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The strongly increasing and strongly decreasing solutions to a system of n nonlinear first order equations are here studied, under the assumption that both the coefficients and the nonlinearities are regularly varying functions. We establish conditions under which such solutions exist and are (all) regularly varying functions, we derive their index of regular variation and establish asymptotic representations. Several applications of the main results are given, involving n-th order nonlinear differential equations, equations with a generalized ϕ-Laplacian, and nonlinear partial differential systems.

1 Introduction

Consider the nonlinear differential system

$$x_i' = \delta a_i(t) F_i(x_{i+1}), \;\; i = 1, \dots, n, \;\; x_{n+1} = x_1, \tag{1}$$

$n \in \mathbb{N}$, $n \geq 2$, where F_i, $i = 1, \dots, n$, are continuous functions defined on $\mathbb{R}$ with $uF_i(u) > 0$ for $u \neq 0$, a_i, $i = 1, \dots, n$, are positive continuous functions defined on $[T, \infty)$, $T \geq 0$, and $\delta \in \{-1, 1\}$. By a solution we mean a vector function $(x_1, \dots, x_n)$ which has its components in $C^1([T, \infty))$ and satisfies (1) on $[T, \infty)$. A solution is called *nonoscillatory* if all its components are eventually of one sign. Because of sign condition on the coefficients, if one component is nonoscillatory, then all components are nonoscillatory and eventually monotone, and therefore they have limit. A nonoscillatory solution is called *positive* all its components are eventually positive.

An extensive literature has been devoted to the study of asymptotic behavior of positive solutions of second order or higher order Emden-Fowler type equations or systems. In [8], Kamo and Usami studied the equation

$$(\Phi_\alpha(x'))' = p(t)\Phi_\beta(x), \tag{2}$$

where $\Phi_\lambda(u) = |u|^\lambda \operatorname{sgn} u$, $\alpha > \beta > 0$, under the condition $p(t) \sim t^\sigma$ as $t \to \infty$. They established asymptotic forms for all positive solutions of (2). In particular, using the so called asymptotic equivalence theorem, they gave conditions guaranteeing that all strongly decreasing solutions x (i.e., $x(t) \to 0$ and $x'(t) \to 0$ as $t \to \infty$) are asymptotically equivalent to a power function. A similar result is proved also for strongly increasing solutions

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x (i.e., $x(t) \to \infty$ and $x'(t) \to \infty$ as $t \to \infty$). See also [17] for related results. Clearly, system (1) reduces to equation (2) when $n = 2$, $a_1(t) = p(t)$, $a_2(t) = 1$, $F_1(u) = \Phi_\beta(u)$ and $F_2(u) = \Phi_{1/\alpha}(u)$, both in case $\delta = 1$ and in case $\delta = -1$.

Equation (2) can be generalized, for instance, by increasing the order of the equation, considering more general differential operator, and assuming that the coefficients and the nonlinearities are regularly varying functions. Indeed, a natural generalization of the condition $p(t) \sim t^\sigma$ is to assume that p is a regularly varying function of index σ (see Section 3 for the definition and properties of regularly varying functions). The theory of regularly varying functions, in combination with some other tools, has been shown to be very useful in the study of asymptotic properties of solutions to differential and difference equations. An important monograph summarizing themes in the research up to the year 2000 is the one by Marić [14]. Many recent interesting contributions, devoted in particular to the study of Emden-Fowler type equations and systems in the framework of regular variation, are due to Kusano, Jaros, Marić, Manojlović and Tanigawa, see for instance [10–13] and the references therein. Some contributions by the authors in this field are [16, 20]. Other important works related to our topic are the papers by Evtukhov et al., see e.g. [1, 5]. As a representative of an application of the theory of regularly varying sequences in difference equations see, for instance, [15]. Some fundamental results on asymptotic properties of nonlinear systems and n-th order nonlinear differential equations in a general setting can be found, e.g., in the monograph [9] by Kiguradze and Chanturia, and in the papers [18, 19] by Naito.

Considering system (1), due to the sign assumptions on the coefficients and on the nonlinearities, positive solutions of (1) have all the components which are eventually increasing if $\delta = 1$, while with $\delta = -1$, the components of a positive solution are all eventually decreasing. When $\delta = 1$, we denote by $\mathcal{IS}$ the set of all the solutions of (1) whose components are all eventually positive increasing. Analogously, when $\delta = -1$, we denote by $\mathcal{DS}$ the set of all the solutions of (1) whose components are all eventually positive decreasing. Our aim is to study the asymptotic behavior of $\mathcal{IS}$-solutions and $\mathcal{DS}$-solutions to system (1) in the extreme cases where all the solution components tend to infinity or tend to zero, respectively. In accordance with the existing literature, solutions in $\mathcal{IS}$ with all components tending to infinity are called *strongly increasing* (or fast growing), and are here denoted by $\mathcal{SIS}$, while solutions in $\mathcal{DS}$ with all components tending to zero are called *strongly decreasing* (or strongly decaying), and are denoted by $\mathcal{SDS}$. A positive solution which is strongly increasing or strongly decreasing is called an *extremal* solution.

One of the main tool used here in order to derive the asymptotics of extremal solutions is the theory of regularly varying functions. We denote by $\mathcal{RV}(\vartheta)$ and by $\mathcal{RV}_0(\vartheta)$ the classes of all functions which are, respectively, regularly varying at infinity of index ϑ, and regularly varying at zero of index ϑ, $\vartheta \in \mathbb{R}$. We assume

$$a_i \in \mathcal{RV}(\sigma_i), \ \ \sigma_i \in \mathbb{R}, \ \ i = 1, \dots, n, \tag{3}$$

and

$$F_i \in \mathcal{RV}(\alpha_i), \ \ \alpha_i \in (0, \infty), \ \ i = 1, \dots, n, \tag{4}$$

when studying the asymptotics of strongly increasing solutions, or (3) and

$$F_i \in \mathcal{RV}_0(\alpha_i), \ \ \alpha_i \in (0, \infty), \ \ i = 1, \dots, n. \tag{5}$$

when the asymptotics of strongly decreasing solutions is considered. Further, in both cases we assume that the indices $\alpha_1, \cdots, \alpha_n$ satisfy

$$\alpha_1 \cdots \alpha_n < 1. \tag{6}$$

System (1) satisfying condition (6) will be called *subhomogeneous* (an alternative terminology is *sub-half-linear*). The opposite (strict) inequality is called *superhomogeneity* (or *super-half-linearity*).

Our main results are Theorem 4.1 and Theorem 4.2, stated in Section 4. In the first one we prove the existence of regularly varying $\mathcal{SIS}$ (respectively $\mathcal{SDS}$) solutions, and establish asymptotic representations. The second theorem, under slightly stronger assumptions on nonlinearities, says that every $\mathcal{SIS}$ (respectively $\mathcal{SDS}$) solution is regularly varying, and it can be seen as an extension of the above Kamo-Usami result; it is worthy to mention that the asymptotic equivalence theorem cannot be used in our general setting. At the same time, Theorem 4.2 can be understood as a complementary information to [10], where the existence of strongly monotone regularly varying solutions is discussed; we now claim that all considered strongly monotone solutions are regularly varying. This result is new even in the case when $a_i(t) \sim t^{\sigma_i}$ as $t \to \infty$. Thanks to the general setting, Theorem 4.1 can be seen as an extension of the quoted existence result [10] in the sense of more general nonlinearities, and enables us to include also equations or systems with generalized Laplacians, see Example 4.6. The main tools in the proof of Theorem 4.1 are the Schauder-Tychonoff fixed point theorem and various properties of regularly varying functions like the Karamata integration theorem or the uniform convergence theorem. The proof of Theorem 4.2 is based on various estimates (including a generalized arithmetic-geometric mean inequality), some matrix analysis, and properties of regularly varying functions.

The paper is organized as follows. After a short section on extremal solutions of (1), in Section 3 we recall several useful properties of regularly varying functions which are used in this work. The main results are presented in Section 4, together with comparisons with the existing literature and with some ideas of possible applications. Finally, the last section is devoted to the proof of preliminary lemmas and of the main results.

2 Extremal solutions

Consider the more general system

$$x_i' = \delta_i a_i(t) F_i(x_{i+1}), \ \ i = 1, \dots, n, \ \ x_{n+1} = x_1, \tag{7}$$

with $\delta_i \in \{-1, 1\}$ for $i = 1, \dots, n$. The condition $\delta_i = 1$ for all $i = 1, \dots, n$ is necessary for (7) to have solution in the class $\mathcal{IS}$. Analogously, the condition $\delta_i = -1$ for all $i = 1, \dots, n$ is necessary for (7) to have $\mathcal{DS} \neq \emptyset$. Therefore in studying positive increasing or decreasing solutions, it is not restrictive to consider system (1).

In order to put our considerations into a broader context, we recall a standard classification of nonoscillatory solutions of higher order equations. Naito in [18, 19] considers the n-th order differential equation

$$D(\gamma_n) D(\gamma_{n-1}) \cdots D(\gamma_1) x + \tilde{\delta} p(t) \Phi_\beta(x) = 0, \tag{8}$$

where $n \geq 2$, $\gamma_1, \dots, \gamma_n, \beta \in (0, \infty)$, $\tilde{\delta} = 1$ or $\tilde{\delta} = -1$, $p(t) > 0$, and $D(\gamma)x = \frac{d}{dt}(\Phi_\gamma(x))$. Equation (8) is a special case of system (1). Indeed, if $F_i = \Phi_{\alpha_i}$, $i = 1, \dots, n$, then (1) can be equivalently written as

$$D_{a_{n-1}^{-\frac{1}{\alpha_{n-1}}}} \left(\frac{1}{\alpha_{n-1}} \right) \cdots D_{a_1^{-\frac{1}{\alpha_1}}} \left(\frac{1}{\alpha_1} \right) D_1(1)\, x_1 = \delta^n a_n(t) \Phi_{\alpha_n}(x_1), \tag{9}$$

where $D_f(\gamma)x = \frac{d}{dt}(f(t)\Phi_\gamma(x))$. Note that a scalar equation viewed as a system may sometimes enable us better understanding of the solution space. Using the substitution $x_1(t) = \Phi_{\gamma_1}(x(t))$ and noticing that $D_1(1)(\Phi_{\gamma_1}(x)) = D(\gamma_1)x$, equation (9) reduces to (8) choosing $\alpha_i = 1/\gamma_{i+1}$, $i = 1, \dots, n-1$, $\alpha_n = \beta/\gamma_1$, $a_i(t) \equiv 1$, $i = 1, \dots, n-1$, $a_n(t) = p(t)$, and $\tilde{\delta} = -\delta^n$. Naito made a basic classification of positive solutions to (8) extending well known results by Kiguradze and Chanturia [9] for the quasilinear equation

