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Questa è la Versione finale referata (Post print/Accepted manuscript) della seguente pubblicazione:

*Original Citation:*

ESTIMATING AREA OF INCLUSIONS IN ANISOTROPIC  
PLATES FROM BOUNDARY DATA / A. Morassi; E. Rosset; S. Vessella. - In: DISCRETE AND CONTINUOUS  
DYNAMICAL SYSTEMS. SERIES S. - ISSN 1937-1632. - STAMPA. - 6:(2013), pp. 501-515.  
[10.3934/dcdss.2013.6.501]

*Availability:*

This version is available at: 2158/813475 since: 2017-10-05T19:37:08Z

*Published version:*

DOI: 10.3934/dcdss.2013.6.501

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## ESTIMATING AREA OF INCLUSIONS IN ANISOTROPIC PLATES FROM BOUNDARY DATA

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**ABSTRACT.** We consider the inverse problem of determining the possible presence of an inclusion in a thin plate by boundary measurements. The plate is made by non-homogeneous linearly elastic material belonging to a general class of anisotropy. The inclusion is made by different elastic material. Under some a priori assumptions on the unknown inclusion, we prove constructive upper and lower estimates of the area of the unknown defect in terms of an easily expressed quantity related to work, which is given in terms of measurements of a couple field applied at the boundary and of the induced transversal displacement and its normal derivative taken at the boundary of the plate.

**1. Introduction.** In this paper we consider an inverse problem in linear elasticity consisting in the identification of an inclusion in a thin plate by boundary measurements. Let  $\Omega$  denote the middle plane of the plate and let  $h$  be its constant thickness. The inclusion  $D$  is modelled as a plane subdomain compactly contained in  $\Omega$ . Suppose we make the following diagnostic test. We take a *reference plate*, i.e. a plate without inclusion, and we deform it by applying a couple field  $\widehat{M}$  at its boundary. Let  $W_0$  be the work exerted in deforming the specimen. Now, we repeat the same experiment on a possibly defective plate. The exerted work generally changes and assumes, say, the value  $W$ . In this paper we want to find constructive estimates, from above and from below, of the *area* of the unknown inclusion  $D$  in terms of the difference  $|W - W_0|$ .

From the mathematical point of view, see [10], [11] the infinitesimal deformation of the defective plate is governed by the fourth order Neumann boundary value problem

$$\operatorname{div}(\operatorname{div}((\chi_{\Omega \setminus D} \mathbb{P} + \chi_D \widetilde{\mathbb{P}}) \nabla^2 w)) = 0, \quad \text{in } \Omega, \quad (1)$$

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2010 *Mathematics Subject Classification.* Primary: 35R30, 35R25, 74K20.

*Key words and phrases.* Inverse problems, elastic plates, inclusions, three spheres inequalities, size estimates, unique continuation.

The second and the third author are supported by MIUR, PRIN grant no. 20089PWTPS.001.

$$(\mathbb{P}\nabla^2 w)n \cdot n = -\widehat{M}_n, \quad \text{on } \partial\Omega, \quad (2)$$

$$\operatorname{div}(\mathbb{P}\nabla^2 w) \cdot n + ((\mathbb{P}\nabla^2 w)n \cdot \tau)_{,s} = (\widehat{M}_\tau)_{,s}, \quad \text{on } \partial\Omega, \quad (3)$$

where  $w$  is the transversal displacement of the plate and  $\widehat{M}_\tau, \widehat{M}_n$  are the twisting and bending components of the assigned couple field  $\widehat{M}$ , respectively. In the above equations  $\chi_D$  denotes the characteristic function of  $D$  and  $n, \tau$  are the unit outer normal and the unit tangent vector to  $\partial\Omega$ , respectively. The plate tensors  $\mathbb{P}, \widetilde{\mathbb{P}}$  are given by

$$\mathbb{P} = \frac{h^3}{12}\mathbb{C}, \quad \widetilde{\mathbb{P}} = \frac{h^3}{12}\widetilde{\mathbb{C}}, \quad (4)$$

where  $\mathbb{C}$  is the elasticity tensor describing the response of the material in the reference plate  $\Omega$ , whereas  $\widetilde{\mathbb{C}}$  denotes the (unknown) corresponding tensor for the inclusion  $D$ . The work exerted by the couple field  $\widehat{M}$  has the expression

$$W = - \int_{\partial\Omega} \widehat{M}_{\tau,s} w + \widehat{M}_n w_{,n}. \quad (5)$$

When the inclusion  $D$  is absent, the equilibrium problem (1)-(3) becomes

$$\operatorname{div}(\operatorname{div}(\mathbb{P}\nabla^2 w_0)) = 0, \quad \text{in } \Omega, \quad (6)$$

$$(\mathbb{P}\nabla^2 w_0)n \cdot n = -\widehat{M}_n, \quad \text{on } \partial\Omega, \quad (7)$$

$$\operatorname{div}(\mathbb{P}\nabla^2 w_0) \cdot n + ((\mathbb{P}\nabla^2 w_0)n \cdot \tau)_{,s} = (\widehat{M}_\tau)_{,s}, \quad \text{on } \partial\Omega, \quad (8)$$

where  $w_0$  is the transversal displacement of the reference plate. The corresponding external work exerted by  $\widehat{M}$  is given by

$$W_0 = - \int_{\partial\Omega} \widehat{M}_{\tau,s} w_0 + \widehat{M}_n w_{0,n}. \quad (9)$$

Our main result (see Theorem 3.1) states that if, for a given  $h_1 > 0$ , the following *fatness-condition*

$$\operatorname{area}(\{x \in D \mid \operatorname{dist}\{x, \partial D\} > h_1\}) \geq \frac{1}{2}\operatorname{area}(D) \quad (10)$$

holds, then

$$C_1 \left| \frac{W - W_0}{W_0} \right| \leq \operatorname{area}(D) \leq C_2 \left| \frac{W - W_0}{W_0} \right|, \quad (11)$$

where the constants  $C_1, C_2$  only depend on the a priori data. Estimates (11) are established under some suitable ellipticity and regularity assumptions on the plate tensor  $\mathbb{C}$  and on the jump  $\widetilde{\mathbb{C}} - \mathbb{C}$ .

Analogous bounds in plate theory for isotropic materials were obtained in [16] and [17] and recently in the context of shallow shells in [9]. The reader is referred to [13], [5], [7] for size estimates of inclusions in the context of the electrical impedance tomography and to [12], [2], [3], [4] for corresponding problems in two and three-dimensional linear elasticity. See also [14] for an application of the size estimates approach in thermography. However, differently from [16] and [17], here we work under very general assumptions on the constitutive properties of the reference plate, which is assumed to be made by nonhomogeneous *anisotropic* elastic material satisfying the *dichotomy condition* (28a)–(28b) only. This choice introduces significant difficulties in obtaining the upper bound for  $\operatorname{area}(D)$ , as we shall discuss shortly. In fact, for isotropic materials, an upper bound of the form

$$\operatorname{area}(D) \leq C_2 \left| \frac{W - W_0}{W_0} \right|^{\frac{1}{p}}, \quad (12)$$

where  $p > 1$  and  $C_2$  are constants only depending on the a priori data, was obtained in [17] for general measurable inclusions. The basic tool which allows to drop the fatness-condition (10) in the isotropic case is the validity of a doubling inequality, which, unfortunately, is not at disposal for anisotropic materials. On the other hand, in view of Alinhac's results [8], it seems hopeless even that the solutions to the plate equation can satisfy a strong unique continuation principle without any a priori assumptions on the anisotropy of the material. Concerning this point, our dichotomy condition basically contains the same assumptions under which the unique continuation property holds for a fourth order elliptic equation in two variables.

