\varepsilon^2}\right)\int_{B_\rho}|U|^2+\frac{\varepsilon^2}{2}\int_{B_\rho}\eta^4\sum_{ij}|\partial^2_{ij}U|^2.\tag{4.20}$$

Choosing $\varepsilon_1=\varepsilon_2=\varepsilon_3=\frac{1}{4}$ and then putting (4.15)–(4.20) together and taking ε sufficiently small, we immediately arrive at the desired estimate (4.13). ∎

We are ready to prove Proposition 4.2.

Proof of Proposition 4.2. It is enough to consider $\rho_0=1$. The general case follows from the scaling argument. It is important to notice that if we define $\tilde{\mathbf{u}}=(\tilde{u}',\tilde{u}_3)$ with

$$\begin{cases}\tilde{u}'=u'-\theta(a,b)^T,\\ \tilde{u}_3=u_3+ax_1+bx_2+c,\end{cases}$$

where $a,b,c\in\mathbb{R}$, then $\mathbf{e}^\theta(\mathbf{u})=\mathbf{e}^\theta(\tilde{\mathbf{u}})$. For any scalar or vector valued function f, we denote $(f)_r(x_0)=\frac{1}{|B_r|}\int_{B_r(x_0)}f(x)dx$. We now set

$$\tilde{U}(x;r)=U(x)-U_r(x_0)-(\nabla U)_r(x_0)\cdot(x-x_0).\tag{4.21}$$

Note that $(x-x_0)_r(x_0)=0$. From the observation above, $\tilde{U}$ combining with $\tilde{u}'=u'+\theta(\nabla u_3)_r$ satisfies (4.2). Hence, we obtain that $\tilde{U}$ solves (4.8) and satisfies

$$\int_{B_r(x_0)}\tilde{U}(x)dx=\int_{B_r(x_0)}\nabla\tilde{U}(x)dx=0.$$

Since $\tilde{U}$ is a solution to (4.8), by interior regularity estimates $\tilde{U}\in H^4_{loc}(\Omega)$. Using Caccippoli's type inequality (4.13), doubling inequality (4.9), and Poincaré inequality,

we have that for every $0 < r \leq \frac{\vartheta}{4}\bar{r}$,

(4.22)
$$\int_{B_{2r}(x_0)} |\nabla^2 U|^2 dx = \int_{B_{2r}(x_0)} |\nabla^2 \tilde{U}|^2 dx \leq \frac{C}{r^4} \int_{B_{4r}(x_0)} |\tilde{U}|^2 dx$$
$$\leq \frac{C}{r^4} \int_{B_r(x_0)} |\tilde{U}|^2 dx \leq C \int_{B_r(x_0)} |\nabla^2 \tilde{U}|^2 dx = C \int_{B_r(x_0)} |\nabla^2 U|^2 dx,$$

where C is a positive constant depending on A_0, A_1, A_2, δ_0 and the ratio $\frac{||\tilde{U}||_{H^{\frac{1}{2}}(\Omega)}}{||\tilde{U}||_{L^2(\Omega)}}$.

To finish the proof, we want to estimate the ratio $\frac{||\tilde{U}||_{H^{\frac{1}{2}}(\Omega)}}{||\tilde{U}||_{L^2(\Omega)}}$ by the boundary data following the lines of arguments in [6, p. 12]. By (4.21), we have that

$$\begin{cases} \displaystyle\int_\Omega |\tilde{U}|^2 dx \leq C \int_\Omega |U|^2 dx + C||U||_{L^\infty(B_r(x_0))} + C||\nabla U||_{L^\infty(B_r(x_0))}, \\ \displaystyle\int_\Omega |\nabla \tilde{U}|^2 dx \leq C \int_\Omega |\nabla U|^2 dx + C||\nabla U||_{L^\infty(B_r(x_0))}, \end{cases} \tag{4.23}$$

where C depends on the diameter of Ω. Using the Sobolev embedding theorem, interior estimates, and the Poincaré inequality with normalization conditions (2.7), (4.3), we can deduce that

$$\begin{cases} \displaystyle\int_\Omega |\tilde{U}|^2 dx \leq C \int_\Omega |\nabla^2 U|^2 dx, \\ \displaystyle\int_\Omega |\nabla \tilde{U}|^2 dx \leq C \int_\Omega |\nabla^2 U|^2 dx. \end{cases} \tag{4.24}$$

Combining (4.24) and the interpolation inequality, we obtain that

$$||\tilde{U}||_{H^{\frac{1}{2}}(\Omega)} \leq C||\tilde{U}||^{1/2}_{L^2(\Omega)}||\nabla \tilde{U}||^{1/2}_{H^1(\Omega)} \leq C \int_\Omega |\nabla^2 U|^2 dx, \tag{4.25}$$

where C depends on A_0, A_1, A_2. ∎

To get the lower bound of $||\tilde{U}||_{L^2(\Omega)}$, we note that Ω contains a ball of radius δ, where $\delta = (1 + \sqrt{1 + A_0^2})^{-1}$ centered at some point $\bar{x}$. Let $t = \delta/\chi$, where χ is the constant in Theorem 2.3. Using the interior estimate, Theorem 2.3, and (4.12), we have

$$\begin{aligned} \int_\Omega |\tilde{U}|^2 dx &\geq C \int_{B_t(\bar{x})} |\tilde{U}|^2 dx \geq Ct^4 \int_{B_{t/2}(\bar{x})} |\nabla^2 U|^2 dx \\ &\geq C \int_{B_{t/2}(\bar{x})} (|\mathbf{e}^\theta(\mathbf{u})|^2 + |\nabla^2 u_3|^2) dx \\ &\geq C \int_\Omega (|\mathbf{e}^\theta(\mathbf{u})|^2 + |\nabla^2 u_3|^2) dx \\ &\geq C \int_\Omega |\nabla^2 U|^2 dx, \end{aligned} \tag{4.26}$$

where C depends on $A_0, A_1, A_2, \delta_0, \gamma_0, \bar{r}$, and

$$\|(\widehat{T}, \widehat{M})\|^2_{(L^2(\partial\Omega))^2 \times H^{1/2}(\partial\Omega)} / \|(\widehat{T}, \widehat{M})\|_{(H^{-1/2}(\partial\Omega))^3}.$$

Finally, estimate (4.10) follows from (4.12), (4.22), (4.25), and (4.26).

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