$$x^{(n)} + \tilde{\delta} p(t) \Phi_\beta(x) = 0. \tag{10}$$

Notice that (8) reduces to (10) when $\gamma_1 = \cdots = \gamma_n = 1$. The classification of solutions x to (8) in [18] is made according to the eventual signs of $D(\gamma_j) \cdots D(\gamma_1)x(t)$, $j = 0, \dots, n-1$ (with the operator being the identity when $j = 0$). For solutions x of (8), the so-called *Kiguradze class of degree* k is denoted by $\mathcal{P}_k$ and defined as

$$\begin{cases} D(\gamma_j) \cdots D(\gamma_1) x(t) > 0 & \text{for } j = 0, 1, \dots, k-1, \\ (-1)^{j-k} D(\gamma_j) \cdots D(\gamma_1) x(t) > 0 & \text{for } j = k, \dots, n-1 \end{cases} \tag{11}$$

for t large, where for $k = 0$ (resp. $k = n$), the first line (resp. the second line) in (11) is omitted. It is not difficult to see that

$$x \in \mathcal{P}_0 \iff (x_1, \dots, x_n) \in \mathcal{DS} \ \text{ and } \ x \in \mathcal{P}_n \iff (x_1, \dots, x_n) \in \mathcal{IS}, \tag{12}$$

where $x_1 = \Phi_{\gamma_1}(x)$, $x_i = \delta\Phi_{\gamma_i}(x'_{i-1})$, $i = 2, \dots, n$. If we denote the set of all eventually positive solutions of (8) by $\mathcal{P}$, then [18, Theorem 1.1] implies that

$$\begin{cases} \mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_3 \cup \dots \cup \mathcal{P}_{n-1} & \text{for } \tilde{\delta} = 1 \text{ and } n \text{ even;} \\ \mathcal{P} = \mathcal{P}_0 \cup \mathcal{P}_2 \cup \dots \cup \mathcal{P}_{n-1} & \text{for } \tilde{\delta} = 1 \text{ and } n \text{ odd;} \\ \mathcal{P} = \mathcal{P}_0 \cup \mathcal{P}_2 \cup \dots \cup \mathcal{P}_{n-2} \cup \mathcal{P}_n & \text{for } \tilde{\delta} = -1 \text{ and } n \text{ even;} \\ \mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_3 \cup \dots \cup \mathcal{P}_{n-2} \cup \mathcal{P}_n & \text{for } \tilde{\delta} = -1 \text{ and } n \text{ odd.} \end{cases} \tag{13}$$

In view of (12) and the equality $\delta^n a_n(t) = -\tilde{\delta}p(t)$, the relations in (13) says that the conditions $\delta = -1$ and $\delta = 1$ are nonrestrictive when studying, respectively, positive decreasing and positive increasing solutions of (7).

3 Basics on regularly varying functions

In this section we recall basic information about regularly varying functions which play an important role in our theory. We recommend the monographs [2, 7] as very good sources of information on the theory of regular variation. A measurable function $f : [a, \infty) \to (0, \infty)$ is called *regularly varying at infinity of index* $\vartheta \in \mathbb{R}$ if

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^{\vartheta} \quad \text{for every } \lambda > 0. \tag{14}$$

Analogously, a measurable function $g : (0, a] \to (0, \infty)$ is called *regularly varying at zero of index* $\vartheta \in \mathbb{R}$ if $f(t) = g(1/t)$ is regularly varying at infinity of index $-\vartheta$. If $\vartheta = 0$, then the function is called slowly varying; we denote the class of all slowly varying functions at infinity by $\mathcal{SV}$; analogously, $\mathcal{SV}_0$ indicates the class of all functions which are slowly varying at zero. Since $g \in \mathcal{RV}_0(\vartheta)$ if and only if $g(1/t) \in \mathcal{RV}(-\vartheta)$, all properties for functions which are regularly varying at zero can be deduced with obvious modifications from the theory of regularly varying functions at infinity.

As an immediate consequence of the definition, we get $f \in \mathcal{RV}(\vartheta)$ if and only if

$$f(t) = t^{\vartheta} L(t), \tag{15}$$

where $L \in \mathcal{SV}$. The above relation shows that the class of regularly varying function extends the class of functions asymptotically equivalent to power functions. Further, to derive the properties of the function $f \in \mathcal{RV}(\vartheta)$, to large extent it is sufficient to study the properties of its slowly varying part $L(t)$. The following holds; the proofs of all the statements can be found in [2, 7]. As usually, $f(t) \sim g(t)$ as $t \to \infty$ means that $\lim_{t\to\infty} f(t)/g(t) = 1$.

Proposition 3.1 *Let $L, L_1, \cdots, L_n \in \mathcal{SV}$. Then*

(i) $L(\lambda t)/L(t) \to 1$ *as* $t \to \infty$, *uniformly on each compact* λ-*set in* $(0, \infty)$.

(ii) *If* $R(x_1, \dots, x_n)$ *is a rational function with positive coefficients, then* $R(L_1, \dots, L_n) \in \mathcal{SV}$. *Moreover,* $L_1 \circ L_2 \in \mathcal{SV}$ *provided* $L_2(t) \to \infty$ *as* $t \to \infty$.

(iii) *If* $\vartheta > 0$, *then* $t^{\vartheta} L(t) \to \infty$, $t^{-\vartheta} L(t) \to 0$ *as* $t \to \infty$.

(iv) *If* $\zeta < -1$, *it holds*

$$\int_t^{\infty} s^{\zeta} L(s)\, ds \sim \frac{1}{-\zeta - 1} t^{\zeta+1} L(t); \tag{16}$$

while, if $\zeta > -1$,

$$\int_a^t s^{\zeta} L(s)\, ds \sim \frac{1}{\zeta + 1} t^{\zeta+1} L(t). \tag{17}$$

Property stated in (i) of the above proposition is often referred to as the *uniform convergence theorem*. Property stated in (iv) is often referred to as *Karamata's theorem; direct half*.

We point out that, as an immediate consequence of (iii), if $f_i \in \mathcal{RV}(\vartheta_i)$, $i = 1, 2$, and $\gamma \in \mathbb{R}$, then

$$f_1 f_2 \in \mathcal{RV}(\vartheta_1 + \vartheta_2) \text{ and } f_1^{\gamma} \in \mathcal{RV}(\gamma\vartheta_1). \tag{18}$$

Further, if $f \in \mathcal{RV}_0(\vartheta)$, $\vartheta > 0$, is monotone, then

$$f^{-1} \in \mathcal{RV}_0(1/\vartheta). \tag{19}$$

Finally, the following property of *almost monotonicity* and *continuity* holds.

Proposition 3.2 *If $f \in \mathcal{RV}(\vartheta)$ with $\vartheta > 0$ ($\vartheta < 0$), then there exists a nondecreasing (nonincreasing) $g \in C \cap \mathcal{RV}(\vartheta)$ such that $f(t) \sim g(t)$ as $t \to \infty$.*

Since the coefficients and nonlinearities in system (1) are assumed to be regularly varying, let us give few examples. For instance, for the coefficients we can have $a_i(t) = t^{\sigma_i} L_i(t)$, with L_i asymptotically equivalent to a positive constant or $L_i(t) = (\ln t)^{\gamma}(\ln(\ln t))^{\delta}$, $\gamma, \delta \in \mathbb{R}$. Concerning the nonlinearities, examples are $F_i(u) = |u|^{\alpha_i} \operatorname{sgn} u$, which belongs to $\mathcal{RV}(\alpha_i) \cap \mathcal{RV}_0(\alpha_i)$, or $F_i(u) = u^{\delta}(A + Bu^{\beta})^{\gamma}$, which belongs to $\mathcal{RV}(\delta + \beta\gamma) \cap \mathcal{RV}_0(\delta)$ if $\delta, \beta, \gamma > 0$. Other, more special, examples are $F_i(u) = u/\sqrt{1+u^2}$ or $F_i(u) = u/\sqrt{1-u^2}$ which are both regularly varying at zero of index 1.

4 Main results

We start this section with conventions and notations used throughout. For eventually positive functions f, g, we denote: $f(t) \asymp g(t)$ if there exist $c_1, c_2 \in (0, \infty)$ such that $c_1 g(t) \le f(t) \le c_2 g(t)$ for large t, and $f(t) = o(g(t))$ if $\lim_{t\to\infty} f(t)/g(t) = 0$. Further, we adopt the usual conventions: $\prod_{j=k}^{k-1} u_j = 1$ and $\sum_{j=k}^{k-1} u_j = 0$.

The subscripts that indicate the components are always to be intended modulo n and not bigger than n, that is

$$u_k = \begin{cases} u_k & \text{if } 1 \le k \le n, \\ u_{k-mn} & \text{if } k > n, \end{cases} \tag{20}$$

where $m \in \mathbb{N}$ is such that $1 \le k - mn \le n$.

The slowly varying component in the representation (15) of $f \in \mathcal{RV}(\vartheta)$ will be denoted by L_f, i.e., $L_f(t) = f(t)/t^{\vartheta}$; similarly for $f \in \mathcal{RV}_0(\vartheta)$.