The first step of the proof of *area estimates* (11) consists in proving that the strain energy of the reference plate stored in the set  $D$  is comparable with the difference between the works exerted by the boundary couple fields in deforming the plate with and without the inclusion. More precisely, we have the following double inequality

$$K_1 \int_D |\nabla^2 w_0|^2 \leq |W - W_0| \leq K_2 \int_D |\nabla^2 w_0|^2, \quad (13)$$

for suitable constants  $K_1, K_2$  only depending on the a priori data (see Lemma 4.1). The proof of these bounds is based on variational considerations and has been obtained in [16] (Lemma 5.1).

The lower bound for  $area(D)$  follows from the right hand side of (13) and from regularity estimates for solutions to the fourth order elliptic equation (6) governing the equilibrium problem in the anisotropic case.

In order to obtain the upper bound for  $area(D)$  from the left hand side of (13), the next issue is to estimate from below  $\int_D |\nabla^2 w_0|^2$ . This task is rather technical and involves quantitative estimates of unique continuation in the form of three spheres inequalities for the hessian  $\nabla^2 w_0$  of the reference solution  $w_0$  to equation (6). It is exactly to this point that the dichotomy condition (28a)–(28b) on the tensor  $\mathbb{C}$  is needed. More precisely, it was shown in [19] that if  $\mathbb{C}$  satisfies the dichotomy condition, then the plate operator of equation (6) can be written as the sum of a product of two second order uniformly elliptic operators with regular coefficients and a third order operator with bounded coefficients. Then, Carleman estimates can be developed to derive a three spheres inequality for  $\nabla^2 w_0$  (see Theorem 6.2 of [19]). The reader is referred to the paper [19] for the necessary background.

The paper is organized as follows. Some basic notation is introduced in Section 2. In Section 3 we state the main result, Theorem 3.1, which is proved in Section 4. Section 5 is devoted to the proof of the *Lipschitz propagation of smallness property* (see Proposition 1), which is used in the proof of Theorem 3.1.

**2. Notation.** We shall denote by  $B_r(P)$  the disc in  $\mathbb{R}^2$  of radius  $r$  and center  $P$ . When representing locally a boundary as a graph, we use the following notation. For every  $x \in \mathbb{R}^2$  we set  $x = (x_1, x_2)$ , where  $x_1, x_2 \in \mathbb{R}$ .

**Definition 2.1.** ( $C^{k,1}$  regularity) Let  $\Omega$  be a bounded domain in  $\mathbb{R}^2$ . Given  $k$ , with  $k \in \mathbb{N}$ , we say that a portion  $S$  of  $\partial\Omega$  is of *class  $C^{k,1}$  with constants  $\rho_0, M_0 > 0$* , if, for any  $P \in S$ , there exists a rigid transformation of coordinates under which we have  $P = 0$  and

$$\Omega \cap B_{\rho_0}(0) = \{x = (x_1, x_2) \in B_{\rho_0}(0) \mid x_2 > \psi(x_1)\},$$

where  $\psi$  is a  $C^{k,1}$  function on  $(-\rho_0, \rho_0)$  satisfying

$$\begin{aligned}\psi(0) &= 0, \\ \nabla\psi(0) &= 0, \quad \text{when } k \geq 1, \\ \|\psi\|_{C^{k,1}(-\rho_0, \rho_0)} &\leq M_0\rho_0.\end{aligned}$$

When  $k = 0$ , we also say that  $S$  is of *Lipschitz class with constants*  $\rho_0, M_0$ .

**Remark 1.** We use the convention to normalize all norms in such a way that their terms are dimensionally homogeneous with their argument and coincide with the standard definition when the dimensional parameter equals one. For instance, given a function  $u : \Omega \mapsto \mathbb{R}$ , where  $\partial\Omega$  satisfies Definition 2.1, we denote

$$\|u\|_{C^{1,1}(\Omega)} = \|u\|_{L^\infty(\Omega)} + \rho_0\|\nabla u\|_{L^\infty(\Omega)} + \rho_0^2\|\nabla^2 u\|_{L^\infty(\Omega)},$$

and

$$\|u\|_{H^2(\Omega)} = \rho_0^{-1} \left( \int_{\Omega} u^2 + \rho_0^2 \int_{\Omega} |\nabla u|^2 + \rho_0^4 \int_{\Omega} |\nabla^2 u|^2 \right)^{\frac{1}{2}},$$

and so on for boundary and trace norms such as  $\|\cdot\|_{H^{\frac{1}{2}}(\partial\Omega)}, \|\cdot\|_{H^{-\frac{1}{2}}(\partial\Omega)}$ .

For any  $r > 0$  we denote

$$\Omega_r = \{x \in \Omega \mid \text{dist}(x, \partial\Omega) > r\}. \quad (14)$$

Given a bounded domain  $\Omega$  in  $\mathbb{R}^2$  such that  $\partial\Omega$  is of class  $C^{k,1}$ , with  $k \geq 1$ , we consider as positive the orientation of the boundary induced by the outer unit normal  $n$  in the following sense. Given a point  $P \in \partial\Omega$ , let us denote by  $\tau = \tau(P)$  the unit tangent at the boundary in  $P$  obtained by applying to  $n$  a counterclockwise rotation of angle  $\frac{\pi}{2}$ , that is

$$\tau = e_3 \times n, \quad (15)$$

where  $\times$  denotes the vector product in  $\mathbb{R}^3$  and  $\{e_1, e_2, e_3\}$  is the canonical basis in  $\mathbb{R}^3$ .

Given any connected component  $\mathcal{C}$  of  $\partial\Omega$  and fixed a point  $P_0 \in \mathcal{C}$ , let us define as positive the orientation of  $\mathcal{C}$  associated to an arclength parameterization  $\varphi(s) = (x_1(s), x_2(s))$ ,  $s \in [0, l(\mathcal{C})]$ , such that  $\varphi(0) = P_0$  and  $\varphi'(s) = \tau(\varphi(s))$ . Here  $l(\mathcal{C})$  denotes the length of  $\mathcal{C}$ .

Throughout the paper, we denote by  $w_{,i}$ ,  $w_{,s}$ , and  $w_{,n}$  the derivatives of a function  $w$  with respect to the  $x_i$  variable, to the arclength  $s$  and to the normal direction  $n$ , respectively, and similarly for higher order derivatives.

We denote by  $\mathbb{M}^2$  the space of  $2 \times 2$  real valued matrices and by  $\mathcal{L}(X, Y)$  the space of bounded linear operators between Banach spaces  $X$  and  $Y$ .

For every pair of real 2-vectors  $a$  and  $b$ , we denote by  $a \cdot b$  the scalar product of  $a$  and  $b$ . For every  $2 \times 2$  matrices  $A, B$  and for every  $\mathbb{L} \in \mathcal{L}(\mathbb{M}^2, \mathbb{M}^2)$ , we use the following notation:

$$(\mathbb{L}A)_{ij} = L_{ijkl}A_{kl}, \quad A \cdot B = A_{ij}B_{ij}, \quad |A| = (A \cdot A)^{\frac{1}{2}}, \quad (16)$$

$$A^{sym} = \frac{1}{2}(A + A^T), \quad (17)$$

where, here and in the sequel, summation over repeated indexes is implied.