For sake of simplicity, we introduce here some constants that repeatedly appear in what follows. We set

$$A_{i,j} = \prod_{k=i}^{j-1} \alpha_k, \text{ for } 1 \le i \le j \le i + n \le 2n.$$

It is easy to check that $A_{i,n+i} = \alpha_1 \cdots \alpha_n$ and, because of our convention, $A_{i,i} = 1$ for all $i = 1, \dots, n$. We emphasize that the convention (20) is used only for simple subscripts and not for double ones. We indicate by $(\nu_1, \dots, \nu_n)$ the unique solution of the linear system

$$\nu_i - \alpha_i \nu_{i+1} = \sigma_i + 1, \quad i = 1, \dots, n, \tag{21}$$

where $\sigma_1, \dots, \sigma_n$ are given real numbers. Notice that the associated matrix is nonsingular thanks to the subhomogeneity condition. Further, let $(h_1, \dots, h_n)$ be the unique solution of

$$|\nu_i| h_i = h_{i+1}^{\alpha_i}, \quad i = 1, \dots, n. \tag{22}$$

Notice that the subhomogeneity condition again plays a key role in its unique solvability. A simple calculation shows that

$$\nu_i = \frac{1}{1 - A_{i,n+i}} \sum_{k=0}^{n-1} (\sigma_{i+k} + 1) A_{i,i+k}, \quad i = 1, \dots, n, \tag{23}$$

and

$$h_i = \left(\prod_{k=0}^{n-1} |\nu_{i+k}|^{-A_{i,i+k}} \right)^{\frac{1}{1-A_{i,n+i}}}, \quad i = 1, \dots, n,$$

for $\nu_1, \dots, \nu_n \neq 0$. If we set

$$L_i(t) = \left(\prod_{j=0}^{n-1} \left(L_{a_{i+j}}(t) L_{F_{i+j}}(t^{\nu_{i+j+1}}) \right)^{A_{i,i+j}} \right)^{\frac{1}{1-A_{i,n+i}}}, \quad i = 1, \dots, n, \tag{24}$$

then $(L_1, \dots, L_n)(t)$ is the unique solution (up to asymptotic equivalence) to the system of the relations

$$L_i(t) \sim L_{a_i}(t) L_{i+1}^{\alpha_i}(t) L_{F_i}(t^{\nu_{i+1}}) \quad \text{as } t \to \infty, \quad i = 1, \dots, n. \tag{25}$$

The proof of this property will be given in Section 5, Lemma 5.2. Here we limit to observe that if $L_{F_1} \equiv \cdots \equiv L_{F_n} \equiv 1$, then $(L_1, \dots, L_n)(t)$ reduces to the unique solution of the system

$$L_i(t) = L_{a_i}(t) L_{i+1}^{\alpha_i}(t), \tag{26}$$

$i = 1, \dots, n$.

In what follows, we assume an additional condition for the slowly varying components of the nonlinearities F_i. In particular, if $F_i, i = 1, \dots, n$, satisfies (5), we assume

$$L_{F_i}(ug(u)) \sim L_{F_i}(u) \quad \text{as } u \to 0+, \quad i = 1, \dots, n, \tag{27}$$

for every $g \in \mathcal{SV}_0$, while if $F_i, i = 1, \dots, n$, satisfies (4), we assume

$$L_{F_i}(ug(u)) \sim L_{F_i}(u) \quad \text{as } u \to \infty, \quad i = 1, \dots, n, \tag{28}$$

for every $g \in \mathcal{SV}$. A wide class of regularly varying functions satisfy (27) or (28). For instance, all the examples of regularly varying nonlinearities F_i which are mentioned in the previous section satisfy these additional conditions.

Now we are ready to present the main results. The first theorem gives sufficient conditions under which system (1) possesses a $\mathcal{SDS}$ or $\mathcal{SIS}$ solution which is regularly varying and provides an asymptotic formula. We point out that F_i does not need to be monotone.

Theorem 4.1 *Let* (3) *hold.*
(i) Assume $\delta = -1$, (5) *and* (27). *If $\nu_i < 0$, $i = 1, \dots, n$, then there exists*

$$(x_1, \dots, x_n) \in \mathcal{SDS} \cap (\mathcal{RV}(\nu_1) \times \cdots \times \mathcal{RV}(\nu_n))$$

and

$$x_i(t) \sim h_i t^{\nu_i} L_i(t) \quad \text{as } t \to \infty, \quad i = 1, \dots, n. \tag{29}$$

(ii) Assume $\delta = 1$, (4) *and* (28). *If $\nu_i > 0$, $i = 1, \dots, n$, then there exists*

$$(x_1, \dots, x_n) \in \mathcal{SIS} \cap (\mathcal{RV}(\nu_1) \times \cdots \times \mathcal{RV}(\nu_n))$$

and (29) *holds.*

As a corollary, we get sufficient conditions for $\mathcal{SDS} \neq \emptyset$ and $\mathcal{SIS} \neq \emptyset$. Note that in part (i) of Theorem 4.1 it is sufficient to assume that the nonlinearities F_i are defined only in a neighborhood of zero, and in part (ii) only in a neighborhood of infinity.

In the second result, we show that, strengthening the assumptions on the nonlinearities F_i, we are able to prove that all $\mathcal{SDS}$ and $\mathcal{SIS}$ solutions are regularly varying.

Theorem 4.2 *Let* (3) *hold and* $F_i = \Phi_{\alpha_i}$ *with* $\alpha_i > 0$, $i = 1, \dots, n$.

(i) If $\delta = -1$ *and* $\nu_i < 0$, $i = 1, \dots, n$, *then* $\mathcal{SDS} \neq \emptyset$ *and for every* $(x_1, \dots, x_n) \in \mathcal{SDS}$, *it holds* $(x_1, \dots, x_n) \in \mathcal{RV}(\nu_1) \times \dots \times \mathcal{RV}(\nu_n)$. *Further,* (29) *holds with* $L_{F_1} \equiv \dots \equiv L_{F_n} \equiv 1$.

(ii) If $\delta = 1$ *and* $\nu_i > 0$, $i = 1, \dots, n$, *then* $\mathcal{SIS} \neq \emptyset$ *and for every* $(x_1, \dots, x_n) \in \mathcal{SIS}$, *it holds* $(x_1, \dots, x_n) \in \mathcal{RV}(\nu_1) \times \dots \times \mathcal{RV}(\nu_n)$. *Further,* (29) *holds with* $L_{F_1} \equiv \dots \equiv L_{F_n} \equiv 1$.

An alternative expression of sufficient conditions in Theorem 4.1 and Theorem 4.2 can be found in Section 5, Lemma 5.6.

Theorem 4.2 can be applied, for instance, to the equations

$$x^{(n)} = (-1)^n p(t) \Phi_\beta(x), \tag{30}$$

and

$$x^{(n)} = p(t) \Phi_\beta(x), \tag{31}$$

where $p(t) = t^\varrho L_p(t)$, $L_p \in \mathcal{SV}$ and $0 < \beta < 1$. Indeed, (30) and (31) can be written as (1) with $\delta = -1$ and $\delta = 1$, respectively, provided

$$F_1 = \dots = F_{n-1} = \mathrm{id}, \;\; F_n = \Phi_\beta, \;\; a_1 = \dots = a_{n-1} = 1, \;\; a_n = p, \;\; x_1 = x.$$

Then

$$\begin{aligned} &\alpha_1 = \dots = \alpha_{n-1} = 1, \;\; \alpha_n = \beta, \;\; L_{F_1} = \dots = L_{F_n} = 1, \\ &\sigma_1 = \dots = \sigma_{n-1} = 0, \;\; \sigma_n = \varrho, \;\; L_{a_1} = \dots = L_{a_{n-1}} = 1, \;\; L_{a_n} = L_p, \end{aligned}$$

and hence,

$$\begin{aligned} \nu_i &= \frac{\varrho + n}{1 - \beta} - (i - 1), \; i = 1, \dots, n, \;\; L_1 = L_p^{\frac{1}{1-\beta}}, \\ h_1 &= \left(\prod_{j=1}^{n} \frac{1 - \beta}{-\varrho - n + (1 - \beta)(j - 1)} \right)^{\frac{1}{1-\beta}} \quad (\text{for } \nu_i < 0), \\ h_1 &= \left(\prod_{j=1}^{n} \frac{1 - \beta}{\varrho + n - (1 - \beta)(j - 1)} \right)^{\frac{1}{1-\beta}} \quad (\text{for } \nu_i > 0). \end{aligned}$$

We have $\nu_n > 0$ iff $\varrho + 1 + \beta(n-1) > 0$, and $\nu_1 < 0$ iff $\varrho + n < 0$. Since $\nu_{i+1} < \nu_i$, $i = 1, \dots, n-1$, it holds that $\varrho + 1 + \beta(n-1) > 0$ implies $\nu_i > 0$, $i = 1, \dots, n$, and $\varrho + n < 0$ implies $\nu_i < 0$, $i = 1, \dots, n$. Thus we get the following corollary.

Corollary 4.3 *(a) If* $\varrho + n < 0$, *then equation* (30) *possesses a (strongly decreasing) solution* x *such that* $\lim_{t\to\infty} x^{(i)}(t) = 0$, $i = 0, \dots, n-1$. *Moreover, for any such a solution there holds*

$$x \in \mathcal{RV}\left(\frac{\varrho + n}{1 - \beta} \right) \tag{32}$$

with

$$x^{1-\beta}(t) \sim t^{\varrho+n} L_p(t) \prod_{j=1}^{n} \frac{1 - \beta}{-\varrho - n + (1 - \beta)(j - 1)} \;\; \text{as } t \to \infty.$$

(b) If $\varrho + 1 + \beta(n-1) > 0$, *then equation* (31) *possesses a (strongly increasing) solution* x *such that* $\lim_{t\to\infty} x^{(i)}(t) = \infty$, $i = 1, \dots, n-1$. *Moreover, for any such a solution there hold* (32) *with*

$$x^{1-\beta}(t) \sim t^{\varrho+n} L_p(t) \prod_{j=1}^{n} \frac{1 - \beta}{\varrho + n - (1 - \beta)(j - 1)} \;\; \text{as } t \to \infty.$$

Remark 4.4 Kusano and Manojlović considered the odd order equation of the form $x^{(2n+1)}+p(t)\Phi_\beta(x)=0$ in [11] and of the form $x^{(2n+1)}-p(t)\Phi_\beta(x)=0$ in [12], with $p\in\mathcal{RV}(\sigma)$, $0<\beta<1$. They studied all possible asymptotic classes of positive solutions and gave overall structure of regularly varying solutions. Jaroš and Kusano in the recent paper [10] studied strongly decreasing solutions resp. strongly increasing solutions of the system $x_i'=-a_i(t)x_{i+1}^{\alpha_i}$ resp. of the system $x_i'=a_i(t)x_{i+1}^{\alpha_i}$, $a_i\in\mathcal{RV}(\sigma_i)$, $i=1,\dots,n$, $\alpha_1\cdots\alpha_n<1$, and, in particular, they established sufficient and necessary conditions for the existence of regularly varying strongly decreasing (resp. strongly increasing) solutions and derived asymptotic formulas. As an application, they examined equations (30) and (31). Results of a similar character can be found in [5] by Evtukhov and Samoilenko, who considered the equation $x^{(n)}=\pm p(t)F(x)$, where F is a regularly varying function of index different from 1 and p is a continuous function. They made a classification of positive solutions of whose behavior is related to regularly varying and rapidly varying functions. Using quite different methods they proved the existence of solutions in such classes.

Our results complement the above ones since, thanks to Theorem 4.2, we can claim that *all* strongly decreasing (strongly increasing) solutions are regularly varying. It means that we got full knowledge of the asymptotics of all strongly monotone solutions. It would be interesting to obtain conditions guaranteeing that also other asymptotic classes of solutions of (1) consist only of regularly (or rapidly) varying functions.