Moreover we say that

$$\tilde{\mathbb{L}} \leq \mathbb{L}, \quad (18)$$

if and only if, for every  $2 \times 2$  symmetric matrix  $A$ ,

$$\tilde{\mathbb{L}}A \cdot A \leq \mathbb{L}A \cdot A. \tag{19}$$

**3. The main result.** Let us consider a thin plate  $\Omega \times [-\frac{h}{2}, \frac{h}{2}]$  with middle surface represented by a bounded domain  $\Omega$  in  $\mathbb{R}^2$  and having uniform thickness  $h$ ,  $h \ll \text{diam}(\Omega)$ . We assume that  $\partial\Omega$  is of class  $C^{1,1}$  with constants  $\rho_0$ ,  $M_0$  and that, for a given positive number  $M_1$ , satisfies

$$\text{area}(\Omega) \equiv |\Omega| \leq M_1 \rho_0^2. \tag{20}$$

We shall assume throughout that the elasticity tensor  $\mathbb{C}$  of the reference plate is known and has cartesian components  $C_{ijkl}$  which satisfy the following symmetry conditions

$$C_{ijkl}(x) = C_{klij}(x) = C_{lki j}(x), \quad i, j, k, l = 1, 2, \text{ a.e. in } \Omega. \tag{21}$$

On the elasticity tensor  $\mathbb{C}$  let us make the following assumptions:

i) *Ellipticity (strong convexity)*

There exists a positive constant  $\gamma$  such that

$$\mathbb{C}A \cdot A \geq \gamma|A|^2, \quad \text{a.e. in } \Omega, \tag{22}$$

for every  $2 \times 2$  symmetric matrix  $A$ .

ii)  *$C^{1,1}$  regularity*

There exists  $M > 0$  such that

$$\sum_{i,j,k,l=1}^2 \sum_{m=0}^2 \rho_0^m \|\nabla^m C_{ijkl}\|_{L^\infty(\mathbb{R}^2)} \leq M. \tag{23}$$

Condition (21) implies that instead of 16 coefficients we actually deal with 6 coefficients and we denote

$$\begin{cases} C_{1111} = A_0, & C_{1122} = C_{2211} = B_0, \\ C_{1112} = C_{1121} = C_{1211} = C_{2111} = C_0, \\ C_{2212} = C_{2221} = C_{1222} = C_{2122} = D_0, \\ C_{1212} = C_{1221} = C_{2112} = C_{2121} = E_0, \\ C_{2222} = F_0, \end{cases} \tag{24}$$

with

$$a_0 = A_0, \quad a_1 = 4C_0, \quad a_2 = 2B_0 + 4E_0, \quad a_3 = 4D_0, \quad a_4 = F_0. \tag{25}$$

Let  $S(x)$  be the following  $7 \times 7$  matrix

$$S(x) = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 & a_4 & 0 \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 & a_4 \\ 4a_0 & 3a_1 & 2a_2 & a_3 & 0 & 0 & 0 \\ 0 & 4a_0 & 3a_1 & 2a_2 & a_3 & 0 & 0 \\ 0 & 0 & 4a_0 & 3a_1 & 2a_2 & a_3 & 0 \\ 0 & 0 & 0 & 4a_0 & 3a_1 & 2a_2 & a_3 \end{pmatrix}, \tag{26}$$

and

$$\mathcal{D}(x) = \frac{1}{a_0} |\det S(x)|. \tag{27}$$

On the elasticity tensor  $\mathbb{C}$  we make the following additional assumption:

iii) *Dichotomy condition*

$$\text{either } \mathcal{D}(x) > 0, \quad \text{for every } x \in \mathbb{R}^2, \tag{28a}$$

$$\text{or } \mathcal{D}(x) = 0, \quad \text{for every } x \in \mathbb{R}^2, \tag{28b}$$

where  $\mathcal{D}(x)$  is defined by (27).

**Remark 2.** Whenever (28a) holds we denote

$$\mu = \min_{\mathbb{R}^2} \mathcal{D}. \tag{29}$$

We emphasize that, in all the following statements, whenever a constant is said to depend on  $\mu$  (among other quantities) it is understood that such dependence occurs *only* when (28a) holds.

Let  $D \times [-\frac{h}{2}, \frac{h}{2}]$  be a possible unknown inclusion in the plate, where  $D$  is a measurable, possibly disconnected subset of  $\Omega$  satisfying

$$\text{dist}(D, \partial\Omega) \geq d_0\rho_0, \tag{30}$$

for some positive constant  $d_0$ .

Concerning the material forming the inclusion, we assume that the corresponding elasticity tensor  $\tilde{\mathbb{C}} = \tilde{\mathbb{C}}(x)$  belongs to  $L^\infty(\Omega, \mathcal{L}(\mathbb{M}^2, \mathbb{M}^2))$  and has Cartesian components which satisfy the symmetry conditions

$$\tilde{C}_{ijkl}(x) = \tilde{C}_{klij}(x) = \tilde{C}_{lki j}(x), \quad i, j, k, l = 1, 2, \text{ a.e. in } \Omega. \tag{31}$$

Moreover, we assume the following *jump conditions* on  $\tilde{\mathbb{C}}$ : either there exist  $\eta_0 > 0$  and  $\eta_1 > 1$  such that

$$\eta_0\mathbb{C} \leq \tilde{\mathbb{C}} - \mathbb{C} \leq (\eta_1 - 1)\mathbb{C}, \quad \text{a.e. in } \Omega, \tag{32}$$

or there exist  $\eta_0 > 0$  and  $0 < \eta_1 < 1$  such that

$$-(1 - \eta_1)\mathbb{C} \leq \tilde{\mathbb{C}} - \mathbb{C} \leq -\eta_0\mathbb{C}, \quad \text{a.e. in } \Omega. \tag{33}$$

Let us assume that the body forces inside the plate are absent and that a couple field  $\widehat{M}$  is acting on the boundary of  $\Omega$ . We shall assume:

$$\widehat{M} \in L^2(\partial\Omega, \mathbb{R}^2), \tag{34}$$

$$\text{supp}(\widehat{M}) \subset \Gamma, \tag{35}$$

where  $\Gamma$  is an open subarc of  $\partial\Omega$ , such that

$$|\Gamma| \leq (1 - \delta_0)|\partial\Omega|, \tag{36}$$

for some positive constant  $\delta_0$ . Moreover, we obviously assume the *compatibility conditions* on the boundary couple field  $\widehat{M}$

$$\int_{\partial\Omega} \widehat{M}_\alpha = 0, \quad \alpha = 1, 2, \tag{37}$$

and that, for a given constant  $F > 0$ ,

$$\frac{\|\widehat{M}\|_{L^2(\partial\Omega, \mathbb{R}^2)}}{\|\widehat{M}\|_{H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2)}} \leq F. \tag{38}$$

Let us notice that, following a standard convention in the theory of plates, we represent the boundary couple field  $\widehat{M}$  in cartesian coordinates as

$$\widehat{M} = \widehat{M}_2 e_1 + \widehat{M}_1 e_2, \quad \text{on } \partial\Omega. \tag{39}$$

The equilibrium problem of the plate with and without inclusion is described by the Neumann problem (1)-(3) and (6)-(8), respectively. Under the above assumptions, the problems (1)-(3) and (6)-(8) have solutions  $w \in H^2(\Omega)$ ,  $w_0 \in H^2(\Omega)$ , respectively. These solutions are uniquely determined by imposing the normalization conditions

$$\int_{\Omega} w = 0, \quad \int_{\Omega} w_{,\alpha} = 0, \quad \alpha = 1, 2, \tag{40}$$

$$\int_{\Omega} w_0 = 0, \quad \int_{\Omega} w_{0,\alpha} = 0, \quad \alpha = 1, 2. \tag{41}$$

We recall that the quantities  $W, W_0$  defined by (5), (9) represent the work exerted by the boundary value couple field  $\widehat{M}$  when the inclusion  $D$  is present or absent, respectively. By the weak formulation of problems (1)-(3) and (6)-(8), the works  $W$  and  $W_0$  coincide with the strain energies stored in the plate, namely

$$W = \int_{\Omega} (\chi_{\Omega \setminus D} \mathbb{P} + \chi_D \widetilde{\mathbb{P}}) \nabla^2 w \cdot \nabla^2 w, \tag{42}$$

$$W_0 = \int_{\Omega} \mathbb{P} \nabla^2 w_0 \cdot \nabla^2 w_0. \tag{43}$$

We are now in position to state the main result of this paper.

**Theorem 3.1.** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^2$ , such that  $\partial\Omega$  is of class  $C^{2,1}$  with constants  $\rho_0, M_0$  and satisfying (20). Let  $D$  be a measurable subset of  $\Omega$  satisfying (30) and*

$$|D_{h_1 \rho_0}| \geq \frac{1}{2} |D|, \tag{44}$$

for a given positive constant  $h_1$ . Let  $\mathbb{P}$  given by (4) satisfy (21), (22), (23) and the dichotomy condition (28a)-(28b). Let  $\widetilde{\mathbb{P}} \in L^\infty(\Omega, \mathcal{L}(\mathbb{M}^2, \mathbb{M}^2))$ , defined by (4), satisfy (31). Let  $\widehat{M} \in L^2(\partial\Omega, \mathbb{R}^2)$  satisfy (35)-(38). If (32) holds, then we have

$$\frac{1}{\eta_1 - 1} C_1^+ \rho_0^2 \frac{W_0 - W}{W_0} \leq |D| \leq \frac{\eta_1}{\eta_0} C_2^+ \rho_0^2 \frac{W_0 - W}{W_0}. \tag{45}$$

If, conversely, (33) holds, then we have

$$\frac{\eta_1}{1 - \eta_1} C_1^- \rho_0^2 \frac{W - W_0}{W_0} \leq |D| \leq \frac{1}{\eta_0} C_2^- \rho_0^2 \frac{W - W_0}{W_0}, \tag{46}$$

where  $C_1^+, C_1^-$  only depend on  $h, M_0, M_1, d_0, \gamma, \mu, M$ , whereas  $C_2^+, C_2^-$  only depend on the same quantities and also on  $\delta_0, h_1$  and  $F$ .

**4. Proof of Theorem 3.1.** The proof of Theorem 3.1 is mainly based on the following key ingredients: energy estimates for the equilibrium problems (1)-(3) and (6)-(8) (Lemma 4.1) and an estimate of continuation from the interior for solutions to the Neumann problem (6)-(8) (Proposition 1).

**Lemma 4.1.** *Let the fourth-order tensor fields  $\mathbb{P}, \widetilde{\mathbb{P}} \in L^\infty(\Omega, \mathcal{L}(\mathbb{M}^2, \mathbb{M}^2))$  given by (4), satisfy the symmetry conditions (21) and (31), respectively. Let  $\widehat{M} \in H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2)$  satisfy (37). Let  $\xi_0, \xi_1, 0 < \xi_0 < \xi_1$ , be such that*

$$\xi_0 |A|^2 \leq \mathbb{P}(x) A \cdot A \leq \xi_1 |A|^2, \quad \text{for a.e. } x \in \Omega, \tag{47}$$

for every symmetric matrix  $A \in \mathbb{M}^2$ , and let the jump  $(\widetilde{\mathbb{P}}(x) - \mathbb{P}(x))$  satisfy either (32) or (33). Let  $w, w_0 \in H^2(\Omega)$  be the weak solutions to the problems (1)-(3), (6)-(8) respectively.

If (32) holds, then we have

$$\frac{\eta_0 \xi_0}{\eta_1} \int_D |\nabla^2 w_0|^2 \leq W_0 - W \leq (\eta_1 - 1) \xi_1 \int_D |\nabla^2 w_0|^2. \tag{48}$$

If, instead, (33) holds, then we have

$$\eta_0 \xi_0 \int_D |\nabla^2 w_0|^2 \leq W - W_0 \leq \frac{1 - \eta_1}{\eta_1} \xi_1 \int_D |\nabla^2 w_0|^2. \tag{49}$$

The proof of the above lemma is given in [16], Lemma 5.1.

**Proposition 1 (Lipschitz propagation of smallness).** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^2$ , such that  $\partial\Omega$  is of class  $C^{2,1}$  with constants  $\rho_0, M_0$  and satisfying (20). Let the fourth order tensor  $\mathbb{P}$  be defined by (4) and satisfying (21), (22), (23) and the dichotomy condition (28a)–(28b). Let  $w_0 \in H^2(\Omega)$  be the unique weak solution of the problem (6)–(8) satisfying (41), with  $\widehat{M} \in L^2(\partial\Omega, \mathbb{R}^2)$  satisfying (35)–(38). There exists  $s > 1$ , only depending on  $\gamma, M, \mu, M_0$  and  $\delta_0$ , such that for every  $\rho > 0$  and every  $\bar{x} \in \Omega_{s\rho}$ , we have*

$$\int_{B_\rho(\bar{x})} |\nabla^2 w_0|^2 \geq \frac{C}{\exp \left[ A \left( \frac{\rho_0}{\rho} \right)^B \right]} \int_\Omega |\nabla^2 w_0|^2, \tag{50}$$

where  $A > 0, B > 0$  and  $C > 0$  only depend on  $h, M_0, M_1, \gamma, \mu, M, \delta_0$  and  $F$ .

**Remark 3.** i) In the above estimate we compute precisely the dependence on the radius  $\rho$  of the constant on the right hand side. It is worth to notice that the exponential character of this dependence is essentially optimal, in view of the character of the unique continuation property described in [19, Corollary 5.4].

ii) The main idea underlying the proof of the above proposition is to propagate smallness from the disc  $B_\rho(\bar{x})$ , by an iterated application of the three spheres inequality over chains of discs. However, since the estimate deteriorates as the number of iterations increases, in order to limit the number of discs in the chains, instead of choosing discs with uniform radius, we take advantage of the regularity of the boundary, by choosing discs with varying radii, tangent to suitable cones near the boundary.

*Proof of Theorem 3.1.* By the hypotheses made on  $\mathbb{P}$ , the inequality (47) is satisfied with  $\xi_0 = \gamma \frac{h^3}{12}, \xi_1 = \frac{h^3}{6} M$ , so that Lemma 4.1 can be applied.

By standard interior regularity estimates (see, for instance, Theorem 8.3 in [16]) and by the Sobolev embedding theorem, we have

$$\|\nabla^2 w_0\|_{L^\infty(D)} \leq \frac{C}{\rho_0^2} \|w_0\|_{H^2(\Omega)}, \tag{51}$$

with  $C$  only depending on  $\gamma, h, M$  and  $d_0$ .

From (51), Poincaré inequality, (47), (43), we have

$$\|\nabla^2 w_0\|_{L^\infty(D)} \leq \frac{C}{\rho_0} W_0^{\frac{1}{2}}, \tag{52}$$

where the constant  $C$  only depends on  $\gamma, h, M, d_0, M_0$  and  $M_1$ .