As an application of Theorem 4.1-(i), we can study positive decreasing solutions of equations in the form

$$(G(x'))'=p(t)F(x), \tag{33}$$

where p,F, and G are continuous functions with $uF(u)>0$, $uG(u)>0$, F,G defined in a neighborhood of zero with G increasing. Equation (33) can be rewritten in the form (1) with $\delta=-1$, putting $x=x_1$ and $-G(x')=x_2$, namely

$$x_1'=-\tilde G(x_2),\quad x_2'=-p(t)F(x_1), \tag{34}$$

with $\tilde G(u)=-G^{-1}(-u)$, $u>0$. In addition, we assume that $p\in\mathcal{RV}(\varrho)$, $F\in\mathcal{RV}_0(\beta)$, and $\tilde G\in\mathcal{RV}_0(1/\alpha)$ with $\alpha>\beta>0$. Therefore, we can apply Theorem 4.1 to obtain the following corollary.

Corollary 4.5 *Assume that*

$$L_F(uh(u))\sim L_F(u),\quad L_{\tilde G}(uh(u))\sim L_{\tilde G}(u)\quad \textit{as } u\to 0+ \tag{35}$$

for every $h\in\mathcal{SV}_0$. If

$$\varrho+1+\alpha<0, \tag{36}$$

then (33) *possesses an eventually positive decreasing solution x such that* $\lim_{t\to\infty}x(t)=\lim_{t\to\infty}x'(t)=0$, $x\in\mathcal{RV}(\nu)$, *and*

$$x^{\alpha-\beta}(t)\sim\frac{1}{-(\varrho+1+\beta\nu)(-\nu)^\alpha}L_{\tilde G}^\alpha(t^{\varrho+1+\beta\nu})L_F(t^\nu)L_p(t)t^{\nu(\alpha-\beta)} \tag{37}$$

as $t\to\infty$, where $\nu=(\alpha+\varrho+1)/(\alpha-\beta)$.

Note that the subhomogeneity condition in the previous setting becomes $\alpha>\beta$. For the numbers ν_1,ν_2 in Theorem 4.1 we have

$$\nu_1=\frac{\varrho+\alpha+1}{\alpha-\beta},\quad \nu_2=\frac{\alpha(\varrho+\beta+1)}{\alpha-\beta}.$$

Condition (36) is equivalent to $\nu_1<0,\nu_2<0$.

Example 4.6 To illustrate the previous corollary, we consider a special case of (33) in the form

$$\left(\frac{x'}{\sqrt{1\pm x'^2}}\right)'=t^\varrho(\ln^\gamma t+g(t))[\Phi_\beta(x)+\Phi_\delta(x)]|\ln x|, \tag{38}$$

where $0<\beta<\delta$, $\beta<1$, $\gamma,\varrho\in\mathbb{R}$, and g is a a continuous function on $[a,\infty)$ with $|g(t)|=o(\ln^\gamma t)$ as $t\to\infty$. Examples of g when $\gamma>0$ are $g(t)=\sin t$ or $g(t)=\ln(\ln t)$. We have $\alpha=1$, $L_{\tilde G}(u)=L_{G^{-1}}(u)=\frac{1}{\sqrt{1\mp u^2}}\sim 1$

as $u \to 0+$, $F(u) = \Phi_\beta(u)[1 + u^{\delta-\beta}]|\ln u| \in \mathcal{RV}_0(\beta)$ with $L_F(u) = [1 + u^{\delta-\beta}]|\ln u| \sim |\ln u|$ as $u \to 0+$, $L_p(t) = \ln^\gamma t + g(t) \sim \ln^\gamma t$ as $t \to \infty$, and $\nu = (\varrho + 2)/(1 - \beta)$. If $\varrho + 2 < 0$, then (38) possesses a positive decreasing solution x such that $x(t) \to 0$, $x \in \mathcal{RV}(\nu)$, and

$$\begin{aligned} x^{1-\beta}(t) &\sim \frac{1}{\nu(\varrho + 1 + \beta\nu)} |\ln t^\nu| (\ln t)^\gamma t^{\nu(1-\beta)} \\ &= \frac{-1}{\varrho + 1 + \beta\nu} (\ln t)^{\gamma+1} t^{\nu(1-\beta)} \end{aligned}$$

as $t \to \infty$.

Note that such a particular case of G as in (38) arises when searching radial solutions of PDE's with the mean curvature operator (the sign '$+$') resp. the relativistic operator (the sign '$-$').

Our results can be useful also to provide information about behavior of radial solutions of the partial differential system

$$\begin{cases} \operatorname{div}(\|\nabla u_1\|^{\lambda_1 - 1} \nabla u_1) &= \varphi_1(\|z\|) G(u_2), \\ \operatorname{div}(\|\nabla u_2\|^{\lambda_2 - 1} \nabla u_2) &= \varphi_2(\|z\|) G(u_3), \\ &\vdots \\ \operatorname{div}(\|\nabla u_k\|^{\lambda_k - 1} \nabla u_k) &= \varphi_k(\|z\|) G(u_1) \end{cases} \tag{39}$$

in an exterior domain $\{z \in \mathbb{R}^N : \|z\| \geq a\}$, $N \geq 2$, where $\lambda_i > 0$, φ_i be positive continuous functions on $[a, \infty)$, and G_i be continuous functions satisfying $uG_i(u) > 0$ for $u \neq 0$. Put $t = \|z\|$, a radial function $(u_1(z), \dots, u_k(z))$ is a solution of (39) if and only if, taking $y_i(\|z\|) = u_i(z)$, the vector function $(y_1(t), \dots, y_k(t))$ satisfies the ordinary differential system

$$\begin{cases} (t^{N-1} \Phi_{\lambda_1}(y_1'))' &= t^{N-1} \varphi_1(t) G_1(y_2), \\ (t^{N-1} \Phi_{\lambda_2}(y_2'))' &= t^{N-1} \varphi_2(t) G_2(y_3), \\ &\vdots \\ (t^{N-1} \Phi_{\lambda_k}(y_k'))' &= t^{N-1} \varphi_k(t) G_k(y_1). \end{cases} \tag{40}$$

Differential systems of the form similar to (39) and (40) were studied from other points of view e.g. in [4,6]; see also the references therein. Some authors use the term "Lane-Emden type" for such systems. System (40) and (39) have been very frequently studied in case $k = 2$, as coupled ones, under various settings, see e.g. [3,4,16,21].

System (40) can be rewritten in the form (1) with $\delta = -1$. Indeed, it is sufficient to set $n = 2k$,

$$a_{2i-1}(t) = \Phi_{1/\lambda_i}(t^{1-N}), \ \ a_{2i}(t) = t^{N-1} \varphi_i(t), \ \ F_{2i-1} = \Phi_{1/\lambda_i}, \ \ F_{2i} = G_i,$$

$$x_{2i-1} = y_i, \ \ x_{2i} = -t^{N-1} \Phi_{\lambda_i}(y_i'),$$

$i = 1, \dots, k$. Assume $\varphi_i \in \mathcal{RV}(\omega_i)$, $G_i \in \mathcal{RV}(\mu_i)$ or $G_i \in \mathcal{RV}_0(\mu_i)$, $\mu_i > 0$, $i = 1, \dots, k$. The subhomogeneity condition then reads as $\mu_1 \cdots \mu_k < \lambda_1 \cdots \lambda_k$ and we have

$$\alpha_{2i-1} = \frac{1}{\lambda_i}, \ \ \alpha_{2i} = \mu_i, \ \ L_{F_{2i}} = L_{G_i}, \ \ \sigma_{2i-1} = \frac{1-N}{\lambda_i}, \ \ \sigma_{2i} = N - 1 + \omega_i,$$

$i = 1, \dots, k$. Then our results can be applied to system (39). The next example provides details in the case $k = 1$.

Example 4.7 Let $k = 1$ and $G_1 \in \mathcal{RV}_0(\mu_1)$ with $L_{G_1}(ug(u)) \sim L_{G_1}(u)$ as $u \to 0+$ for all $g \in \mathcal{SV}_0$. Since the numbers ν_1, ν_2 in Theorem 4.1 are now given by

$$\nu_1 = \frac{\lambda_1 + \omega_1 + 1}{\lambda_1 - \mu_1}, \ \ \nu_2 = \frac{\mu_1}{\lambda_1 - \mu_1} \left(\frac{\lambda_1}{\mu_1}(N + \omega_1) - N + 1 + \lambda_1 \right),$$

we get the following result. If $N+\omega_1 < \min\{N-1-\lambda_1, \mu_1(N-1-\lambda_1)/\lambda_1\}$, then (39) possesses a positive strongly decreasing radial solution u satisfying

$$\lim_{\|z\|\to\infty} \frac{u(z)}{\|z\|^\nu L_{G_1}^\tau(\|z\|^\nu) L_{\varphi_1}^\tau(\|z\|)} = (-\nu)^{-\lambda_1\tau}(-N-\omega_1-\mu_1\nu)^{-\tau}, \tag{41}$$

where $\nu = \tau(\lambda_1+\omega_1+1)$ and $\tau = 1/(\lambda_1-\mu_1)$. If, in addition, $L_{G_1}=1$, then every positive strongly decreasing radial solution u of (39) satisfies (41) by Theorem 4.2. Analogous results can be derived for positive strongly increasing radial solutions of (39) writing (40) in the form (1) with $\delta = 1$.

We finish this section with noting that our results can simply be used for examining extremal negative solutions. Indeed, let $(x_1,\dots,x_n)$ be a negative solution of (1). Set $u_i = -x_i$, $i=1,\dots,n$. Then $(u_1,\dots,u_n)$ is a positive solution of the system $u_i' = \delta a_i(t)\tilde{F}_i(u_{i+1})$, $i=1,\dots,n$, where $\tilde{F}_i(u) = -F_i(-u)$. Instead of (4) or (5), we assume $\tilde{F}_i \in \mathcal{RV}(\alpha_i)$ or $\tilde{F}_i \in \mathcal{RV}_0(\alpha_i)$, respectively.

5 Proof of the main results

In order to prove Theorem 4.1 and Theorem 4.2, we need some technical lemmas. The first two lemmas analyze conditions (27), (28) and show how they lead to the unique solvability (up to asymptotic equivalence) in the class $\mathcal{SV}$ of relation (25).