The lower bound for  $|D|$  in (45), (46) follows from the right hand side of (48), (49) and from (52).

Next, let us prove the upper bound for  $|D|$  in (45), (46).

Let  $\epsilon = \min\{\frac{2d_0}{s}, \frac{h_1}{\sqrt{2}}\}$ , where  $s$  is as in Proposition 1. Let us cover  $D_{h_1\rho_0}$  with internally non overlapping closed squares  $Q_l$  of side  $\epsilon\rho_0$ , for  $l = 1, \dots, L$ . By the choice of  $\epsilon$  the squares  $Q_l$  are contained in  $D$ . Let  $\bar{l}$  be such that  $\int_{Q_{\bar{l}}} |\nabla^2 w_0|^2 = \min_l \int_{Q_l} |\nabla^2 w_0|^2$ . Noticing that  $|D_{h_1\rho_0}| \leq L\epsilon^2\rho_0^2$ , we have

$$\int_D |\nabla^2 w_0|^2 \geq \int_{\bigcup_{l=1}^L Q_l} |\nabla^2 w_0|^2 \geq L \int_{Q_{\bar{l}}} |\nabla^2 w_0|^2 \geq \frac{|D_{h_1\rho_0}|}{\rho_0^2 \epsilon^2} \int_{Q_{\bar{l}}} |\nabla^2 w_0|^2. \quad (53)$$

Let  $\bar{x}$  be the center of  $Q_{\bar{l}}$ . From (47), (53), estimate (50) with  $\rho = \frac{\epsilon}{2}\rho_0$ , from (43) and by our hypothesis (44) we have

$$\int_D |\nabla^2 w_0|^2 \geq \frac{K|D|}{\rho_0^2} W_0, \quad (54)$$

where  $K$  is a positive constant only depending on  $\gamma, h, M, M_0, M_1, d_0, \delta_0, h_1$  and  $F$ . The upper bound for  $|D|$  in (45), (46) follows from the left hand side of (48), (49) and from (54).  $\square$

**5. Proof of Proposition 1.** Let us premise the following Lemmas.

**Proposition 2** (Three Spheres Inequality). *Let  $\Omega$  be a domain in  $\mathbb{R}^2$ , and let the plate tensor  $\mathbb{P}$  given by (4) satisfies (21), (22), (23) and the dichotomy condition (28a)–(28b). Let  $u \in H^2(\Omega)$  be a weak solution to the equation*

$$\operatorname{div}(\operatorname{div}(\mathbb{P}\nabla^2 u)) = 0, \quad \text{in } \Omega. \quad (55)$$

For every  $r_1, r_2, r_3, \bar{r}, 0 < r_1 < r_2 < r_3 \leq \bar{r}$ , and for every  $x \in \Omega_{\bar{r}}$  we have

$$\int_{B_{r_2}(x)} |\nabla^2 u|^2 \leq C \left( \int_{B_{r_1}(x)} |\nabla^2 u|^2 \right)^\delta \left( \int_{B_{r_3}(x)} |\nabla^2 u|^2 \right)^{1-\delta}, \quad (56)$$

where  $C > 0$  and  $\delta, 0 < \delta < 1$ , only depend on  $\gamma, M, \mu, \frac{r_3}{r_2}$  and  $\frac{r_3}{r_1}$ .

A proof of the above proposition can be easily obtained by Theorem 6.5 in [19].

In order to prove Proposition 1, we need the estimate stated in the following Lemma (for the proof see [16], Lemma 7.1).

**Lemma 5.1.** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^2$ , such that  $\partial\Omega$  is of class  $C^{2,1}$  with constants  $\rho_0, M_0$ . Let the fourth order tensor  $\mathbb{P}$  be defined by (4) and satisfying (21), (22) and (23). Let  $w_0 \in H^2(\Omega)$  be the unique weak solution of the problem (6)–(8) satisfying (41), with  $\widehat{M} \in H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2)$  satisfying (35)–(37). We have*

$$\|\widehat{M}\|_{H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2)} \leq C \|\nabla^2 w_0\|_{L^2(\Omega)}, \quad (57)$$

where  $C$  is a positive constant only depending on  $M_0, M_1, \delta_0$  and  $M$ .

**Lemma 5.2.** *Let the hypotheses of Proposition 1 be satisfied. There exists  $\tilde{\rho} > 0$ , only depending on  $M_0, M_1, \delta_0, \gamma, M, \mu$  and  $F$ , such that for every  $r \leq \tilde{\rho}$  we have*

$$\frac{\int_{\Omega_r} |\nabla^2 w_0|^2}{\int_{\Omega} |\nabla^2 w_0|^2} \geq \frac{1}{2}. \quad (58)$$

*Proof.* Let us set

$$\frac{\int_{\Omega_r} |\nabla^2 w_0|^2}{\int_{\Omega} |\nabla^2 w_0|^2} = 1 - \frac{\int_{\Omega \setminus \Omega_r} |\nabla^2 w_0|^2}{\int_{\Omega} |\nabla^2 w_0|^2}. \quad (59)$$

By Hölder inequality

$$\|\nabla^2 w_0\|_{L^2(\Omega \setminus \Omega_r)}^2 \leq |\Omega \setminus \Omega_r|^{\frac{1}{2}} \|\nabla^2 w_0\|_{L^4(\Omega \setminus \Omega_r)}^2, \quad (60)$$

and by Sobolev inequality [1]

$$\|\nabla^2 w_0\|_{L^4(\Omega)}^2 \leq C \|\nabla^2 w_0\|_{H^{\frac{1}{2}}(\Omega)}^2, \quad (61)$$

we have

$$\|\nabla^2 w_0\|_{L^2(\Omega \setminus \Omega_r)}^2 \leq \frac{C}{\rho_0^4} |\Omega \setminus \Omega_r|^{\frac{1}{2}} \|w_0\|_{H^{\frac{5}{2}}(\Omega)}^2, \quad (62)$$

where  $C$  only depends on  $M_0, M_1$ . We recall that, by the variational formulation of the problem (6)–(8), the function  $w_0$  satisfies

$$\|w_0\|_{H^2(\Omega)} \leq C \rho_0^2 \|\widehat{M}\|_{H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2)}, \quad (63)$$

where  $C > 0$  only depends on  $h, M_0, M_1$  and  $\gamma$ . Now, by using the following regularity estimate (see [18] for a proof)

$$\|w_0\|_{H^3(\Omega)} \leq C \rho_0^2 \|\widehat{M}\|_{H^{\frac{1}{2}}(\partial\Omega, \mathbb{R}^2)}, \quad (64)$$

where  $C > 0$  only depends on  $h, M_0, M_1, \gamma$  and  $M$ . By interpolating (63) and (64), we get

$$\|w_0\|_{H^{\frac{5}{2}}(\Omega)} \leq C \rho_0^2 \|\widehat{M}\|_{L^2(\partial\Omega, \mathbb{R}^2)}, \quad (65)$$

where  $C$  only depends on  $h, M_0, M_1, \gamma$  and  $M$ .

Moreover

$$|\Omega \setminus \Omega_r| \leq Cr, \quad (66)$$

with  $C$  only depending on  $M_0$  and  $M_1$ , see for details (A.3) in [5]. From (62), (65) and (66) we have

$$\int_{\Omega \setminus \Omega_r} |\nabla^2 w_0|^2 \leq C \rho_0^2 r^{\frac{1}{2}} \|\widehat{M}\|_{L^2(\partial\Omega, \mathbb{R}^2)}, \quad (67)$$

where  $C$  only depends on  $M_0, M_1, \gamma, M$ . Finally, by (59), (67) and (57) we obtain (58).  $\square$

*Proof of Proposition 1.* It is not restrictive to assume  $\rho_0 = 1$ .