Lemma 5.1 *If* (27) *holds, then*

$$L_{F_i}(t^\nu h(t)) \sim L_{F_i}(t^\nu) \tag{42}$$

as $t\to\infty$ *for any* $\nu<0$ *and* $h\in\mathcal{SV}$. *Analogously, condition* (28) *implies that* (42) *holds as* $t\to\infty$ *for any* $\nu>0$ *and* $h\in\mathcal{SV}$.

P r o o f. Taking the substitution $u=t^\nu$, relation (42) is transformed into

$$L_{F_i}(uh(u^{1/\nu})) \sim L_{F_i}(u), \quad u\to 0+. \tag{43}$$

Since $g(u) := h(u^{1/\nu}) \in \mathcal{RV}_0(0) = \mathcal{SV}_0$, condition (27) implies that (43) holds. Similarly we can show that (28) implies (42) for any $\nu>0$. □

Lemma 5.2 *Let* $\nu_i<0$ *for* $i=1,\dots,n$. *Then the system of asymptotic relations* (25) *has a unique solution (up to asymptotic equivalence) belonging to the set* $\mathcal{SV}$, *and this solution is given by* (24). *Further, the system of asymptotic relations*

$$\tilde{L}_i(t) \sim \frac{1}{|\nu_i|} L_{a_i}(t)\tilde{L}_{i+1}^{\alpha_i}(t) L_{F_i}(t^{\nu_{i+1}}) \quad \text{as } t\to\infty, \quad i=1,\dots,n, \tag{44}$$

has the unique solution (up to asymptotic equivalence) $\tilde{L}_i(t) = h_i L_i(t)$, *where* h_i *satisfies* (22), *for all* $i=1,\dots,n$. *Analogous statements hold under the assumption* $\nu_i>0$, $i=1,\dots,n$.

P r o o f. The first statement follows from the fact that system (25) can be written as

$$\begin{aligned} L_i(t) &\sim L_{a_i}(t)L_{F_i}(t^{\nu_{i+1}})\left(L_{a_{i+1}}(t)L_{F_{i+1}}(t^{\nu_{i+2}})L_{i+2}^{\alpha_{i+1}}(t)\right)^{\alpha_i} \sim \cdots \sim \\ &\sim \prod_{j=0}^{n-1}\left(L_{a_{i+j}}(t)L_{F_{i+j}}(t^{\nu_{i+j+1}})\right)^{A_{i,i+j}} L_{i+n}^{A_{i,i+n}}(t) \end{aligned}$$

as $t\to\infty$, $i=1,\dots,n$, and taking into account that $L_{i+n}(t)=L_i(t)$. The second statement is immediate from the definition of h_i, $i=1,\dots,n$. Indeed, (22) and (25) result

$$\tilde{L}_i(t) \sim \frac{h_{i+1}^{\alpha_i}}{|\nu_i|} L_{a_i}(t)L_{i+1}^{\alpha_i}(t)L_{F_i}(t^{\nu_{i+1}}) = \frac{1}{|\nu_i|}L_{a_i}(t)\tilde{L}_{i+1}^{\alpha_i}(t)L_{F_i}(t^{\nu_{i+1}}).$$

□

The next lemma provides some properties of the constants ν_i, $i = 1, \dots, n$, defined by (21).

Lemma 5.3 *Let $(p_1, \dots, p_n)$ be the unique solution of the system*

$$\begin{cases} p_1 + \cdots + p_n = 1, \\ \alpha_i p_i + p_{i+2} = p_{i+1}(\alpha_{i+1} + 1), \quad i = 1, 2, \dots, n-2, n. \end{cases} \tag{45}$$

Then

$$p_i = \frac{1 + \sum_{k=i+1}^{n+i-1} A_{k,n+i}}{n + \sum_{k=1}^{n}(A_{k,k+1} + A_{k,k+2} + \cdots + A_{k,k+n-1})} > 0, \quad i = 1, \dots, n, \tag{46}$$

and the following identity holds

$$\nu_1 + \cdots + \nu_n = \frac{(\sigma_1 p_1 + \cdots + \sigma_n p_n + 1)}{1 - \xi}, \tag{47}$$

where ξ is defined by $p_1 + \cdots + p_{n-2} + p_{n-1}(\alpha_{n-1} + 1)$ and satisfies $\xi < 1$. Further, if $L_{F_1} \equiv \cdots \equiv L_{F_n} \equiv 1$, then

$$L_1(t) \cdots L_n(t) = \left(L_{a_1}^{p_1}(t) \cdots L_{a_n}^{p_n}(t)\right)^{\frac{1}{1-\xi}} \tag{48}$$

P r o o f. It is easy to check that $(p_1, \dots, p_n)$ given by (46) solves (45). Positivity of p_i is clear and the uniqueness can readily be seen when writing (45) in a matrix form. The inequality $\xi < 1$ is equivalent to the subhomogeneity assumption. A series of routine and tedious computations (where we can expand the explicit expressions for ν_i and p_i, and compare corresponding summands in the resulting formulas) shows identities (47) and (48). $\square$

Define

$$B_{i,j} = \begin{cases} A_{i,j} & \text{for } 1 \le i < j \le n, \\ \prod_{k=i}^{j-1+n} \alpha_k & \text{for } 1 \le j \le i \le j + n - 1 \le 2n - 1. \end{cases}$$

It is easy to verify that the following relations hold

$$\begin{aligned} B_{i,i} &= \alpha_1 \cdots \alpha_n, \\ B_{i,j} B_{j,i} &= \alpha_1 \cdots \alpha_n = B_{i,i}, \\ B_{i,j} B_{j,\ell} &= B_{i,\ell}, \end{aligned}$$

where $i, j, \ell \in \{1, \dots, n\}$, and for the last equality we assume $i < j < \ell$ or $\ell < i < j$ or $j < \ell < i$. Now let

$$\varrho_{i,j} = \nu_i - B_{i,j}\nu_j, \quad i, j = 1, \dots, n. \tag{49}$$

In the subsequent lemmas we derive several properties of the constants $\varrho_{i,j}$, $i, j = 1, \dots, n$.

Lemma 5.4 *Let $\nu_i < 0$ for $i = 1, \dots, n$. Then $\varrho_{i,j}$, $i, j = 1, \dots, n$, satisfy the following relations:*

(i) $\varrho_{i,i} < 0$ *for* $i = 1, \dots, n$.
(ii) *If there exist $i, j \in \{1, \dots, n\}$, $i \ne j$, such that $\varrho_{i,j} \ge 0$, then $\varrho_{j,i} < 0$.*
(iii) *If there exist $i, j, \ell \in \{1, \dots, n\}$, with $i < j < \ell$, such that $\varrho_{i,j} < 0$ and $\varrho_{j,\ell} \le 0$, then $\varrho_{i,\ell} < 0$.*
(iv) *If there exist $i, j, \ell \in \{1, \dots, n\}$, with $j < i < \ell$, such that $\varrho_{i,j} < 0$ and $\varrho_{\ell,j} \ge 0$, then $\varrho_{i,\ell} < 0$.*

P r o o f. (i) The claim follows from $B_{i,i} < 1$ which is in fact the subhomogeneity condition.
(ii) We have $\varrho_{j,i} = \nu_j - B_{j,i}\nu_i \le \nu_j - B_{j,i}B_{i,j}\nu_j = (1 - B_{j,j})\nu_j < 0$.
(iii) We have $\varrho_{i,\ell} = \nu_i - B_{i,\ell}\nu_\ell < B_{i,j}\nu_j - B_{i,\ell}\nu_\ell = B_{i,j}(\nu_j - B_{j,\ell}\nu_\ell) \le 0$.
(iv) We have $\varrho_{i,\ell} = \nu_i - B_{i,\ell}\nu_\ell < B_{i,j}\nu_j - B_{i,\ell}\nu_\ell = B_{i,\ell}(B_{\ell,j}\nu_j - \nu_\ell) \le 0$. $\square$

Lemma 5.5 *Let $\gamma_{i,j}$, $i, j = 1, \dots, n$, $n \ge 2$, be any numbers which obey the rules (i)-(iv) in Lemma 5.4. Then the matrix $(\gamma_{i,j})_{1 \le i,j \le n}$ has at least one column whose elements are negative.*

P r o o f. The proof can be done by induction with respect to n. For $n = 2$ the statement is clearly valid. Assume that the claim holds for $n-1$, i.e., for the leading principal submatrix $(\gamma_{i,j})_{1\le i,j\le n-1}$ of the matrix $(\gamma_{i,j})_{1\le i,j\le n}$. Thus, there exists $m \in \{1,\dots,n-1\}$ such that $\gamma_{i,m} < 0$ for $i = 1,\dots,n-1$. If $\gamma_{n,m} < 0$, then the statement is proved. If $\gamma_{n,m} \ge 0$, then $\gamma_{m,n} < 0$ by (ii) and we get: The inequalities $\gamma_{i,m} < 0$ for $i = 1,\dots,m-1$ and $\gamma_{m,n} < 0$ yield $\gamma_{i,n} < 0$ for $i = 1,\dots,m-1$ by (iii). The inequalities $\gamma_{i,m} < 0$ for $i = m+1,\dots,n-1$ and $\gamma_{n,m} \ge 0$ yield $\gamma_{i,n} < 0$ for $i = m+1,\dots,n-1$ by (iv). By (i), we get $\gamma_{n,n} < 0$. Altogether we obtain $\gamma_{i,n} < 0$ for $i = 1,\dots,n$. The assertion follows. □

Lemma 5.6 *The following equivalence holds:* $\nu_i < 0$ (> 0) *for all* $i = 1,\dots,n$ *if and only if* $\varrho_{i,m} < 0$ (> 0) *for all* $i = 1,\dots,n$ *and some* $m \in \{1,\dots,n\}$.

P r o o f. We prove only the case with negative ν_i and $\varrho_{i,m}$. The other case uses similar arguments; in particular, Lemma 5.4 and Lemma 5.5, which find applications here, can be proved in an analogous way also when the inequalities are reversed.

If. Let $\varrho_{i,m} < 0$ for all $i = 1,\dots,n$ and some $m \in \{1,\dots,n\}$. Then, from (49), $\nu_i < B_{i,m}\nu_m$ for $i = 1,\dots,n$. From the subhomogeneity condition, $1 > B_{m,m}$, and thus $\nu_m < 0$. This then clearly implies that $\nu_i < 0$ for all $i = 1,\dots,n$.