Set

$$\begin{aligned} \vartheta_0 &= \arctan \frac{1}{M_0}, \\ s &= \frac{5 + \sin \vartheta_0 + \sqrt{\sin^2 \vartheta_0 + 30 \sin \vartheta_0 + 25}}{2 \sin \vartheta_0}, \\ \chi &= \frac{s \sin \vartheta_0}{5} = \frac{5 + \sin \vartheta_0 + \sqrt{\sin^2 \vartheta_0 + 30 \sin \vartheta_0 + 25}}{10}, \\ \vartheta_1 &= \arcsin \frac{1}{s}. \end{aligned}$$

Let us notice that  $s > 1, \chi > 1$  and  $\vartheta_1 > 0$  only depend on  $M_0$ .

Given  $z \in \mathbb{R}^2, \xi \in \mathbb{R}^2, |\xi| = 1, \vartheta > 0$ , we shall denote by

$$C(z, \xi, \vartheta) = \{x \in \mathbb{R}^2 \text{ s. t. } \frac{(x-z) \cdot \xi}{|x-z|} > \cos \vartheta\}, \quad (68)$$

the open cone having vertex  $z$ , axis in the direction  $\xi$  and width  $2\vartheta$ .

**Step 1** For every  $\rho, 0 < \rho \leq \rho_1 = \frac{1}{16s}$ , and for every  $x \in \Omega$  satisfying  $s\rho < \text{dist}(x, \partial\Omega) \leq \frac{1}{4}$ , there exists  $\tilde{x} \in \Omega$  such that

- i)  $B_{5\chi\rho}(x) \subset C(\tilde{x}, \frac{x-\tilde{x}}{|x-\tilde{x}|}, \vartheta_0) \cap B_{\frac{1}{8}}(\tilde{x}) \subset \Omega$ ,
- ii) the discs  $B_\rho(x)$  and  $B_{\chi\rho}(x_2)$  are internally tangent to  $C(\tilde{x}, \frac{x-\tilde{x}}{|x-\tilde{x}|}, \vartheta_1)$ , where  $x_2 = x + (\chi + 1)\rho \frac{x-\tilde{x}}{|x-\tilde{x}|}$ .

The proof of this step has merely geometrical character and has been given in [15], Proof of Proposition 3.1. Up to a rigid motion, we may assume that  $\frac{x-\tilde{x}}{|x-\tilde{x}|} = e_2$ , where  $(e_1, e_2)$  is the canonical basis of  $\mathbb{R}^2$

Set

$$\begin{aligned} r_1 &= \rho, & r_k &= \chi r_{k-1} = \chi^{k-1} \rho, & k &\geq 2, \\ x_1 &= x, & x_k &= x_{k-1} + (r_{k-1} + r_k)e_2, & k &\geq 2. \end{aligned}$$

For every  $k \in \mathbb{N}$ ,  $B_{r_k}(x_k)$  is internally tangent to the cone  $C(\tilde{x}, e_2, \vartheta_1)$  and  $B_{5\chi r_k}(x_k)$  is internally tangent to the cone  $C(\tilde{x}, e_2, \vartheta_0)$ . Moreover, we have that  $B_{5r_k}(x_k) \subset B_{\frac{1}{8}}(\tilde{x})$  if and only if

$$k - 1 \leq \frac{\log \left\{ \frac{\chi-1}{6\chi-4} \left( \frac{1}{8\rho} - s + 1 + \frac{2}{\chi-1} \right) \right\}}{\log \chi}. \tag{69}$$

In order to ensure that  $B_{5r_k}(x_k) \subset B_{\frac{1}{8}}(\tilde{x})$  holds at least for  $k = 1, 2$ , let us assume also that  $\rho \leq \rho_2 = \frac{1}{8(6\chi+s+1)}$ . Let us define

$$k(\rho) = \left\lceil \frac{\log \left\{ \frac{\chi-1}{6\chi-4} \left( \frac{h_0}{8\rho} - s + 1 + \frac{2}{\chi-1} \right) \right\}}{\log \chi} \right\rceil + 1, \tag{70}$$

where  $h_0, 0 < h_0 < 1$ , only depending on  $M_0$ , is such that  $\Omega_h$  is connected for every  $h < h_0$  (see Prop. 5.5 in [6]) and  $\lceil \cdot \rceil$  denotes the integer part of a real number. We have that  $B_{5r_{k(\rho)}}(x_{k(\rho)}) \subset B_{\frac{1}{8}}(\tilde{x}) \cap \Omega$  and  $B_{5\chi r_j}(x_j) \subset B_{\frac{1}{8}}(\tilde{x}) \cap \Omega$  for every  $j = 1, \dots, k(\rho) - 1$ .

Moreover let  $\rho \leq \rho_3 = \frac{h_0}{16s}$ . We have

$$k(\rho) \geq \frac{\log \frac{\tau}{\rho}}{\log \chi}, \tag{71}$$

with  $\tau = \frac{(\chi-1)h_0}{16(6\chi-4)}$ . Assuming also that  $\rho \leq \rho_4 = \frac{(\chi-1)h_0}{16}$ , and noticing that  $\frac{\chi-1}{6\chi-4} \leq \frac{1}{5}$ , we have

$$k(\rho) \leq \frac{\log \frac{h_0}{20\rho}}{\log \chi} + 1. \tag{72}$$

From (71) and (72), it follows that, for  $\rho \leq \bar{\rho} = \min\{\rho_1, \rho_2, \rho_3, \rho_4\}$ ,

$$\frac{\tau}{\chi} \leq r_{k(\rho)} = \chi^{k(\rho)-1} \rho \leq \frac{h_0}{20}. \tag{73}$$

**Step 2** There exists  $\bar{\rho} > 0$ , only depending on  $\gamma, M, \mu$  and  $M_0$ , such that for every  $\rho, 0 < \rho \leq \bar{\rho}$ , and for every  $x \in \Omega$  such that  $s\rho < \text{dist}(x, \partial\Omega) \leq \frac{1}{4}$ ,

$$\frac{\int_{B_{r_{k(\rho)}}(x_{k(\rho)})} |\nabla^2 w_0|^2}{\int_{\Omega} |\nabla^2 w_0|^2} \leq C \left( \frac{\int_{B_\rho(x)} |\nabla^2 w_0|^2}{\int_{\Omega} |\nabla^2 w_0|^2} \right)^{\delta_x^{k(\rho)-1}}, \tag{74}$$

$$\frac{\int_{B_\rho(x)} |\nabla^2 w_0|^2}{\int_{\Omega} |\nabla^2 w_0|^2} \leq C \left( \frac{\int_{B_{r_{k(\rho)}}(x_{k(\rho)})} |\nabla^2 w_0|^2}{\int_{\Omega} |\nabla^2 w_0|^2} \right)^{\delta_x^{k(\rho)-1}}, \tag{75}$$

where  $C > 1$ ,  $\delta \in (0, 1)$ , only depend on  $\gamma$ ,  $M$  and  $\mu$  whereas  $\delta_\chi \in (0, 1)$ , only depends on  $\gamma$ ,  $M$ ,  $\mu$  and  $M_0$ .