Only if. Assume $\nu_i < 0$ for all $i = 1,\dots,n$. Lemma 5.5 applied to $(\varrho_{i,j})_{1\le i,j\le n}$ yields that there exists $m \in \{1,\dots,n\}$ such that $\varrho_{i,m} < 0$ for all $i = 1,\dots,n$. □

Lemma 5.7 *The numbers* $\varrho_{i,j}$, $i,j = 1,\dots,n$, *defined by* (49), *satisfy the relations*

$$\begin{cases} \varrho_{j+k-1,j} = \sigma_{j+k-1} + 1 + \alpha_{j+k-1}\varrho_{j+k,j}, & k = 1,\dots,n-1, \\ \varrho_{j+n-1,j} = \sigma_{j+n-1} + 1, \end{cases} \tag{50}$$

$j = 1,\dots,n$.

P r o o f. First note that (21) can be written as

$$\begin{pmatrix} 1 & -\alpha_1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -\alpha_2 & 0 & \cdots & 0 \\ 0 & 0 & 1 & -\alpha_3 & \cdots & 0 \\ \vdots & & & & & \\ -\alpha_n & 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \\ \vdots \\ \nu_n \end{pmatrix} = \begin{pmatrix} \sigma_1 + 1 \\ \sigma_2 + 1 \\ \sigma_3 + 1 \\ \vdots \\ \sigma_n + 1 \end{pmatrix}.$$

This system can equivalently be written as any of the following n systems with $(n+1)\times(n+1)$ matrix $\mathcal{C}_j$

$$\mathcal{C}_j \begin{pmatrix} \nu_j \\ \nu_{j+1} \\ \vdots \\ \nu_{j+n-1} \\ \nu_j \end{pmatrix} = \begin{pmatrix} \sigma_j + 1 \\ \sigma_{j+1} + 1 \\ \vdots \\ \sigma_{j+n-1} + 1 \\ \nu_j \end{pmatrix}, \quad \text{that is} \quad \begin{pmatrix} \nu_j \\ \nu_{j+1} \\ \vdots \\ \nu_{j+n-1} \\ \nu_j \end{pmatrix} = \mathcal{C}_j^{-1} \begin{pmatrix} \sigma_j + 1 \\ \sigma_{j+1} + 1 \\ \vdots \\ \sigma_{j+n-1} + 1 \\ \nu_j \end{pmatrix},$$

where

$$\mathcal{C}_j = \begin{pmatrix} 1 & -\alpha_j & 0 & \cdots & 0 & 0 \\ 0 & 1 & -\alpha_{j+1} & \cdots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & 1 & -\alpha_{j+n-1} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

$j = 1,\dots,n$, and the inverse of $\mathcal{C}_j$ is

$$\mathcal{C}_j^{-1} = \begin{pmatrix} A_{j,j} & A_{j,j+1} & A_{j,j+2} & \cdots & A_{j,j+n-1} & A_{j,j+n} \\ 0 & A_{j+1,j+1} & A_{j+1,j+2} & \cdots & A_{j+1,j+n-1} & A_{j+1,j+n} \\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & A_{j+n-1,j+n-1} & A_{j+n-1,j+n} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

$j = 1, \ldots, n$. Let $\tilde{\mathcal{C}}_j$ denote the leading principal submatrix of $\mathcal{C}_j^{-1}$ (which is formed by deleting the last row and the last column), $j = 1, \ldots, n$. Then

$$\begin{pmatrix} \nu_j - A_{j,j+n}\nu_j \\ \nu_{j+1} - A_{j+1,j+n}\nu_j \\ \vdots \\ \nu_{j+n-1} - A_{j+n-1,j+n}\nu_j \end{pmatrix} = \tilde{\mathcal{C}}_j \begin{pmatrix} \sigma_j + 1 \\ \sigma_{j+1} + 1 \\ \vdots \\ \sigma_{j+n-1} + 1 \end{pmatrix}, \tag{51}$$

$j = 1, \ldots, n$. The claim now follows observing that (49) can be equivalently written as

$$\varrho_{j+k-1,j} = \nu_{j+k-1} - B_{j+k-1,j}\,\nu_j = \nu_{j+k-1} - A_{j+k-1,j+n}\,\nu_j,$$

$j, k = 1, \ldots, n$, and from the fact that (50) is equivalent to

$$\begin{pmatrix} \varrho_{j,j} \\ \varrho_{j+1,j} \\ \vdots \\ \varrho_{j+n-1,j} \end{pmatrix} = \tilde{\mathcal{C}}_j \begin{pmatrix} \sigma_j + 1 \\ \sigma_{j+1} + 1 \\ \vdots \\ \sigma_{j+n-1} + 1 \end{pmatrix}, \tag{52}$$

$j = 1, \ldots, n$. □

Now we are ready to prove our main theorems.

Proof of Theorem 4.1. (i) First we prove that (1) has at least one solution $\mathbf{x} = (x_1, \ldots, x_n) \in \mathcal{SDS}$. Since (5) is valid, from Proposition 3.2 for every $i = 1, \ldots, n$ there exists $\tilde{F}_i \in C(\mathbb{R}) \cap \mathcal{RV}_0(\alpha_i)$, nondecreasing, such that $\tilde{F}_i(u) \sim F_i(u)$ as $u \to 0$. Thus there exists $u_0 > 0$ such that

$$\frac{1}{\sqrt[4]{2}}\,\tilde{F}_i(u) \le F_i(u) \le \sqrt[4]{2}\,\tilde{F}_i(u), \quad \forall u \in [0, u_0], i = 1, \ldots, n. \tag{53}$$

Notice that $\tilde{F}_i$ satisfies conditions (24) and (26) if F_i does, for $i = 1, \ldots, n$. Let $(k_1, \ldots, k_n)$, be the unique solution of the linear system $k_i - \alpha_i k_{i+1} = 1$. System (21) reduces to this if $\sigma_i = 0$ for all $i = 1, \cdots, n$, and therefore, from (23), it results $k_i > 0$ for all i. Let $\tilde{L}_i(t) = h_i L_i(t)$, $i = 1, \ldots, n$, see Lemma 5.2. Now, taking into account that $\nu_i < 0$, $i = 1, \ldots, n$, properties iii) and iv) in Proposition 3.1, assumptions (42) and (44) imply that t_0 sufficiently large exists such that

$$2^{k_i}\tilde{L}_i(t)t^{\nu_i} \le u_0, \tag{54}$$

$$L_{\tilde{F}_i}(2^{k_{i+1}}\tilde{L}_{i+1}(t)t^{\nu_{i+1}}) \le \sqrt[4]{2}\,L_{\tilde{F}_i}(t^{\nu_{i+1}}), \tag{55}$$

$$L_{\tilde{F}_i}(2^{-k_{i+1}}\tilde{L}_{i+1}(t)t^{\nu_{i+1}}) \ge \frac{1}{\sqrt[4]{2}}\,L_{\tilde{F}_i}(t^{\nu_{i+1}}) \tag{56}$$

$$\frac{1}{\sqrt[4]{2}}\,\tilde{L}_i(t) \le \frac{1}{|\nu_i|}\,L_{a_i}(t)\tilde{L}_{i+1}^{\alpha_i}(t)L_{\tilde{F}_i}(t^{\nu_{i+1}}) \le \sqrt[4]{2}\,\tilde{L}_i(t), \tag{57}$$

$$\frac{1}{\sqrt[4]{2}}\,t^{\nu_i}\tilde{L}_i(t) \le |\nu_i|\int_t^\infty s^{\nu_i-1}\tilde{L}_i(s)\,ds \le \sqrt[4]{2}\,t^{\nu_i}\tilde{L}_i(t), \tag{58}$$

for all $t \in [t_0, \infty)$, and $i = 1, \ldots, n$. Let $\Omega \subset (C[t_0, \infty))^n = C[t_0, \infty) \times \cdots \times C[t_0, \infty)$ be the set defined as

$$\Omega = \{\mathbf{x} = (x_1, \ldots, x_n) :\, x_i \in C[t_0, \infty),\, 2^{-k_i}\tilde{L}_i(t)t^{\nu_i} \le x_i(t) \le 2^{k_i}\tilde{L}_i(t)t^{\nu_i},\, i = 1, \ldots, n\},$$

and let $\mathcal{T} : \Omega \to (C[t_0, \infty))^n$ be the operator defined by $\mathcal{T}\mathbf{x} = (T_1x_2, T_2x_3, \ldots, T_nx_1)$, with

$$(T_ix_{i+1})(t) = \int_t^\infty a_i(s)F_i(x_{i+1}(s))\,ds, \quad i = 1, \ldots, n.$$

First of all notice that $\mathcal{T}$ is well defined in Ω. Indeed, for all $\mathbf{x} \in \Omega$ we have

$$0 \le a_i(t)F_i(x_{i+1}(t)) \le \sqrt[4]{2}\, a_i(t)\tilde{F}_i(x_{i+1}(t)) \le \sqrt[4]{2}\, a_i(t)\tilde{F}_i(2^{k_{i+1}}\tilde{L}_{i+1}(t)t^{\nu_{i+1}}),$$

where we used (53). Since $\tilde{F}_i \in \mathcal{RV}_0(\alpha_i)$ and $a_i \in \mathcal{RV}(\sigma_i)$, the last term in the above inequality belongs to the class $\mathcal{RV}(\sigma_i + \alpha_i\nu_{i+1}) = \mathcal{RV}(\nu_i - 1)$, see (21), with $\nu_i < 0$, and therefore it is integrable on $[t_0, \infty)$. In particular, taking into account (55), (57), and (58), for every $t \ge t_0$ it holds

$$\begin{aligned}
(T_i x_{i+1})(t) &\le \sqrt[4]{2} \int_t^\infty a_i(s)\tilde{F}_i(2^{k_{i+1}}\tilde{L}_{i+1}(s)s^{\nu_{i+1}})\, ds \\
&= 2^{\alpha_i k_{i+1} + \frac{1}{4}} \int_t^\infty s^{\sigma_i} L_{a_i}(s)\tilde{L}_{i+1}^{\alpha_i}(s) s^{\alpha_i\nu_{i+1}} L_{\tilde{F}_i}(2^{k_{i+1}}\tilde{L}_{i+1}(s)s^{\nu_{i+1}})\, ds \\
&= 2^{k_i - \frac{3}{4}} \int_t^\infty s^{\nu_i - 1} L_{a_i}(s)\tilde{L}_{i+1}^{\alpha_i}(s) L_{\tilde{F}_i}(2^{k_{i+1}}\tilde{L}_{i+1}(s)s^{\nu_{i+1}})\, ds \\
&\le 2^{k_i - \frac{1}{2}} \int_t^\infty s^{\nu_i - 1} L_{a_i}(s)\tilde{L}_{i+1}^{\alpha_i}(s) L_{\tilde{F}_i}(s^{\nu_{i+1}})\, ds \\
&\le 2^{k_i - \frac{1}{4}}\, |\nu_i| \int_t^\infty s^{\nu_i - 1}\tilde{L}_i(s)\, ds \le 2^{k_i} t^{\nu_i}\tilde{L}_i(t).
\end{aligned}$$