*Proof of Step 2.* Let  $\rho \leq \bar{\rho} = \min\{\rho_1, \rho_2, \rho_3, \rho_4\}$ . Let us apply the three spheres inequality (56) to the discs of center  $x_j$  and radii  $r_j, 3\chi r_j, 4\chi r_j$ , for  $j = 1, \dots, k(\rho) - 1$ . Since  $B_{r_{j+1}}(x_{j+1}) \subset B_{3\chi r_j}(x_j)$ , for  $j = 1, \dots, k(\rho) - 1$ , we have

$$\int_{B_{r_{j+1}}(x_{j+1})} |\nabla^2 w_0|^2 \leq C \left( \int_{B_{r_j}(x_j)} |\nabla^2 w_0|^2 \right)^{\delta_\chi} \left( \int_{B_{4\chi r_j}(x_j)} |\nabla^2 w_0|^2 \right)^{1-\delta_\chi}, \tag{76}$$

with  $C > 1$  and  $\delta_\chi, 0 < \delta_\chi < 1$ , only depending on  $\gamma$ ,  $M$ ,  $\mu$  and  $M_0$  which we may rewrite as

$$\frac{\int_{B_{r_{j+1}}(x_{j+1})} |\nabla^2 w_0|^2}{\int_\Omega |\nabla^2 w_0|^2} \leq C \left( \frac{\int_{B_{r_j}(x_j)} |\nabla^2 w_0|^2}{\int_\Omega |\nabla^2 w_0|^2} \right)^{\delta_\chi}. \tag{77}$$

By iterating (77) over  $j = 1, \dots, k(\rho) - 1$ , (74) follows. Similarly, by applying the three spheres inequality to the discs  $B_{r_j}(x_j), B_{3r_j}(x_j), B_{4r_j}(x_j)$  for  $j = 2, \dots, k(\rho)$  and noticing that  $B_{r_j}(x_{j-1}) \subset B_{3r_j}(x_j)$  we can repeat the above argument obtaining (75).  $\square$

**Step 3** There exists  $\rho^*$ , only depending on  $\gamma$ ,  $M$ ,  $\mu$ ,  $M_0$ ,  $M_1$ ,  $\delta_0$  and  $F$ , such that for every  $\rho \leq \rho^*$  and for every  $\bar{x} \in \Omega_{s\rho}$  we have

$$\frac{\int_{B_\rho(y)} |\nabla^2 w_0|^2}{\int_\Omega |\nabla^2 w_0|^2} \leq C \left( \frac{\int_{B_\rho(\bar{x})} |\nabla^2 w_0|^2}{\int_\Omega |\nabla^2 w_0|^2} \right)^{\delta_\chi^{A_1 + B_1 \log \frac{1}{\rho}}}, \quad \forall y \in \Omega_{s\rho}, \tag{78}$$

where  $C > 1$ ,  $B_1$  only depends on  $\gamma$ ,  $M$ ,  $\mu$  and  $M_0$ , whereas  $A_1$  only depends on  $\gamma$ ,  $M$ ,  $\mu$ ,  $M_0$  and  $M_1$ .

*Proof of Step 3.* First we consider the case  $\bar{x} \in \Omega_{s\rho}$  satisfying  $\text{dist}(\bar{x}, \partial\Omega) \leq \frac{1}{4}$ . Let us take  $\rho \leq \bar{\rho}$ . Since, by (73),  $5r_{k(\rho)} \leq \frac{h_0}{4}$ , it follows that  $\Omega_{5r_{k(\rho)}}$  is connected.

Let  $y \in \Omega$  such that  $s\rho < \text{dist}(y, \partial\Omega) \leq \frac{h_0}{4}$  and let  $\sigma$  be an arc in  $\Omega_{5r_{k(\rho)}}$  joining  $\bar{x}_{k(\rho)}$  to  $y_{k(\rho)}$ . Let us define  $\{x_i\}$ ,  $i = 1, \dots, L$ , as follows:  $x_1 = \bar{x}_{k(\rho)}$ ,  $x_{i+1} = \sigma(t_i)$ , where  $t_i = \max\{t \text{ s. t. } |\sigma(t) - x_i| = 2r_{k(\rho)}\}$  if  $|x_i - y_{k(\rho)}| > 2r_{k(\rho)}$ , otherwise let  $i = L$  and stop the process. By construction, the discs  $B_{r_{k(\rho)}}(x_i)$  are pairwise disjoint,  $|x_{i+1} - x_i| = 2r_{k(\rho)}$ , for  $i = 1, \dots, L - 1$ ,  $|x_L - y_{k(\rho)}| \leq 2r_{k(\rho)}$ . Hence we have

$$L \leq \frac{M_1}{\pi r_{k(\rho)}^2}. \tag{79}$$

By an iterated application of the three spheres inequality (56) over the discs of center  $x_i$  and radii  $r_{k(\rho)}, 3r_{k(\rho)}, 4r_{k(\rho)}$ , we obtain

$$\frac{\int_{B_{r_{k(\rho)}}(y_{k(\rho)})} |\nabla^2 w_0|^2}{\int_\Omega |\nabla^2 w_0|^2} \leq C \left( \frac{\int_{B_{r_{k(\rho)}}(\bar{x}_{k(\rho)})} |\nabla^2 w_0|^2}{\int_\Omega |\nabla^2 w_0|^2} \right)^{\delta^L}, \tag{80}$$

where  $C > 1$  only depends on  $\gamma$ ,  $M$  and  $\mu$ .

By applying (74) for  $x = \bar{x}$  and (75) for  $x = y$ , we have

$$\frac{\int_{B_\rho(y)} |\nabla^2 w_0|^2}{\int_\Omega |\nabla^2 w_0|^2} \leq C \left( \frac{\int_{B_\rho(\bar{x})} |\nabla^2 w_0|^2}{\int_\Omega |\nabla^2 w_0|^2} \right)^{\delta_\chi^{k(\rho)-1} \delta^{k(\rho)+L-1}}, \tag{81}$$

where  $C > 1$  only depends on  $\gamma, M, \mu$  and  $M_0$ .

The above estimate holds for every  $y \in \Omega$  satisfying  $s\rho < \text{dist}(y, \partial\Omega) \leq \frac{h_0}{4}$ . Next, let  $y \in \Omega$  satisfying  $\text{dist}(y, \partial\Omega) > \frac{h_0}{4}$ . Since  $B_{5r_{k(\rho)}}(\bar{x}_{k(\rho)}) \subset B_{\frac{1}{8}}(\tilde{x}) \subset \Omega$  we have

$$\text{dist}(\bar{x}_{k(\rho)}, \partial\Omega) \geq 5r_{k(\rho)}, \tag{82}$$

and by (73),

$$\text{dist}(y, \partial\Omega) > \frac{h_0}{4} \geq 5r_{k(\rho)}. \tag{83}$$

Recalling that  $\Omega_{5r_{k(\rho)}}$  is connected, we can consider an arc in  $\Omega_{5r_{k(\rho)}}$  joining  $\bar{x}_{k(\rho)}$  to  $y$  and mimic the arguments just seen above over a chain of  $\tilde{L}$  discs of center  $x_j \in \Omega_{5r_{k(\rho)}}$  and radii  $r_{k(\rho)}, 3r_{k(\rho)}, 4r_{k(\rho)}$ , where

$$\tilde{L} \leq \frac{M_1}{\pi r_{k(\rho)}^2}. \tag{84}$$

By an iterated application of the three spheres inequality and by applying (74) for  $x = \bar{x}$  we have

$$\frac{\int_{B_\rho(y)} |\nabla^2 w_0|^2}{\int_\Omega |\nabla^2 w_0|^2} \leq C \left( \frac{\int_{B_\rho(\bar{x})} |\nabla^2 w_0|^2}{\int_\Omega |\nabla^2 w_0|^2} \right)^{\delta_\chi^{k(\rho)-1} \delta^{\tilde{L}}}, \tag{85}$$

where  $C > 1$  only depends on  $\gamma, M, \mu$  and  $M_0$ . By (85), (81), (72), (79), (84) and since  $\delta_\chi < \delta$ , we obtain (78).