Analogously, from (56), (57), and (58), for every $t \ge t_0$ it holds

$$\begin{aligned}
(T_i x_{i+1})(t) &\ge 2^{-\frac{1}{4}} \int_t^\infty a_i(s)\tilde{F}_i(2^{-k_{i+1}}\tilde{L}_{i+1}(s)s^{\nu_{i+1}})\, ds \\
&= 2^{-k_i + \frac{3}{4}} \int_t^\infty s^{\nu_i - 1} L_{a_i}(s)\tilde{L}_{i+1}^{\alpha_i}(s) L_{\tilde{F}_i}(2^{-k_{i+1}}\tilde{L}_{i+1}(s)s^{\nu_{i+1}})\, ds \\
&\ge 2^{-k_i + \frac{1}{2}} \int_t^\infty s^{\nu_i - 1} L_{a_i}(s)\tilde{L}_{i+1}^{\alpha_i}(s) L_{\tilde{F}_i}(s^{\nu_{i+1}})\, ds \\
&\ge 2^{-k_i + \frac{1}{4}}\, |\nu_i| \int_t^\infty s^{\nu_i - 1}\tilde{L}_i(s)\, ds \ge 2^{-k_i} t^{\nu_i}\tilde{L}_i(t).
\end{aligned}$$

Thus $\mathcal{T}$ maps Ω into itself. In order to apply the Schauder-Tychonoff fixed point theorem, we have to show that $\mathcal{T}$ is completely continuous in Ω. Since $\mathcal{T}(\Omega) \subseteq \Omega$, elements in $\mathcal{T}(\Omega)$ are equibounded on $[t_0, \infty)$; further, the inequality

$$0 \ge (T_i x_{i+1})'(t) \ge -\sqrt[4]{2}\, a_i(t)\tilde{F}_i(2^{k_{i+1}}\tilde{L}_{i+1}(t)t^{\nu_{i+1}})$$

which is valid for $t \ge t_0$, $i = 1, \ldots, n$, and for every $(x_1, \ldots, x_n) \in \Omega$, ensure that elements in $\mathcal{T}(\Omega)$ are equicontinuous. The relative compactness of $\mathcal{T}(\Omega)$ follows from the Ascoli-Arzelà theorem. To prove the continuity of $\mathcal{T}$ in Ω we have to show that for any sequence $\mathbf{x}^m = (x_1^m, \ldots, x_n^m)$ in Ω which converges to $\bar{\mathbf{x}} = (\bar{x}_1, \ldots, \bar{x}_n) \in \Omega$ as $m \to \infty$ uniformly on any compact subset of $[t_0, \infty)$, it holds $(\mathcal{T}\mathbf{x}^m)(t) \to (\mathcal{T}\bar{\mathbf{x}})(t)$ as $m \to \infty$ uniformly on compact subsets of $[t_0, \infty)$. This fact is a direct consequence of the Lebesgue dominated convergence theorem. Since all the assumptions of the Schauder-Tychonoff fixed point theorem are fulfilled, the operator $\mathcal{T}$ has (at least) a fixed point $\mathbf{x} \in \Omega$. This fixed point $\mathbf{x} = (x_1, \ldots, x_n)$ is a positive solution of (1), and from the definition of the set Ω it follows that $x_i(t) \to 0$ as $t \to \infty$, $i = 1, \ldots, n$, i.e., $\mathbf{x} \in \mathcal{SDS}$.

Now we prove that $x_i \in \mathcal{RV}(\nu_i)$, $i = 1, \ldots, n$. Since $\mathbf{x} \in \Omega$, we have that $x_i(t) \asymp t^{\nu_i}\tilde{L}_i(t)$ as $t \to \infty$, $i = 1, \ldots, n$. Taking into account that $\tilde{L}_i(\lambda t)/\tilde{L}_i(t) \to 1$ as $t \to \infty$ for every $\lambda > 0$, $i = 1, \ldots, n$, we can find $m_i, M_i \in (0, \infty)$, $i = 1, \ldots, n$, such that

$$m_i \le \tau_i(t) \le M_i, \text{ where } \tau_i(t) := \frac{x_i(\lambda t)}{x_i(t)}, \quad i = 1, \ldots, n, \tag{59}$$

for $t \geq t_0$, and so $\liminf_{t\to\infty} \tau_i(t) =: \underline{\Lambda}_i \in (0,\infty)$, $\limsup_{t\to\infty} \tau_i(t) =: \overline{\Lambda}_i \in (0,\infty)$. From the uniform convergence theorem for $\mathcal{SV}_0$ functions, we get

$$\begin{aligned}\left|\frac{L_{F_i}(x_{i+1}(\lambda t))}{L_{F_i}(x_{i+1}(t))} - 1\right| &= \left|\frac{L_{F_i}(\tau_{i+1}(t)x_{i+1}(t))}{L_{F_i}(x_{i+1}(t))} - 1\right| \\ &\leq \sup_{\xi\in[m_i,M_i]} \left|\frac{L_{F_i}(\xi x_{i+1}(t))}{L_{F_i}(x_{i+1}(t))} - 1\right| = o(1)\end{aligned}$$

as $t \to \infty$. Thus,

$$\begin{aligned}\underline{\Lambda}_i &\geq \liminf_{t\to\infty} \frac{\lambda x_i'(\lambda t)}{x_i'(t)} = \liminf_{t\to\infty} \frac{\lambda a_i(\lambda t)x_{i+1}^{\alpha_i}(\lambda t)}{a_i(t)x_{i+1}^{\alpha_i}(t)} \cdot \frac{L_{F_i}(x_{i+1}(\lambda t))}{L_{F_i}(x_{i+1}(t))} \\ &\geq \lambda^{1+\sigma_i}\left(\liminf_{t\to\infty} \frac{x_{i+1}(\lambda t)}{x_{i+1}(t)}\right)^{\alpha_i} \geq \lambda^{1+\sigma_i}\left(\liminf_{t\to\infty} \frac{\lambda x_{i+1}'(\lambda t)}{x_{i+1}'(t)}\right)^{\alpha_i} \\ &= \lambda^{1+\sigma_i}\left(\liminf_{t\to\infty} \frac{\lambda a_{i+1}(\lambda t)x_{i+2}^{\alpha_{i+1}}(\lambda t)}{a_{i+1}(t)x_{i+2}^{\alpha_{i+1}}(t)} \cdot \frac{L_{F_{i+1}}(x_{i+2}(\lambda t))}{L_{F_{i+1}}(x_{i+2}(t))}\right)^{\alpha_i} \\ &\geq \lambda^{1+\sigma_i+\alpha_i(1+\sigma_{i+1})}\left(\liminf_{t\to\infty} \frac{\lambda x_{i+2}'(\lambda t)}{x_{i+2}'(t)}\right)^{\alpha_i\alpha_{i+1}} \geq \dots \\ &\geq \lambda^{1+\sigma_i+A_{i,i+1}(1+\sigma_{i+1})+A_{i,i+2}(1+\sigma_{i+2})+\dots+A_{i,i+n-1}(1+\sigma_{i+n-1})}\left(\liminf_{t\to\infty} \tau_{i+n}(t)\right)^{A_{i,i+n}} \\ &= \lambda^{\nu_i(1-A_{i,i+n})}\underline{\Lambda}_i^{A_{i,i+n}},\end{aligned} \tag{60}$$

where we used (23). Realizing now that $A_{i,i+n} = \alpha_1 \cdots \alpha_n < 1$, we obtain $\underline{\Lambda}_i \geq \lambda^{\nu_i}$. Similarly we get $\overline{\Lambda}_i \leq \lambda^{\nu_i}$. This implies that there exists the limit $\lim_{t\to\infty} x_i(\lambda t)/x_i(t)$ and it is equal to λ^{ν_i}. Since λ was arbitrary, we get $x_i \in \mathcal{RV}(\nu_i)$, $i = 1,\dots,n$.

Finally, we establish asymptotic formula (29). We have $x_i(t) = t^{\nu_i}\bar{L}_i(t)$, $i = 1,\dots,n$, where $\bar{L}_i \in \mathcal{SV}$ has to be determined. Then, taking into account (21), (27), and (16), it results

$$\begin{aligned}t^{\nu_i}\bar{L}_i(t) &= \int_t^\infty a_i(s)F_i(x_{i+1}(s))\,ds = \int_t^\infty s^{\sigma_i}L_{a_i}(s)x_{i+1}^{\alpha_i}(s)L_{F_i}(x_{i+1}(s))\,ds \\ &= \int_t^\infty s^{\nu_i-1}L_{a_i}(s)\bar{L}_{i+1}^{\alpha_i}(s)L_{F_i}\left(s^{\nu_{i+1}}\bar{L}_{i+1}(s)\right)\,ds \\ &\sim \int_t^\infty s^{\nu_i-1}L_{a_i}(s)\bar{L}_{i+1}^{\alpha_i}(s)L_{F_i}\left(s^{\nu_{i+1}}\right)\,ds \sim \frac{1}{|\nu_i|}t^{\nu_i}L_{a_i}(t)\bar{L}_{i+1}^{\alpha_i}(t)L_{F_i}\left(t^{\nu_{i+1}}\right)\end{aligned}$$

as $t \to \infty$, $i = 1,\dots,n$. Hence, $\bar{L}_i(t) \sim \tilde{L}_i(t) = h_iL_i(t)$, $i = 1,\dots,n$, see Lemma 5.2, and (29) follows.

The proof of (ii) is similar and hence omitted. Just note that relation (17) finds its extensive application here. □

The following lemma is needed to prove Theorem 4.2.