Now let us consider the case  $\bar{x} \in \Omega_{s\rho}$  satisfying  $\text{dist}(\bar{x}, \partial\Omega) > \frac{1}{4}$ . Let  $\rho \leq \bar{\rho}$  and notice that  $B_{s\rho}(\bar{x}) \subset B_{\frac{1}{16}}(\bar{x})$ . Hence, given any point  $\tilde{x}$  such that  $|\bar{x} - \tilde{x}| = s\rho$ , we have that  $B_{\frac{1}{8}}(\tilde{x}) \subset \Omega$ . Therefore we can mimic the construction in Steps 1 and 2, finding a point  $\bar{x}_{k(\rho)} \in \Omega_{5r_{k(\rho)}}$ , with  $k(\rho)$  satisfying (71), (72) and  $r_{k(\rho)}$  satisfying (73), such that the following inequality holds

$$\frac{\int_{B_{r_{k(\rho)}}(\bar{x}_{k(\rho)})} |\nabla^2 w_0|^2}{\int_\Omega |\nabla^2 w_0|^2} \leq C \left( \frac{\int_{B_\rho(\bar{x})} |\nabla^2 w_0|^2}{\int_\Omega |\nabla^2 w_0|^2} \right)^{\delta_\chi^{k(\rho)-1}} \tag{86}$$

with  $C > 1$  only depending on  $\gamma, M$  and  $M_0$ .

Let  $y \in \Omega_{s\rho}$  such that  $\text{dist}(y, \partial\Omega) \leq \frac{1}{4}$ . By the same arguments seen above, we have

$$\frac{\int_{B_\rho(y)} |\nabla^2 w_0|^2}{\int_\Omega |\nabla^2 w_0|^2} \leq C \left( \frac{\int_{B_\rho(\bar{x})} |\nabla^2 w_0|^2}{\int_\Omega |\nabla^2 w_0|^2} \right)^{\delta_\chi^{k(\rho)-1} \delta^{k(\rho)+L-1}}, \tag{87}$$

where  $C > 1$  only depends on  $\gamma, M, \mu, M_0$ , and  $L$  satisfies (79).

Let  $y \in \Omega_{s\rho}$  such that  $\text{dist}(y, \partial\Omega) > \frac{1}{4}$ . By repeating the arguments above, we have

$$\frac{\int_{B_\rho(y)} |\nabla^2 w_0|^2}{\int_\Omega |\nabla^2 w_0|^2} \leq C \left( \frac{\int_{B_\rho(\bar{x})} |\nabla^2 w_0|^2}{\int_\Omega |\nabla^2 w_0|^2} \right)^{\delta_\chi^{k(\rho)-1} \delta^{\tilde{L}}}, \tag{88}$$

where  $\tilde{L}$  satisfies (84) and  $C > 1$  only depends on  $\gamma, M, \mu$  and  $M_0$ .

From (87), (88), (72), (79), (84), and recalling that  $\delta_\chi < \delta$ , we obtain (78).

Let us cover  $\Omega_{(s+1)\rho}$  with internally nonoverlapping closed squares of side  $l = \frac{2\rho}{\sqrt{2}}$ . Any such square is contained in a disc of radius  $\rho$  and center at a point of  $\Omega_{s\rho}$  and the number of such squares is dominated by

$$N = \frac{M_1}{2\rho^2}. \tag{89}$$

Therefore, from (78) and (89), we have

$$\int_{B_\rho(\bar{x})} |\nabla^2 w_0|^2 \geq \int_\Omega |\nabla^2 w_0|^2 \left( \frac{C'\rho^2 \int_{\Omega_{(s+1)\rho}} |\nabla^2 w_0|^2}{\int_\Omega |\nabla^2 w_0|^2} \right)^{\delta_\chi^{-A_1 - B_1 \log \frac{1}{\rho}}}, \tag{90}$$

where  $B_1$  and  $C' > 0$  only depend on  $\gamma, M, \mu$  and  $M_0$ , whereas  $A_1$  only depends on  $\gamma, M, \mu$  and  $M_0$  and  $M_1$ .

By Lemma 5.2, assuming also  $\rho \leq \frac{\tilde{\rho}}{s+1}$ , where  $\tilde{\rho}$  has been introduced in Lemma 5.2 and only depends  $\gamma, M, \mu, M_0, M_1, \delta_0, F$  we have

$$\int_{B_\rho(\bar{x})} |\nabla^2 w_0|^2 \geq (\tilde{C}\rho^2)^{\delta_\chi^{-A_1 - B_1 \log \frac{1}{\rho}}} \int_\Omega |\nabla^2 w_0|^2, \tag{91}$$

where  $\tilde{C} > 0$  only depends on  $\gamma, M, \mu, M_0, M_1$  and  $\delta_0$ . Let us take  $\rho \leq \tilde{C}$ . Noticing that  $|\log \rho| \leq \frac{1}{\rho}$ , for every  $\rho > 0$ , and that  $\tilde{\rho} < 1$ , by straightforward computations we obtain that (50) holds with  $A = 3 \exp(A_1 |\log \delta_\chi|)$ ,  $B = |\log \delta_\chi| B_1 + 1$  for every  $\rho \leq \rho^*$  with  $\rho^* = \min\{\tilde{\rho}, \frac{\tilde{\rho}}{s+1}, \tilde{C}\}$ ,  $\rho^*$  only depending on  $\gamma, M, \mu, M_0, M_1, \delta_0$ , and  $F$ .  $\square$

**Conclusion.** We have seen that (50) holds for every  $\rho \leq \rho^*$  and for every  $\bar{x} \in \Omega_{s\rho}$ , where  $\rho^*$  only depends on  $\gamma, M, \mu, M_0, M_1, \delta_0$  and  $F$ .

If  $\rho > \rho^*$  and  $\bar{x} \in \Omega_{s\rho} \subset \Omega_{s\rho^*}$ , then we have

$$\int_{B_\rho(\bar{x})} |\nabla^2 w_0|^2 \geq \int_{B_{\rho^*}(\bar{x})} |\nabla^2 w_0|^2 \geq C^* \int_\Omega |\nabla^2 w_0|^2, \tag{92}$$

where  $C^*$  only depends on  $\gamma, M, \mu, M_0, M_1, \delta_0$  and  $F$ . Since  $\bar{x} \in \Omega_{s\rho}$ , we have that

$$\text{diam}(\Omega) \geq 2s\rho, \tag{93}$$

and, on the other hand,

$$\text{diam}(\Omega) \leq C_2, \tag{94}$$

with  $C_2$  only depending on  $M_0$  and  $M_1$ , so that

$$\frac{2s}{C_2} \leq \frac{1}{\rho}. \tag{95}$$

By (92) and (95), we have

$$\int_{B_\rho(\bar{x})} |\nabla^2 w_0|^2 \geq \frac{C}{\exp\left[A\left(\frac{1}{\rho}\right)^B\right]} \int_\Omega |\nabla^2 w_0|^2, \tag{96}$$

with  $C = C^* \exp\left[A\left(\frac{2s}{C_2}\right)^B\right]$ .  $\square$

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