Lemma 5.8 *Let* $u_1,\dots,u_n > 0$ *and* $p_1,\dots,p_n \geq 0$ *with* $p_1 + \dots + p_n = 1$. *Then*

$$\sum_{i=1}^n u_i \geq \prod_{i=1}^n u_i^{p_i}. \tag{61}$$

P r o o f. Inequality (61) follows from the generalized arithmetic mean-geometric mean inequality

$$\frac{1}{p}\sum_{i=1}^n p_iu_i \geq \left(\prod_{i=1}^n u_i^{p_i}\right)^{\frac{1}{p}},$$

which holds for any $p = p_1 + \cdots + p_n > 0$, by taking $p = 1$, and the fact that

$$\sum_{i=1}^{n} p_i u_i \le \max\{p_1, \ldots, p_n\} \sum_{i=1}^{n} u_i \le \sum_{i=1}^{n} u_i.$$

□

Proof of Theorem 4.2. (i) Take any $(x_1, \ldots, x_n) \in \mathcal{SDS}$, which indeed exists by Theorem 4.1. From (1), x_i satisfies the integral equation

$$x_i(t) = \int_t^\infty a_i(s) x_{i+1}^{\alpha_i}(s)\, ds, \tag{62}$$

$i = 1, \ldots, n$. From Lemma 5.6, there exists $m \in \{1, 2, \ldots, n\}$ such that $\varrho_{i,m} < 0$ for all $i = 1, \ldots, n$, where the constants $\varrho_{i,j}$ are defined by (49) and satisfy also (50). Iterating (62), starting from $i = m$ we get

$$\begin{aligned} x_m(t) &= \int_t^\infty a_m(s_1) x_{m+1}^{\alpha_m}(s_1)\, ds_1 = \int_t^\infty a_m(s_1) \left(\int_{s_1}^\infty a_{m+1}(s_2) x_{m+2}^{\alpha_{m+1}}(s_2)\, ds_2 \right)^{\alpha_m} ds_1 \\ &= \cdots = \int_t^\infty a_m(s_1) \left(\int_{s_1}^\infty a_{m+1}(s_2) \times \right. \\ &\quad \times \left. \left(\ldots \left(\int_{s_{n-1}}^\infty a_{m+n-1}(s_n) x_{m+n}^{\alpha_{m+n-1}}(s_n)\, ds_n \right)^{\alpha_{m+n-2}} \ldots \right)^{\alpha_{m+1}} ds_2 \right)^{\alpha_m} ds_1. \end{aligned}$$

Since $x_{m+n} = x_m$ is eventually decreasing, $x_{m+n}(s_n) \le x_m(t)$ for $s_n \ge t$, t being sufficiently large. Further, $a_{m+n-1}(t) = t^{\sigma_{m+n-1}} L_{a_{m+n-1}}(t)$, with $\sigma_{m+n-1} = \varrho_{m+n-1,m} - 1 < -1$, see (50) and Lemma 5.6. Thus we can apply (16) obtaining the existence of a positive constant k_n such that

$$\int_{s_{n-1}}^\infty a_{m+n-1}(s_n) x_{m+n}^{\alpha_{m+n-1}}(s_n)\, ds_n \le k_n x_m^{\alpha_{m+n-1}}(s_{n-1}) s_{n-1}^{\sigma_{m+n-1}+1} L_{a_{m+n-1}}(s_{n-1}).$$

We can proceed in a similar way for all the iterated integrals; note that (50) and Lemma 5.6 assure that we can apply (16) at each step. We obtain that $k \in (0, \infty)$ exists such that for t large

$$\begin{aligned} x_m(t) &\le x_m^{\alpha_m \cdots \alpha_{m+n-1}}(t) \times \\ &\quad \times \int_t^\infty s_1^{\sigma_m} L_{a_m}(s_1) \left(\ldots \left(\int_{s_{n-1}}^\infty s_n^{\sigma_{m+n-1}} L_{a_{m+n-1}}(s_n)\, ds_n \right)^{\alpha_{m+n-2}} \ldots \right)^{\alpha_m} ds_1 \\ &\le k x_m^{A_{m,m+n}}(t) L_{a_m}(t) L_{a_{m+1}}^{A_{m,m+1}}(t) \cdots L_{a_{m+n-1}}^{A_{m,m+n-1}}(t) t^{\nu_m(1-A_{m,m+n})} \end{aligned}$$

where we used the equality $\nu_m(1-A_{m,m+n}) = \sigma_m + 1 + A_{m,m+1}(\sigma_{m+1}+1) + \cdots + A_{m,m+n-1}\,(\sigma_{m+n-1}+1)$, see (23). Since $A_{m,m+n} = \alpha_1 \cdots \alpha_n < 1$, from (24) with $L_{F_1} \equiv \cdots \equiv L_{F_n} \equiv 1$, there exists $d_m \in (0, \infty)$ such that $x_m(t) \le d_m t^{\nu_m} L_m(t)$ for large t.

Now we show that $x_i(t) \le d_i t^{\nu_i} L_i(t)$ for large t and for all $i = 1, \ldots, n$, with $d_i > 0$. From the estimation for $x_m = x_{m+n}$, we can now easily get the estimation for x_{m+n-1}. Recall that (21) and (26) hold. From (62) and (16), in view of $\nu_m < 0$, we have

$$\begin{aligned} x_{m+n-1}(t) &= \int_t^\infty a_{m+n-1}(s) x_m^{\alpha_{m+n-1}}(s)\, ds \\ &\le d_m^{\alpha_{m+n-1}} \int_t^\infty s^{\sigma_{m+n-1}+\alpha_{m+n-1}\nu_m} L_{a_{m+n-1}}(s) L_m^{\alpha_{m+n-1}}(s)\, ds \\ &\le d_{m+n-1} t^{\nu_{m+n-1}} L_{a_{m+n-1}}(t) L_m^{\alpha_{m+n-1}}(t) = d_{m+n-1} t^{\nu_{m+n-1}} L_{m+n-1}(t) \end{aligned}$$

for large t, where d_{m+n-1} is a suitable positive constant. Repeating this process, and taking into account the modulo n convention, it results $x_i(t) \le d_i t^{\nu_i} L_i(t)$ for large t and for all $i = 1, \dots, n$.

Next we derive lower estimates for x_i's. For brevity we sometimes omit arguments. Again take any solution $(x_1, \dots, x_n) \in \mathcal{SDS}$. Then

$$\begin{aligned}-(x_1x_2\cdots x_n)' &= -\sum_{i=1}^n x_i' x_{i+1}\cdots x_{i+n-1} = \sum_{i=1}^n a_i x_{i+1}^{\alpha_i} x_{i+1}\cdots x_{i+n-1}\\ &= \sum_{i=1}^n H_i, \quad \text{where } H_i = a_i x_{i+1}^{\alpha_i} \prod_{k=1;k\ne i}^n x_k.\end{aligned}$$

Consider the (positive) numbers $p_1, \dots, p_n$ defined in Lemma 5.3. System (45) is equivalent to the system

$$\begin{cases} p_1 + \cdots + p_n = 1,\\ p_2 + p_3 + \cdots + p_{n-1} + p_n(\alpha_n + 1) = p_1(\alpha_1 + 1) + p_3 + \cdots + p_n,\\ p_1(\alpha_1+1) + p_3 + \cdots + p_n = p_1 + p_2(\alpha_2+1) + p_4 + \cdots + p_n,\\ \quad\vdots\\ p_1 + \cdots + p_{n-3} + p_{n-2}(\alpha_{n-2}+1) + p_n = p_1 + \cdots + p_{n-2} + p_{n-1}(\alpha_{n-1}+1).\end{cases} \tag{63}$$

From Lemma 5.8 we get that

$$\begin{aligned}-(x_1x_2\cdots x_n)' &= \sum_{i=1}^n H_i \ge \prod_{i=1}^n H_i^{p_i}\\ &= a_1^{p_1}a_2^{p_2}\cdots a_n^{p_n} x_1^{p_2+p_3+\cdots+p_{n-1}+p_n(\alpha_n+1)} x_2^{p_1(\alpha_1+1)+p_3+\cdots+p_n}\times\\ &\quad\times\cdots x_{n-1}^{p_1+p_2+\cdots+p_{n-2}(\alpha_{n-2}+1)+p_n} x_n^{p_1+p_2+\cdots+p_{n-2}+p_{n-1}(\alpha_{n-1}+1)}.\end{aligned}$$

Observe that except of the first equality all sides of the equalities in (63) are mutually equal and denote any of them by ξ; this is the same ξ as in Lemma 5.3. Then, from the last estimate, we get

$$-(x_1\cdots x_n)' \ge a_1^{p_1}\cdots a_n^{p_n}(x_1\cdots x_n)^{\xi}. \tag{64}$$

By Lemma 5.3, we have $\xi < 1$. Dividing (64) by $(x_1\cdots x_n)^\xi$ and integrating from t to ∞, we obtain

$$(x_1(t)\cdots x_n(t))^{1-\xi} \ge (1-\xi)\int_t^\infty a_1^{p_1}(s)\cdots a_n^{p_n}(s)\,ds. \tag{65}$$

From the upper estimates for x_i's we have

$$x_1(t)\cdots x_n(t) \le l_1 x_i(t) t^{\sum_{k=1;k\ne i}^n \nu_k} \prod_{k=1;k\ne i}^n L_k(t) \tag{66}$$

for large t, where $i \in \{1, \dots, n\}$ and l_1 is some positive number. Taking into account that $\sum_{i=1}^n \sigma_i p_i < -1$, see (47), from (16) there exists $l_2 \in (0, \infty)$ such that

$$\int_t^\infty a_1^{p_1}(s)\cdots a_n^{p_n}(s)\,ds \ge l_2 t^{\sigma_1p_1+\cdots+\sigma_np_n+1} L_{a_1}^{p_1}(t)\cdots L_{a_n}^{p_n}(t) \tag{67}$$

for large t. Combining (65), (66), and (67), we find $c_i \in (0, \infty)$ such that $x_i(t) \ge c_i t^{\bar\nu_i}\bar L_i(t)$ for large t, where

$$\bar\nu_i = \frac{1}{1-\xi}(\sigma_1p_1 + \cdots + \sigma_np_n + 1) - \sum_{k=1;k\ne i}^n \nu_k \quad\text{and}\quad \bar L_i = \frac{\left(L_{a_1}^{p_1}\cdots L_{a_n}^{p_n}\right)^{\frac{1}{1-\xi}}}{\prod_{k=1;k\ne i}^n L_k}.$$

Identities (47) and (48) now imply $\bar\nu_i = \nu_i$ and $\bar L_i(t) = L_i(t)$, $i = 1, \dots, n$. Thus we have proved $x_i(t) \asymp t^{\nu_i}L_i(t)$ as $t \to \infty$, $i = 1, \dots, n$. This implies (59). The property $x_i \in \mathcal{RV}(\nu_i)$, $i = 1, \dots, n$, and asymptotic formula (29) follow by the same arguments as those used in the second part of the proof of Theorem 4.1-(i).

(ii) This statement was proved in [20]. Note that from Lemma 5.6, under the assumptions of Theorem 4.2-(ii), the additional condition assumed in [20] is automatically satisfied. □

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